Stability of Periodic Linear Systems by a Perturbation Method

GENE W. ARNOLD

Ferris State College,
Big Rapids, Michigan 49307

Submitted by J. P. LaSalle

INTRODUCTION

Floquet's theorem implies for a linear system \( \dot{x} = A(t)x \), where \( A(t) \) is periodic, the existence of a nonsingular matrix \( P(t) \) so that \( x = P(t)y \) transforms the original system into a system \( \dot{y} = By \) with constant coefficients. It will be shown, in the following development, how to construct, by a perturbation method and for a special class of matrices \( A(t) \), the matrices \( P(t) \) and \( B \) to any desired order of \( \varepsilon \).

The resulting expansion for \( B \) will then be used to solve the problem of determining stability boundaries for Hill's equation. Any future reference to Hill's equation will denote the equation,

\[
\ddot{x} + (\lambda + \varepsilon Q(t))x = 0.
\]

Here \( \lambda > 0 \) is real, \( \varepsilon \) is a real parameter and \( Q(t) \) is a real periodic function of period \( \pi \). Any solution \( x(t) \) of (1) is said to be stable if it is bounded for all time \( t > 0 \). Finally, the problem of determining stability boundaries (or intervals) is that of determining for what values of \( \lambda \) and \( \varepsilon \) the solutions of (1) will be stable (bounded).

Before the above problem can be approached, it is necessary to first present a development of the perturbation method which will be used.

In what follows, capital letters are assumed to be linear operators on a normed linear space \( T \). \( A + B \) and \( A \cdot B \) are defined pointwise and \( \mathcal{L}_0 \) is defined to be the set of all such linear operators which are continuous with respect to the norm: \( |A| = \sup_{|x| \leq 1} |A(x)| \). Further suppose that \( \cdot, \cdot \) is a bracket operator on \( \mathcal{L}_0 \), so that \( \mathcal{L}_0 \) is a Lie algebra. \( adA \) is an operator from \( \mathcal{L}_0 \to \mathcal{L}_0 \) defined by \( adA(X) = [A,X] \). \( e^{adA} \) is defined by \( e^{adA}X = \sum_{n=0}^{\infty} (adA)^nX/n! \), which converges if \( A \) and \( X \) are bounded, and there is a real constant, \( c \), such that \( ||[A, B]|| \leq c |A| |B| \), for all \( A \) and \( B \in \mathcal{L}_0 \).

Now define a new Lie algebra, \( \mathcal{L}_0 \), consisting of continuous functions from \( R \) into \( \mathcal{L}_0 \), which are periodic in \( t \) with period \( \pi \). Adjoin an element \( H_0 \) to

268
and extend \([ \cdot, \cdot ]\) linearly so that \([H_0, A] = \dot{A} = (d/dt)A\), for those elements \(A \in \mathcal{L}\) belonging to the dense subalgebra of \(C^1\) functions. Clearly the null space of \(adH_0\) is \(\mathcal{L}_0\), and the space of functions in \(\mathcal{L}\) satisfying \(\int_0^T A(t) \, dt = 0\), is a space complementary to the null space. It will be shown how to construct elements \(S\) and \(K\) in \(\mathcal{L}\) such that

\[
e^{iadS}(H_0 + \varepsilon V) = H_0 + \varepsilon K, \quad [H_0, K] = 0 \in \mathcal{L},
\]  

that is \(K \in \mathcal{L}_0\), and \(S\) is a \(C^1\) function in \(\mathcal{L}\).

In a later section it will be shown that the \(S\) and \(K\) so constructed will enable Hill's equation to be transformed to a linear equation of the form \(\dot{y} = By\) where \(B\) is a constant \(2 \times 2\) matrix.

The following is due to Kummer [8]. Set

\[
X = \frac{1}{\varepsilon} (e^{iadS} - ieadS - 1)H_0 + \frac{1}{\varepsilon} (e^{iadS} - 1)V.
\]  

Then (2) is equivalent to

\[
 iadH_0 S + K - V + \varepsilon X,
\]  
as can be easily seen by substitution of \(X\) as defined in Eq. (3) into Eq. (4).

