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The signless Laplacian spread[☆]

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ABSTRACT

The signless Laplacian spread of G is defined as $SQ(G) = \mu_1(G) - \mu_n(G)$, where $\mu_1(G)$ and $\mu_n(G)$ are the maximum and minimum eigenvalues of the signless Laplacian matrix of G, respectively. This paper presents some upper and lower bounds for $SQ(G)$. Moreover, the unique unicyclic graph with maximum signless Laplacian spread among the class of connected unicyclic graphs of order n is determined.

1. Introduction

In this paper, $G = (V, E)$ is an undirected simple graph with $|V| = n$ and $|E| = m$. Sometimes, we refer to G as an (n, m) graph. Let $d(u)$ denote the degree of u. Specially, $\Delta = \Delta(G)$ and $\delta = \delta(G)$ denote the maximum and minimum degree of vertices of G, respectively. If $d(u) = 1$, then we call u a pendant vertex of G. Suppose the degree of vertex v_i equals d_i for $i = 1, 2, \ldots, n$. Throughout this paper, we enumerate the degrees in non-increasing order, i.e., $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$. As usual, K_n, $K_{1,n-1}$ denote the complete graph and star of order n, respectively.

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Let $A(G)$ denote the adjacency matrix of G. Since $A(G)$ is symmetric, the eigenvalues of $A(G)$ can be arranged as follows:

$$\rho_1(G) \geqslant \rho_2(G) \geqslant \cdots \geqslant \rho_n(G).$$

The *adjacency spread* of the graph G is defined as (see [10]):

$$SA(G) = \rho_1(G) - \rho_n(G).$$

Let $D(G)$ be the diagonal matrix whose (i, i)-entry is d_i. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, and the *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$. Sometimes, $Q(G)$ is also called the unoriented Laplacian matrix of G (see, e.g. [7,24]).

It is well known that $L(G)$ is positive semidefinite so that its eigenvalues can be arranged as follows:

$$\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_{n-1}(G) \geqslant \lambda_n(G) = 0,$$

where $\lambda_{n-1}(G) > 0$ if and only if G is connected and is called the *algebraic connectivity* of the graph G. Let $\kappa(G)$ denote the *vertex connectivity* of G. If $G \not\cong K_n$, by Fiedler's famous inequality it follows that $\lambda_{n-1}(G) \leqslant \kappa(G)$. Because $\lambda_n(G) = 0$, the *Laplacian spread* of the graph G, denoted by $SL(G)$, is defined as [9]

$$SL(G) = \lambda_1(G) - \lambda_{n-1}(G).$$

The adjacency spread of a graph has received much attention. In [22], Petrović determined all connected graphs with adjacency spread at most 4. In [10,17], some lower and upper bounds for the adjacency spread of a graph were given. After that, the maximum adjacency spreads among all unicyclic graphs and all bicyclic graphs of given order n were determined in [8,23], respectively. However, the Laplacian spread seems less well-known because it was introduced somewhat later [9]. Up to now, there are only very limited results on the Laplacian spread. Firstly, the maximum and minimum Laplacian spreads among all trees of given order were identified in [9], and the maximum Laplacian spread among all unicyclic graphs was determined in [13]. After that, the four trees (resp. the three unicyclic graphs), which share the second to fifth (resp. the second to fourth) largest Laplacian spread among the trees (resp. connected unicyclic graphs) of order n were given in [14].

The matrix $Q(G)$ is symmetric and nonnegative, and, when G is connected, it is irreducible. If M is the $n \times m$ vertex-edge incidence matrix of the (n, m) graph G, then $Q(G) = MM^t$. Thus $Q(G)$ is positive semidefinite and its eigenvalues can be arranged as:

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) \geqslant 0.$$

Research on signless Laplacian matrices has become popular recently. Some properties of $Q(G)$ were studied in [1–3], all unicyclic graphs with first to 16th largest signless Laplacian spectral radii (namely, $\mu_1(G)$) in the class of connected unicyclic graphs of order n were identified in [6,18,25], and all bicyclic graphs with first to 11th largest signless Laplacian spectral radii in the class of connected bicyclic graphs of order n were identified in [7,15,26]. Recently, we determined the first to fourth largest signless Laplacian spectral radii among the class of connected tricyclic graphs of order n in [15].