It will now be shown how to construct explicitly \(S, X\) and \(K\) as power series in \(\varepsilon\).

Suppose now,

\[
S = \sum_{n=0}^{\infty} S^{(n)} \varepsilon^n,
\]  

\[
K = \sum_{n=0}^{\infty} K^{(n)} \varepsilon^n,
\]  

and

\[
X = \sum_{n=0}^{\infty} X^{(n)} \varepsilon^n,
\]

then by substitution in (4) and equating coefficients of similar powers of \(\varepsilon\), the following is obtained.

\[
iadH_0 S^{(0)} + K^{(0)} = V,
\]  

and

\[
iadH_0 S^{(n)} + K^{(n)} = X^{(n-1)}.
\]
If \(^{\top}\) indicates projection onto \(\mathcal{N}\), the null space of \(adH_0\), and if further, \(^{\top}\) indicates projection onto a complementary space, then it is seen that

\[
K^{(n)} = \hat{X}^{(n-1)},
\]
\[
S^{(n)} = (iadH_0)^{-1} \hat{X}^{(n-1)}
\]

is the unique solution of \(iadH_0 S^{(n)} = \hat{X}^{(n-1)}\) satisfying \(S^{(n)} = 0\). Further \(S^{(n)}\) is a \(C^1\) function of \(t\) since \((iadH_0)^{-1}\) is the integration operator. Applying these recursion formulas; \(K^{(0)}, S^{(0)}, X^{(0)}, K^{(1)}, S^{(1)}, X^{(1)}, \ldots\) can all be calculated in sequence. Explicitly:

\[
K^{(0)} = \hat{\nu},
\]
\[
S^{(0)} = (iadH_0)^{-1}(V - \hat{\nu}),
\]
\[
X^{(0)} = iadS^{(0)} \left( \frac{V + \hat{\nu}}{2} \right),
\]

and

\[
K^{(1)} = \frac{i}{2} adS^{(0)} \nu,
\]

since \([S^{(0)}, \hat{\nu}] = 0\). Further

\[
X^{(1)} = \frac{i}{2} (adS^{(0)} \hat{X}^{(0)} + adS^{(1)} \hat{\nu}) + \frac{i}{2} adS^{(1)} \nu - \frac{1}{12} (adS^{(0)})^2 \hat{\nu},
\]

so that

\[
K^{(2)} = \hat{X}^{(1)} = \frac{i}{2} adS^{(1)} \nu - \frac{1}{12} (adS^{(0)})^2 \hat{\nu}
\]

or equivalently,

\[
K^{(2)} = \frac{1}{2} (iadS^{(0)})^2 \hat{\nu} + \frac{1}{3} (iadS^{(0)})^2 \hat{\nu}.
\]

Equations (12), (13) and (15) give explicitly the zeroth, first, and second order terms of \(K\). In principle this procedure could be used to calculate \(K\) to any arbitrary order of \(\varepsilon\).
Consider the system
\[ Z = i\mathcal{A}Z, \quad \text{where} \quad \mathcal{A} = i \begin{pmatrix} -B & -\bar{A} \\ A & B \end{pmatrix} \tag{16} \]
with \( A \) a complex and \( B \) a real valued, continuous, periodic function of \( t \), both with period \( \pi \), the expansions developed in the previous section are valid for all \( t > 0 \), and further that as higher order terms are calculated, a better approximation is indeed obtained.