Motivated by the definition of $SA(G)$ and $SL(G)$, we define the *signless Laplacian spread* of the graph G, denoted by $SQ(G)$, as

$$SQ(G) = \mu_1(G) - \mu_n(G).$$

The following result will be useful in the sequel

Proposition 1.1 [3]. *If G is connected, then $\mu_n(G) = 0$ if and only if G is bipartite. Moreover, if G is bipartite, then $Q(G)$ and $L(G)$ share the same eigenvalues.*

By Proposition 1.1, it immediately follows that

Proposition 1.2. *If G is a bipartite graph, then $\lambda_1(G) = \mu_1(G) = SQ(G)$.*

A graph G is called *k-regular* if $d_1 = \cdots = d_n = k$. If G is k-regular, it is easy to see that $\mu_1(G) = \rho_1(G) + k$ and $\mu_n(G) = \rho_n(G) + k$. Thus, we have

Proposition 1.3. *If G is regular, then* $SA(G) = SQ(G)$.

In this paper, we obtain some upper and lower bounds for $SQ(G)$, and determine the unique unicyclic graph with maximum signless Laplacian spread among the class of connected unicyclic graphs of order n.

2. Main results

We recall the notation of *majorization* (see [20]). Suppose $(x) = (x_1, x_2, \dots, x_n)$ and $(y) = (y_1, y_2, \dots, y_n)$ are two non-increasing sequences of real numbers, we say (x) is majorized by (y), denoted by $(x) \trianglelefteq (y)$, if and only if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, and $\sum_{i=1}^{j} x_i \leqslant \sum_{i=1}^{j} y_i$ for all $j = 1, 2, \dots, n$.

Proposition 2.1. *Let G be a graph with signless Laplacian spectrum* $(\mu) = (\mu_1, \mu_2, \dots, \mu_n)$ *and degree sequence* $(d) = (d_1, d_2, \dots, d_n)$. *Then,* $(d) \trianglelefteq (\mu)$.

Proof. It is well known that (see, e.g., [19, p. 218]) the spectrum of a positive semidefinite Hermitian matrix majorizes its main diagonal (when both are rearranged in non-increasing order). □

Remark 1. Let $(\widehat{d}) = (d_1 + 1, d_2, \dots, d_{n-1}, d_n - 1)$ and $(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_n)$. Grone (see [11]) proved that if G has at least one edge, then $(\widehat{d}) \trianglelefteq (\lambda)$. Unfortunately, it is not correct that $(\widehat{d}) \trianglelefteq (\mu)$. For example, if G is a connected non-bipartite graph with at least one pendant vertex, then $d_1 + 1 + d_2 + \cdots + d_{n-1} = 2m > \mu_1 + \mu_2 + \cdots + \mu_{n-1}$ because $\mu_n > 0$ by Proposition 1.1.

Corollary 2.1. *If* δ *is the minimum degree of vertices of graph G, then* $\mu_n \leqslant \delta$.

Proof. On the contrary, assume $\delta < \mu_n$. By Proposition 2.1 it follows that $(d) \trianglelefteq (\mu)$, then $d_1 + d_2 + \cdots + d_{n-1} \leqslant \mu_1 + \mu_2 + \cdots + \mu_{n-1}$. This implies that $2m = d_1 + d_2 + \cdots + d_n < \mu_1 + \mu_2 + \cdots + \mu_n = 2m$, a contradiction. □

Let $m(v) = \sum_{u \in N(v)} d(u)/d(v)$. The next result gives upper and lower bounds for $\mu_1(G)$.