We define now the real Lie algebra \( su(1, 1) \) and the Lie group \( SU(1, 1) \). \( su(1, 1) \) is defined to be the linear space of \( 2 \times 2 \) matrices, \( \mathcal{A} \), having the form \((-ix \, ix \, i \), where \( x \) is a real and \( z \) a complex-valued function of \( t \), with the bracket operator \([\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}, \mathcal{A} \in su(1, 1) \) if and only if \( \mathcal{A} \) is a complex \( 2 \times 2 \) matrix satisfying \( \mathcal{A}^*\sigma + \sigma \mathcal{A} = 0 \) and \( \text{Tr} \mathcal{A} = 0 \), where \( \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

\( SU(1, 1) \) is defined to be the group, with respect to matrix multiplication, of \( 2 \times 2 \) matrices \( \mathcal{H} \) satisfying \( \mathcal{H}^*\sigma \mathcal{H} = \sigma \), and \( \det \mathcal{H} = 1 \). We will list now some useful properties of \( su(1, 1) \) and \( SU(1, 1) \). \( \mathcal{H} \in SU(1, 1) \) if and only if \( \mathcal{H} \) has the form \( \begin{pmatrix} b & a \\ \bar{a} & \bar{b} \end{pmatrix} \) with \( |b|^2 - |a|^2 = 1 \) and further, if \( \mathcal{A} \in su(1, 1) \) then \( e^t \mathcal{A} \in SU(1, 1) \).

So that the method of the previous section can be applied to Eq. (16), we define \( \mathscr{L} \) to be the set of all expressions of the form \( A \oplus B \) where \( B \) is a real valued and \( A \) a complex valued continuous function of period \( \pi \). Operations are defined on \( \mathscr{L} \) as follows:

\[(A \oplus B) + (C \oplus D) = (A + C) \oplus (R + D),\]
\[\lambda(A \oplus B) = \lambda A \oplus \lambda B\]

for \( \lambda \) a real number and

\[i[A \oplus B, C \oplus D] = 2 \left\{ i \begin{vmatrix} A & B \\ C & D \end{vmatrix} \oplus \text{Im} \bar{A}C \right\} .\]

With the above operations \( \mathscr{L} \) is a real Lie algebra. Define a mapping from \( su(1, 1) \) into \( \mathscr{L} \) by

\[i \begin{pmatrix} -B & -\bar{A} \\ A & B \end{pmatrix} \leftrightarrow A \oplus B,\]

then direct calculation yields

\[-[\mathcal{A}, \mathcal{B}] \leftrightarrow i[A \oplus B, C \oplus D],\]
where

\[ \mathcal{A} = i \begin{pmatrix} -B & -\bar{A} \\ A & B \end{pmatrix} \quad \text{and} \quad \mathcal{B} = i \begin{pmatrix} -D & -\bar{C} \\ C & D \end{pmatrix} \]

so that \( \leftrightarrow \) is a one-to-one homomorphism from \( su(1,1) \) into \( \mathcal{L}_0 \). If we associate with the element \( H_0 \), adjoined to \( \mathcal{L} \), and defined by \( i[H_0, A \oplus B] = \dot{A} \oplus \dot{B} \), for \( A \) and \( B \in C^1 \), the element \( w \) adjoined to the Lie algebra of continuous functions from \( R \) into \( su(1,1) \), then

\[ -[w, \mathcal{A}] = i[H_0, A \oplus B] \]

so that

\[ -[w, \mathcal{A}] = [\mathcal{A}, w] = \mathcal{A}. \]

The proof of convergence of the perturbation method as applied to (16) is actually based on the following results, due to Floquet, modified to fit the Lie algebra \( su(1,1) \).

**Theorem 1.** Every fundamental matrix solution \( X(t) \) of

\[ \dot{x} = A(t)x \quad \text{with} \quad A(t + \pi) = A(t), \tag{17} \]

where \( A(t) \) is a continuous \( n \times n \) matrix, has the form

\[ X(t) = P(t)e^{Bt}, \]

where \( P(t), B \) are \( n \times n \) matrices, \( P(t + \pi) = P(t) \), and \( B \) is constant.

**Proof.** Suppose \( X(t) \) is a fundamental solution of (17), then \( X(t + \pi) \) is also a fundamental solution, so

\[ X(t + \pi) = X(t) \cdot C, \]

where \( C \) is a nonsingular matrix.

Let \( B \) be a matrix such that

\[ C = e^{\pi B}, \]

that is \( \pi B \) is a logarithm of \( C \). Let \( P(t) = X(t) e^{-Bt} \). Then

\[ P(t + \pi) = X(t + \pi) e^{-B(t + \pi)} = X(t) e^{B\pi} e^{-B(t + \pi)} = P(t). \]

**Corollary 1.** There exists a nonsingular transformation of period \( \pi \) which transforms (17) into an equation with constant coefficients.
Proof. Suppose $P(t)$ and $B$ are as in Theorem 1. Let $x = P(t)y$, then

$$\dot{x} = \dot{P}y + P\dot{y} = APy$$

so

$$\dot{y} = P^{-1}(AP - \dot{P})y$$

and since

$$P = Xe^{-Bt}, \quad \dot{P} = AP - PB$$

so that

$$P^{-1}(AP - \dot{P}) = B,$$

which is constant.

The convergence of the perturbation expansion developed in the previous section will now be proven for systems having the form (16).

Lemma 1. If $Z(t)$ is a fundamental solution of (16) with $Z(0) \in SU(1,1)$ then $Z(t) \in SU(1,1)$ for all $t$.

Proof. Since $\mathcal{A}^*\sigma + \sigma \mathcal{A} = 0$, it follows that

$$\dot{Z} = -\sigma \mathcal{A}^* \sigma Z$$

or

$$(\sigma \dot{Z}) = -\mathcal{A}^*(\sigma Z),$$

which is the adjoint of (16). Whence $(\sigma Z)^*Z = C$, where $C$ is a constant matrix. If $Z_0(0) = I \in SU(1,1)$, then $Z_0^*\sigma Z_0 = \sigma$ for all $t$ and

$$\det(Z_0) = \exp \int_0^t \text{tr} \mathcal{A}(s) \, ds = 1$$

so that $Z_0 \in SU(1,1)$ for all $t$. If $Z$ is any other fundamental solution with $Z(0) \in SU(1,1)$ then

$$Z(t) = Z_0(t) Z(0) \in SU(1,1) \quad \text{for all } t.$$ 

Let $Z_0(t, \varepsilon)$ be the fundamental solution of (16) which satisfies $Z_0(0, \varepsilon) = Z_0(t, 0) = I$. Any fundamental solution of (16) with property $Z(t, 0) = I$ may be written as

$$Z(t, \varepsilon) = Z_0(t, \varepsilon) e^{-\varepsilon^2 Z(t)}.$$
where \( \mathcal{L}(\varepsilon) \in \mathfrak{su}(1, 1) \) is analytic in \( \varepsilon \) at \( \varepsilon = 0 \), further \( Z(t, \varepsilon) \) is analytic in \( \varepsilon \) at \( \varepsilon = 0 \) for each fixed \( t \).

**Lemma 2.** There exist matrices \( \mathcal{S}(t, \varepsilon), \mathcal{L}(\varepsilon), \mathcal{R}(\varepsilon) \) all members of \( \mathfrak{su}(1, 1) \) and all analytic functions of \( \varepsilon \) at \( \varepsilon = 0 \), where \( \mathcal{S}(t + \pi, \varepsilon) = \mathcal{S}(t, \varepsilon) \) is \( C^1 \) in \( t \) and \( \mathcal{L}(\varepsilon) \) and \( \mathcal{R}(\varepsilon) \) are independent of \( t \), such that

\[
e^{-\varepsilon I(t, \varepsilon)} Z_0(t, \varepsilon) e^{-\varepsilon \mathcal{L}(\varepsilon)} = y(t, \varepsilon)
\]  

(18)

satisfies

\[
y'(t, \varepsilon) = \varepsilon \mathcal{R}(\varepsilon) y(t, \varepsilon)
\]  

(19)

and

\[
\int_0^\pi \mathcal{S}(t, \varepsilon) \, dt = 0.
\]

It will now be shown that Lemma 2 implies convergence of the perturbation expansion of the previous section for \( \varepsilon \) sufficiently small.