Proposition 2.2. *Let G be a connected graph on n* $(n \geqslant 2)$ *vertices. Then,*

$$min\{d(v) + m(v) : v \in V(G)\} \leqslant \mu_1(G) \leqslant max\{d(v) + m(v) : v \in V(G)\},$$

where equality holds in either of these inequalities if and only if G is regular or semi-regular bipartite.

Proof. The upper bound is given in [5]. The lower bound can be proved in a similar way. For details, one can refer to [5]. □

Remark 2. If G is a connected bipartite graph, by Proposition 1.2 we can conclude that the bounds for $\mu_1(G)$ in Proposition 2.2 are also bounds for $SQ(G)$. Thus, Proposition 2.2 also gives bounds for $SQ(G)$ when G is a connected bipartite graph.

Lemma 2.1 [20,21]. *If G is a graph with at least one edge, then* $\mu_1 \geqslant \lambda_1 \geqslant \Delta + 1$. *If G is connected, the first equality holds if and only if G is bipartite, the second equality holds if and only if* $\Delta = n - 1$.

The next result gives bounds for $SQ(G)$ when G is a connected graph.

Theorem 2.1. *If G is a connected graph with maximum degree* Δ *and minimum degree* $\delta (\delta > 0)$, *then*

$$\Delta + 1 - \delta < SQ(G) \leqslant max\{d(v) + m(v) : v \in V(G)\},$$

where the upper bound holds if and only if G is regular bipartite or semi-regular bipartite.

Proof. Lemma 2.1 and Corollary 2.1 imply the strict lower bound. The upper bound follows from Proposition 2.2. By Propositions 1.1 and 2.2, the upper bound holds if and only if G is regular bipartite or semi-regular bipartite. □

Remark 3. By Proposition 2.2 and Corollary 2.1, we have $SQ(G) \geqslant min\{d(v) + m(v) : v \in V(G)\} - \delta$.

Let $\kappa'(G)$ denote the *edge connectivity* of G. It is well known that $\kappa(G) \leqslant \kappa'(G) \leqslant \delta(G)$. If $G \ncong K_n$, by Fiedler's inequality, then $\lambda_{n-1}(G) \leqslant \delta(G)$. Thus, by Lemma 2.1 it follows that

Remark 4. If G is a connected graph and $G \ncong K_n$, then $SL(G) \geqslant \Delta + 1 - \delta$.

A *semi-edge walk* (see [3]) of length k in an undirected graph G is an alternating sequence $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ of vertices $v_1, v_2, \ldots, v_{k+1}$ and edges $e_1, e_2, \ldots, e_k$ such that for any $i = 1, 2, \ldots, k$ the vertices v_i and v_{i+1} are end-vertices (not necessarily distinct) of the edge e_i.

Lemma 2.2 [3]. *The (i,j)-entry of the matrix $Q(G)^k$ is equal to the number of semi-edge walks of length k starting at vertex i and terminating at vertex j.*

The *distance* $d(u, v)$ between vertices u and v of a connected graph G is equal to the length of (number of edges in) a shortest path that connects u and v. The *diameter* of G, denoted by $\gamma(G)$, is $\gamma(G) = max\{d(u, v) : u, v \in V(G)\}$.

Proposition 2.3. *Let G be a connected graph with diameter $\gamma(G)$. If $Q(G)$ has exactly k distinct eigenvalues, then $\gamma(G) + 1 \leqslant k$.*

Proof. The proof of this result is similar with the corresponding theorem for the adjacency matrix. Assume the contrary holds, i.e., $\gamma(G) \geqslant k$. Then, there exist two vertices of G, say v_i, v_j, such that $d(v_i, v_j) = k$. Suppose the minimum polynomial of $Q(G)$ is $m_{Q(G)}(x)$. Since $Q(G)$ has exactly k distinct eigenvalues, then $m_{Q(G)}(x) = x^k + a_1x^{k-1} + \cdots$.

By Lemma 2.2, the (i,j)-entry of $Q(G)^k$ is positive. But the (i,j)-entry of $Q(G)^l$ is 0 for $1 \leqslant l < k$. This implies that $m_{Q(G)}(Q(G)) \neq O_n$ (O_n is the null matrix with all entries being 0), a contradiction. Thus, $\gamma(G) + 1 \leqslant k$. □

Let $M_1 = \sum_{i=1}^n d_i^2$. In [16], we have showed that

Lemma 2.3 [16]. *If G is a connected (n, m) graph, then $\mu_1(G) \geqslant \frac{M_1}{m}$, where the equality holds if and only if G is a regular graph or a bipartite semi-regular graph.*

Theorem 2.2. *If G is a connected (n, m) graph and $n \geqslant 2$, then*

$$SQ(G) \geqslant \frac{M_1}{m} - \sqrt{\frac{2m^3 + m^2M_1 - M_1^2}{(n-1)m^2}},$$

where equality holds if and only if $G \cong K_n$.

Proof. Because $tr(Q^2) = \sum_{i=1}^n \mu_i^2$, it follows that

$$(n-1)\mu_n^2 + \mu_1^2 \leqslant \sum_{i=1}^n \mu_i^2 = \sum_{i=1}^n d_i + \sum_{i=1}^n d_i^2 = 2m + M_1. \quad (1)$$

Thus, $0 \leqslant \mu_n \leqslant \sqrt{\frac{2m+M_1-\mu_1^2}{n-1}}$, from which we can conclude that

$$SQ(G) \geqslant \mu_1 - \sqrt{\frac{2m + M_1 - \mu_1^2}{n-1}}. \quad (2)$$

Let $f(x) = x - \sqrt{\frac{2m+M_1-x^2}{n-1}}$. It is easy to see that $f(x)$ is an increasing function when $x > 0$. By Lemma 2.3 and inequality (2), we have

$$SQ(G) \geqslant \mu_1 - \sqrt{\frac{2m + M_1 - \mu_1^2}{n-1}} \geqslant \frac{M_1}{m} - \sqrt{\frac{2m^3 + m^2 M_1 - M_1^2}{(n-1)m^2}}.$$

If equality holds, then equality must be taken in inequality (1). This implies that $\mu_2 = \cdots = \mu_n$. By Proposition 2.3, the diameter of G is 1. Thus, $G \cong K_n$.

Conversely, if $G \cong K_n$, then $\mu_1 = 2n - 2$, and $\mu_2 = \cdots = \mu_n = n - 2$. It is easy to check that equality holds in Theorem 2.2. $\square$

Given an n by n matrix $M_{n\times n}$ and an ordered partition $(X_1, \ldots, X_m)$ of the ordered set $\{1, 2, \ldots, n\}$, $M_{n\times n}$ can be presented as a partitioned matrix:

$$M_{n\times n} = \begin{pmatrix} M_{1,1} & \cdots & M_{1,m} \\ \cdots & & \cdots \\ M_{m,1} & \cdots & M_{m,m} \end{pmatrix},$$

where $M_{i,j}$ has X_i as the set of its row numbers and X_j as the set of its column numbers. We always use Q_M hereafter to denote the *quotient matrix* of the partitioned matrix $M_{n\times n}$, which is defined to be the m by m matrix whose entries are the average row sums of the blocks M_{ij}; that is, the (i, j)-entry of Q_M is obtained by dividing the sum of all row sums of $M_{i,j}$ by $|X_i|$, where $1 \leqslant i, j \leqslant m$.

Consider two sequences of real numbers: $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$, and $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_m$ with $m < n$. The second sequence is said to *interlace* the first one whenever $\alpha_i \geqslant \beta_i \geqslant \alpha_{n-m+i}$ for $i = 1, 2, \ldots, m$.

Lemma 2.4 [12]. *Suppose Q_M is the quotient matrix of a symmetric partitioned matrix M. Then, the eigenvalues of Q_M interlace the eigenvalues of M.*

Let $G = (V, E)$, if $\emptyset \neq V_1 \subseteq V(G)$, by the *average degree* of V_1, say d_0, we mean that $d_0 = \sum_{v\in V_1} d(v)/|V_1|$.