Let \( Z(t, \varepsilon) = Z_0(t, \varepsilon) e^{-\varepsilon \mathcal{L}(\varepsilon)} \), then by (18) and (19)

\[
(e^{-\varepsilon I})' Z + \varepsilon e^{-\varepsilon I} CTZ = \varepsilon \mathcal{R} e^{-\varepsilon I} Z.
\]

Further, since \( Z \) is invertible,

\[
(e^{-\varepsilon I})' e^{\varepsilon I} + \varepsilon e^{-\varepsilon I} \mathcal{R} e^{\varepsilon I} = \varepsilon \mathcal{R}.
\]  

(20)

Also

\[
(e^{-\varepsilon I})' e^{\varepsilon I} = -\varepsilon \int_0^1 e^{-\varepsilon u I} \mathcal{S} e^{\varepsilon u I} \, du
\]

\[
= -\varepsilon \int_0^1 e^{-\varepsilon u I} \mathcal{S} \, du
\]

\[
= -\varepsilon \int_0^1 e^{-\varepsilon u I} \mathcal{S} \mathcal{R} w \, du
\]

\[
= e^{-\varepsilon \mathcal{R} w} - w.
\]

So that

\[
(e^{-\varepsilon I})' e^{\varepsilon I} = e^{-\varepsilon \mathcal{R} w} - w.
\]  

(21)

Thus from (20) and (21),

\[
e^{-\varepsilon \mathcal{R} w} (w + \varepsilon \mathcal{R}) = w + \varepsilon \mathcal{R},
\]
or equivalently,

\[ e^{i\varepsilon\Phi}(H_0 + \varepsilon V) = H_0 + \varepsilon K, \]

where \( K \) corresponds to the matrix \( \mathcal{B} \). Since \( \mathcal{S} \) and \( \mathcal{B} \) are analytic in \( \varepsilon \), so are \( S \) and \( K \), and further, since \( \int_0^\pi \mathcal{S}(t, \varepsilon) \, dt = 0 \), it follows that \( S = 0 \), i.e., the projection of \( S \) onto the null space of \( adH_0 \) is \( 0 \oplus 0 \). Finally, since \( S \) and \( K \) are analytic functions of \( \varepsilon \) at \( \varepsilon = 0 \), their power series expansions are convergent for \( |\varepsilon| \) sufficiently small.

Lemma 2 will now be proven. If \( \mathcal{A}(\varepsilon) \), analytic at \( \varepsilon = 0 \), is a matrix satisfying \( \mathcal{A}(0) = I \), define

\[
\log \mathcal{A}(\varepsilon) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\mathcal{A}(\varepsilon) - I)^k
\]

for \( |\varepsilon| \) sufficiently small.

Further, define

\[
\mathcal{P}(t, \varepsilon) = \log Z(t, \varepsilon).
\]

Since \( Z(t, \varepsilon) \) is a fundamental solution of \( \dot{Z} = \varepsilon \mathcal{A}Z \), then \( Z(t, \varepsilon) \) is a \( C^1 \) function of \( t \), so that \( T(t, \varepsilon) \) is also a \( C^1 \) function of \( t \). Further since \( Z(0, \varepsilon) = I \), it follows that

\[
Z_0(t + \pi, \varepsilon) Z_0^{-1}(\pi, \varepsilon) = Z_0(t, \varepsilon).
\]

Whence, a direct calculation shows that \( T(t, \varepsilon) \) is periodic in \( t \), with period \( \pi \), and that

\[
e^{\varepsilon T(t, \varepsilon)} Z_0(t, \varepsilon) = e^{\varepsilon \Phi(\varepsilon)}.
\]

If \( \mathcal{S} \) is a constant matrix, then

\[
e^{\varepsilon T(t, \varepsilon)} Z_0(t, \varepsilon) e^{-\varepsilon \mathcal{A}} = e^{\varepsilon T(t, \varepsilon)} e^{\varepsilon \Phi(\varepsilon)} e^{-\varepsilon \mathcal{A}} = e^{\exp(\varepsilon \mathcal{A})e^{\varepsilon \Phi(\varepsilon)}e^{-\varepsilon \mathcal{A}}} = e^{\varepsilon \Phi(\varepsilon)},
\]

where \( \mathcal{B}(\varepsilon) = e^{i\mathcal{S}} \Phi(\varepsilon) \).