Theorem 2.3. *Let G be a connected (n, m) graph with $n \geqslant 2$. Suppose G contains a nonempty set T of t independent vertices, the average degree of which is d_0. Then,*

$$SQ(G) \geqslant \frac{1}{n-t}\sqrt{(nd_0)^2 + 8(m - td_0)(2m - nd_0)}.$$

Proof. The t independent vertices give rise to a partition of $Q(G)$ with quotient matrix $B = \begin{pmatrix} d_0 & d_0 \\ \frac{td_0}{n-t} & \frac{4m-3td_0}{n-t} \end{pmatrix}$. The eigenvalues of B are

$$\beta_1, \beta_2 = \frac{4m + nd_0 - 4td_0}{2(n-t)} \pm \frac{1}{2(n-t)}\sqrt{(nd_0)^2 + 8(m - td_0)(2m - nd_0)}.$$

By Lemma 2.4, $\mu_1 \geqslant \beta_1 \geqslant \beta_2 \geqslant \mu_n$, which implies the required inequality. $\square$

Remark 5. If G is k-regular, then $nd_0 = 2m$ and Theorem 2.3 gives $SA(G) = SQ(G) \geqslant \frac{nk}{n-t}$, where t and d_0 are denoted as in Theorem 2.3. Solving for t gives Hoffman's bound on t when G is k-regular:

$$t \leqslant \frac{n|\rho_n(G)|}{k - \rho_n(G)}.$$

Thus, Theorem 2.3 may be regarded as a generalization of Hoffman's bound to irregular graphs.

There are many graphs for which the bound in Theorem 2.3 is attained. For if G is k-regular, the bound is attained if and only if G has a set T of independent vertices that attains Hoffman's bound. Also, if $G = G(X, Y)$ is bipartite and T is either of its two vertex parts, then $m - td_0 = 0$ and Theorem 2.3 gives $\mu_1(G) = SQ(G) \geqslant \frac{nm}{t(n-t)}$. Here, $rank(B) = 1$ in the proof of Theorem 2.3, and equality holds in the bound if and only if G is semi-regular.

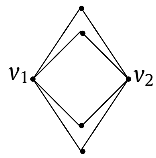

Fig. 1. The graph H.

By Theorem 2.3, it immediately follows that

Corollary 2.2. *Let p be the number of pendant vertices of G. If G is a connected (n, m) graph with $n > p \geqslant 1$, then*

$$SQ(G) \geqslant \frac{1}{n-p}\sqrt{n^2 + 8(m-p)(2m-n)}.$$

Equality holds, for example, if $G \cong K_{1,n-1}$ and $p = n - 1$.

If $d(u) = \Delta$, then u is also an independent set of G. By Theorem 2.3, we have

Corollary 2.3. *If G is a connected (n, m) graph and $n \geqslant 2$, then*

$$SQ(G) \geqslant \frac{1}{n-1}\sqrt{(n\Delta)^2 + 8(m-\Delta)(2m-n\Delta)}.$$

Equality holds, for example, if $G \cong K_n$.

By the proof of Theorem 2.3, we have the following remark.

Remark 6. If G is a connected (n, m) graph and contains $t (1 \leqslant t < n)$ independent vertices, the average degree of which is d_0, then

$$\mu_1 \geqslant \frac{4m + nd_0 - 4td_0}{2(n-t)} + \frac{1}{2(n-t)}\sqrt{(nd_0)^2 + 8(m-td_0)(2m-nd_0)}.$$

Also, with the same method as Corollary 2.3, we have

Remark 7. If G is a connected (n, m) graph and $n \geqslant 2$, then

$$\mu_1(G) \geqslant \frac{4m + n\Delta - 4\Delta}{2(n-1)} + \frac{1}{2(n-1)}\sqrt{(n\Delta)^2 + 8(m-\Delta)(2m-n\Delta)}.$$