By the Campbell–Hausdorff formula, there exists an analytic mapping \( \mathcal{H} : su(1, 1) \times su(1, 1) \to su(1, 1) \) such that

\[
e^{i(\mathcal{T}, \mathcal{A})} = e^{\mathcal{T}} e^{\mathcal{A}} = I + \mathcal{A} + \mathcal{B} + \mathcal{A} \mathcal{B} + \frac{\mathcal{A}^2}{2} + \frac{\mathcal{B}^2}{2} + \cdots
\]
so that

\[ N(\alpha, \beta) = \log(e^{\alpha}e^{\beta}) = \alpha + \beta + \frac{1}{2}[\alpha, \beta] + \cdots \]

through terms of the second order.

If \( \tilde{N}(\alpha, \beta, \varepsilon) \) is defined by

\[ \tilde{N}(\alpha, \beta, \varepsilon) = \frac{1}{\varepsilon} N(e^\alpha, e^\beta) \]

then

\[ \tilde{N}(\alpha, \beta, \varepsilon) = \alpha + \beta + P(\varepsilon). \]

Define \( T(t, \varepsilon) \) to be the solution of

\[ \int_0^t \tilde{N}(\varepsilon, T(t, \varepsilon), \varepsilon) \, dt = 0. \]

Such a \( T(t, \varepsilon) \) exists for \(|\varepsilon|\) sufficiently small by the implicit function theorem since \( \tilde{N} \) is analytic at \( \varepsilon = 0 \).

Finally, define

\[ S(t, \varepsilon) = \tilde{N}(\varepsilon, T(t, \varepsilon), \varepsilon). \]

Then, since \( \tilde{N}(\alpha, \beta, \varepsilon) \) is analytic and \( T(t, \varepsilon) \) is \( C^1 \) in \( t \), \( S(t, \varepsilon) \) is a \( C^1 \) function of \( t \). Now,

\[ e^{t \varepsilon} S(t, \varepsilon) Z_0(t, \varepsilon) e^{-t \varepsilon} = e^{t \varepsilon} S(t, \varepsilon) \]

by (22), since

\[ e^{t \varepsilon} S(t, \varepsilon) = e^{t \varepsilon} e^{T(t, \varepsilon)}. \]

Thus \( S(t, \varepsilon), \varepsilon \) and \( T(t, \varepsilon) \), having the properties required by the lemma, exist.

**Application to Hill’s Equation**

It will be shown in this section how Hill’s equation, written in the form

\[ \ddot{x} + (n^2 + \varepsilon q(t))x = 0, \quad (23) \]
where \( n \) is a positive integer and \( q(t) \) is a real function of \( t \), may be transformed into a linear system:

\[
\dot{z} = i\epsilon \begin{pmatrix} -B & -A \\ A & B \end{pmatrix} z,
\]

(24)

where \( B \) is a real valued, and \( A \) a complex valued, periodic, function of \( t \). Further it will be shown how the \( S \) and \( K \) of the previous section may be calculated explicitly.

We will suppose in all further discussion that \( q(t+\pi) = q(t) \), that \( q(t) \) possesses a continuous first derivative, and that

\[
q(t) = \sum_{n=-\infty}^{\infty} q_n e^{2i\epsilon n t}, \quad \text{with } q_0 = \delta.
\]

In order to write (23) in the form (24), first set \( y = \dot{x} \), then successively define

\[
z = \sqrt{\frac{n}{2}} x + i \sqrt{\frac{1}{2n}} y,
\]

\[
\tilde{z} = \sqrt{\frac{n}{2}} x - i \sqrt{\frac{1}{2n}} y,
\]

and

\[
\tilde{\xi} = e^{-i\epsilon nt} \tilde{z}.
\]