Let G be a connected (n, m) graph. Suppose G contains $t (1 \leqslant t < n)$ independent vertices, the average degree of which is d_0. Then, the t independent vertices give rise to a partition of $L(G)$ with quotient matrix $B = \begin{pmatrix} d_0 & -d_0 \\ \frac{-td_0}{n-t} & \frac{td_0}{n-t} \end{pmatrix}$. It can be proved analogously with Theorem 2.3 that

Remark 8. If G is a connected (n, m) graph and contains $t (1 \leqslant t < n)$ independent vertices, the average degree of which is d_0, then $\lambda_1 \geqslant \frac{nd_0}{n-t}$. In particular, if G is a connected k-regular graph, then $\lambda_1 \geqslant \frac{nk}{n-t}$. Equality holds, for example, if $G \cong K_n$.

As shown in the next example, the bounds in Remarks 7 and 8 are sometimes better than the bounds in Lemma 2.1.

Example 2.1. Let H be the graph as shown in Fig. 1. Clearly, $T = \{v_1, v_2\}$ is an independent vertex set, and $d_0 = 4$. By Remark 8, it follows that $\lambda_1(H) \geqslant \frac{nd_0}{n-t} = 6 > 5 = \Delta + 1$. Actually, $\lambda_1(H) = 6$. Thus, the bound in Remark 8 can be attained. If we replace Δ by 4 in Remark 7, then we have $\mu_1(G) > 5.78$, which is also better than $\mu_1(G) \geqslant \Delta + 1 = 5$ in Lemma 2.1.

Proposition 2.4. *Suppose G has two induced subgraphs G_1 and G_2, where G_i has n_i vertices and e_i edges for $i = 1, 2$, $V(G_1) \cap V(G_2) = \emptyset$ and $n_1 + n_2 = n$. Let $a_1 = \sum_{v \in V_1} d(v)/n_1$ and $a_2 = \sum_{v \in V_2} d(v)/n_2$ then*

$$SQ(G) \geqslant \sqrt{\left(a_1 + a_2 + \frac{2e_1}{n_1} + \frac{2e_2}{n_2}\right)^2 - 16\left(\frac{a_2 e_1}{n_1} + \frac{a_1 e_2}{n_2}\right)}.$$

Proof. Note that $Q(G)$ has B as its quotient matrix, where $B = \begin{pmatrix} a_1 + \frac{2e_1}{n_1} & a_1 - \frac{2e_1}{n_1} \\ a_2 - \frac{2e_2}{n_2} & a_2 + \frac{2e_2}{n_2} \end{pmatrix}$. Obviously, B has two eigenvalues

$$\beta_1, \beta_2 = \frac{1}{2}\left(a_1 + a_2 + \frac{2e_1}{n_1} + \frac{2e_2}{n_2} \pm \sqrt{\left(a_1 + a_2 + \frac{2e_1}{n_1} + \frac{2e_2}{n_2}\right)^2 - 16\left(\frac{a_2 e_1}{n_1} + \frac{a_1 e_2}{n_2}\right)}\right).$$

Then Lemma 2.4 implies the result. □

The join of two vertex disjoint graphs G_1, G_2 is the graph $G_1 \vee G_2$ obtained from their union by including all edges between the vertices in G_1 and the vertices in G_2.

Corollary 2.4. *Suppose $G = G_1 \vee G_2$, where each G_i is a graph with n_i vertices and e_i edges for $i = 1, 2$. Then,*

$$SQ(G) \geqslant \sqrt{\left(n + \frac{4e_1}{n_1} + \frac{4e_2}{n_2}\right)^2 - 16\left(e_1 + e_2 + \frac{4e_1 e_2}{n_1 n_2}\right)}.$$

Equality holds, for example, if $G \cong K_n$.

Proof. Note that $G = G_1 \vee G_2$, then $a_1 - \frac{2e_1}{n_1} = n_2$ and $a_2 - \frac{2e_2}{n_2} = n_1$. By Proposition 2.4, the conclusion follows. When $G \cong K_n$, it is readily checked that equality holds because $SQ(K_n) = n$. □

Recall that the Cartesian product $G \,\square\, H = F(V, E)$ of graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ has vertex set $V = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) is adjacent in F if and only if $u_1 = v_1, u_2 v_2 \in E_2$ or $u_2 = v_2, u_1 v_1 \in E_1$. Let $A \bigotimes B$ denote the Kronecker product of matrix $A_{m \times m}$ and $B_{n \times n}$ (the definition can be found in [4, p. 250]).

Lemma 2.5 [4]. *Suppose the eigenvalues of $A_{m \times m}$ and $B_{n \times n}$ are $s_1, s_2, \ldots, s_m$ and $l_1, l_2, \ldots, l_n$, respectively. Then, the eigenvalues of $A \bigotimes I_n + I_m \bigotimes B$ are*

$$s_i + l_j, \quad \text{where } i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n.$$

Since $Q(G \,\square\, H) = Q(G) \bigotimes I_{|V(H)|} + I_{|V(G)|} \bigotimes Q(H)$, a straightforward application of Lemma 2.5 yields the following result for the eigenvalues of $Q(G \,\square\, H)$.

Proposition 2.5. *The eigenvalues of $Q(G \,\square\, H)$ are*

$$\mu_i(G) + \mu_j(H), \text{ where } 1 \leqslant i \leqslant |V(G)|, 1 \leqslant j \leqslant |V(H)|.$$

By Propositions 1.1 and 2.5, we have

Remark 9. Suppose $F = G \,\square\, H$, where G and H are connected, then F is bipartite if and only if both G and H are bipartite.

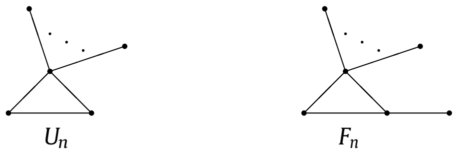

Fig. 2. The unicyclic graphs U_n and F_n.

Theorem 2.4. *Suppose G and H are two connected graphs, then*

$$SQ(G \,\Box\, H) = SQ(G) + SQ(H).$$

Proof. As a consequence of Proposition 2.5, we have

$$\mu_1(G \,\Box\, H) = \mu_1(G) + \mu_1(H), \mu_n(G \,\Box\, H) = \mu_n(G) + \mu_n(H).$$

Thus, $SQ(G \,\Box\, H) = SQ(G) + SQ(H)$ follows. □

In the following, let $\mathcal{U}_n$ denote the class of connected unicyclic graphs of order n. Let U_n, F_n be the unicyclic graphs as shown in Fig. 2.

Lemma 2.6 [18]. *If $n \geqslant 8$ and $G \in \mathcal{U}_n \setminus \{U_n\}$, then $\mu_1(G) \leqslant \mu_1(F_n)$, where equality holds if and only if $G \cong F_n$.*

Theorem 2.5. *If $n \geqslant 8$ and $G \in \mathcal{U}_n \setminus \{U_n\}$, then $SQ(U_n) > SQ(G)$.*

Proof. Let $\Phi(G, x)$=det$(xI - Q(G))$ denote the signless Laplacian characteristic polynomial of G. By a straightforward computation, we have

$$\Phi(U_n, x) = (x-1)^{n-3}\varphi_1(x), \tag{3}$$

$$\Phi(F_n, x) = (x-1)^{n-5}\varphi_2(x), \tag{4}$$

where $\varphi_1(x) = x^3 - (n+3)x^2 + 3nx - 4$, and $\varphi_2(x) = x^5 - (n+5)x^4 + (6n+3)x^3 - (9n-1)x^2 + (3n+8)x - 4$. By Lemma 2.6, we only need to prove that $SQ(U_n) > \mu_1(F_n)$ because $\mu_1(G) \leqslant \mu_1(F_n)$ and $\mu_n(G) \geqslant 0$.