Then (24) is transformed into

\[
\begin{pmatrix} \dot{\xi} \\ \dot{\tilde{\xi}} \end{pmatrix} = \frac{i\epsilon q(t)}{2n} \begin{pmatrix} -1 & -e^{2i\epsilon nt} \\ e^{-2i\epsilon nt} & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix}
\]

(25)

which has the form of (24) with

\[
B = q(t)/2n \quad \text{and} \quad A = q(t) e^{-2i\epsilon nt/2n}.
\]

With \( \mathcal{L} \) and \( i[A \oplus B, C \oplus D] \) defined as in the previous section, let \( \mathcal{N}_{H_0} \) be the null space of the linear operator \( adH_0(\cdot) = [H_0, \cdot] \) and let \( \mathcal{S} \) be the complementary subspace of functions in \( \mathcal{L} \) satisfying

\[
\int_0^\pi A \oplus B \, dt = 0 \oplus 0,
\]

so that

\[
\mathcal{L} = \mathcal{N}_{H_0} \oplus \mathcal{S}.
\]
Also, let $\langle \cdot \rangle$ be projection onto $\mathcal{H}_0$ and let $\langle \cdot \rangle$ be projection onto $\mathcal{P}$. Since $iadH_0(\cdot)$ is differentiation, it is apparent that $\mathcal{H}_0$ will consist of constant functions.

Note that if $q(t)$ is periodic with period $\pi$, and has the Fourier series representation

$$q(t) = \sum_{n=-\infty}^{\infty} q_n e^{2i\pi n t},$$

then

$$\hat{q}(t) = q_0 \quad \text{and} \quad \tilde{q}(t) = \sum_{n \neq 0} q_n e^{2i\pi n t}.$$

If $K \in \mathcal{H}_0$ and $\mathcal{S}$, are now determined such that

$$e^{iadS}(H_0 + \epsilon V) = H_0 + \epsilon K \quad \text{with} \quad K = K_A \oplus K_B,$$

then Hill’s equation will be equivalent to

$$\begin{pmatrix}
\hat{\xi} \\
\hat{z}
\end{pmatrix} = i\epsilon \begin{pmatrix}
K_B & \tilde{K}_A \\
K_A & K_B
\end{pmatrix} \begin{pmatrix}
\xi \\
\zeta
\end{pmatrix}.$$  \hspace{1cm} (26)

It will now be shown how $K$ can be calculated to the second order.

From (25)

$$V = \frac{1}{2n} (e^{-2i\pi t}q(t) \oplus q(t)).$$

Let $q(t) = \sum_{k=-\infty}^{\infty} q_k e^{2i\pi kt}$, with $q_{-k} = q_k$, so that $q(t)$ is real, and with $q_0 = \delta$. Thus

$$V = \frac{1}{2n} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (q_{k+n} e^{2i\pi kt} \oplus q_k e^{2i\pi kt})$$

and so

$$K^{(0)} = \hat{V} = \frac{1}{2n} (q_n \oplus \delta).$$ \hspace{1cm} (27)

Continuing

$$S^{(0)} = (iadH_0)^{-1} \hat{V},$$

$$S^{(0)} = -\frac{1}{4n} \sum_{k \neq 0} \frac{1}{k} (iq_{k+n} e^{2i\pi kt} \oplus iq_ke^{2i\pi kt}).$$
Since \( K^{(1)} = (i/2)[S^{(0)}, V] \) by Eq. (13), it is necessary to calculate \((i/2)[S^{(0)}, V] \).

\[
\frac{i}{2} [S^{(0)}, V] = -\frac{1}{8n^2} \sum_{m \neq 0}^{\infty} \sum_{k = -\infty}^{\infty} \frac{1}{m} \left[ i q_{m+n} e^{2imt} + i q_m e^{2ikt}, q_{k+n} e^{2ikt} + q_k e^{2ikt} \right].
\]