Since $\varphi_1(0) = -4 < 0$, $\varphi_1(0.2) = 0.56n - 4.112 > 0$, $\varphi_1(n) = -4 < 0$ and $\varphi_1(n+1) = n^2 - n - 6 > 0$, by equality (3) it follows that $0 < \mu_n(U_n) < 0.2$ and $n < \mu_1(U_n) < n+1$. Thus,

$$SQ(U_n) = \mu_1(U_n) - \mu_n(U_n) > n - 0.2.$$

Since $\varphi_2(0) = -4 < 0, \varphi_2(0.3) > 0.2439n - 1.468 > 0, \varphi_2(1) = 4 - n < 0, \varphi_2(2) = 2n - 8 > 0$, $\varphi_2(6) = 2024 - 306n < 0$ and $\varphi_2(n - 0.2) = 0.8n^4 - 5.44n^3 + 5.272n^2 + 7.1184n - 5.59232 > 0$, then $6 < \mu_1(F_n) < n - 0.2$by equality (4). Thus,

$$\mu_1(F_n) < n - 0.2 < SQ(U_n).$$

This completes the proof of this result. □

3. Concluding remarks

A number of questions have been left unresolved. Here, we present some of them for further study.

Problem A. If G is regular, by Proposition 1.3 $SA(G) = SQ(G)$. By examining the spectra of $A(G)$ and $Q(G)$ for graphs on five vertices (for instance, see [2, pp. 273–275] and [3]), we see that the inequality $SA(G) \leqslant SQ(G)$ often holds. But for the graph W_1 shown in Fig. 3, we have $SA(W_1) > 5.744 > 5.657 > SQ(W_1)$. It is natural to consider when the strict inequality $SA(G) > SQ(G)$ is necessary and when it is sufficient.

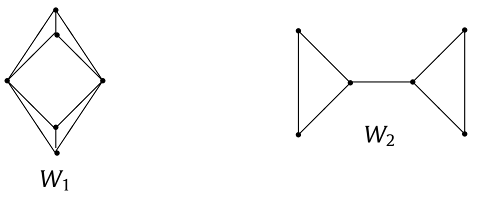

Fig. 3. The graphs W_1 and W_2.

Problem B. Proposition 1.2 implies that $SL(G) \leqslant SQ(G)$ always holds for bipartite graphs G. But for the graph W_2 depicted in Fig. 3, $SL(W_2) > 4.123 > 4 = SQ(W_2)$. Thus, we could also consider conditions for the inequality $SL(G) > SQ(G)$ to hold.

Problem C. In Theorem 2.5, we determine the unicyclic graph with maximum signless Laplacian spread among all connected unicyclic graphs of order n. But the graphs which share the maximum signless Laplacian spread among all connected graphs of order n are still unknown.

Let K_n^1 be the graph on n vertices obtained by attaching a pendant vertex to K_{n-1}. Then, $SQ(K_n^1) = \sqrt{4n^2 - 20n + 33}$. So, $SQ(K_n^1) \leqslant 2n - 4$ when $n \geqslant 5$. A computer run on connected graphs G of order n for $3 \leqslant n \leqslant 8$ indicates that if $n \neq 4$ then $SQ(G) \leqslant SQ(K_n^1)$ and that, when $n = 6, 7, 8$, equality is attained only when $G = K_n^1$. Note that if G is disconnected, then it is straightforward to check that $SQ(G) \leqslant 2n - 4$ and the equality is attained only if $G = K_{n-1} + K_1$, the complete graph on $n - 1$ vertices together with a single isolated vertex. Because of the computer run and because K_n^1 is obtained from the disconnected maximizer $K_{n-1} + K_1$ by adding a single edge, it seems likely that $SQ(G) \leqslant SQ(K_n^1)$ for connected graphs G of order $n \geqslant 5$.

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