Or,

\[
\frac{i}{2} [S^{(0)}, V] = \frac{1}{8n^2} \sum_{m \neq 0}^{\infty} \sum_{k = -\infty}^{\infty} \frac{1}{m} \left\{ \begin{array}{c} q_{m+n} q_m \\ q_{-m+n} q_m \end{array} \right\} \cdot e^{2i(k+m)t} \oplus \text{Re} \bar{q}_{m+n} q_{k+n} e^{2i(k+m)t}.
\]

Thus,

\[
K^{(1)} = \frac{i}{2} [S^{(0)}, V] = \frac{1}{8n^2} \sum_{m \neq 0}^{\infty} \sum_{k = -\infty}^{\infty} \frac{1}{m} \left\{ \begin{array}{c} q_{m+n} q_m \\ q_{-m+n} q_m \end{array} \right\} + \text{Re} \bar{q}_{m+n} q_{k+n} e^{2i(k-m)t}.
\]

By using Eq. (13) and calculating \((\text{iad}S^{(0)})^2 \tilde{V}\) and \((\text{iad}S^{(0)})^2 \tilde{V}\), \(K^{(2)}\) can be calculated in a manner similar to the calculation of \(K^{(1)}\). The result is given by

\[
K^{(2)} = K_A^{(2)} + K_B^{(2)},
\]

\[
K_A^{(2)} = \frac{1}{16n^3} \sum_{k \neq 0} \frac{1}{k^2} \left[ \begin{array}{c} q_{k+n} q_{k-n} \\ q_{-k+n} q_n \end{array} \right] + \frac{1}{24n^3} \sum_{k \neq 0 \atop m \neq 0 \atop k+m \neq 0} \frac{1}{k(k+m)} \left[ \begin{array}{c} q_{k+m+n} q_{k-m+n} q_{m+n} q_{m-n} \\ q_{-k-m} q_{k-n} q_{m+n} q_m \end{array} \right],
\]

and

\[
K_B^{(2)} = \text{Re} \left[ \begin{array}{c} \frac{1}{16n^3} \sum_{k \neq 0} \frac{1}{k^2} \left\{ \begin{array}{c} q_{k+n} q_k \\ q_n \delta \end{array} \right\} \\ + \frac{1}{24n^3} \sum_{k \neq 0 \atop m \neq 0 \atop k+m \neq 0} \frac{1}{k(k+m)} \left\{ \begin{array}{c} q_{k+m+n} q_{k-m+n} q_{m+n} q_m \end{array} \right\} \right].
\]
Therefore, if

\[ K^{(0)} = K_A^{(0)} \oplus K_B^{(0)}, \quad K^{(1)} = K_A^{(1)} \oplus K_B^{(1)} \]

and

\[ K^{(2)} = K_B^{(2)} \oplus K_B^{(2)} \]

are calculated then

\[ K = K_A \oplus K_B 
\]

\[ = (K_A^{(0)} + K_A^{(1)} \varepsilon + K_A^{(2)} \varepsilon^2) \oplus (K_B^{(0)} + K_B^{(1)} \varepsilon + K_B^{(2)} \varepsilon^2) \]

to the second order.

Thus we have explicit formulas for calculating to the second order the constant coefficient matrix of the linear system to which Hill's equation is equivalent. Use of this will be made in the following section to calculate the regions of stability for Hill's equation.

**Regions of Stability**

It will now be shown how the previous approximation to \( K \) leads to a determination of the regions of stability of Hill's equation in the \( \delta - \varepsilon \) plane.

The formulas developed in the previous section constructed the element

\[ K = K_A \oplus K_B \quad \text{in } \mathcal{M}_{H_0} \]

through terms of the second order in \( \varepsilon \) such that Hill's equation was equivalent to

\[
\begin{pmatrix}
\dot{\xi} \\
\dot{\eta}
\end{pmatrix} = i\varepsilon \begin{pmatrix}
-K_B \\
K_A
\end{pmatrix} \begin{pmatrix}
K_A \\
K_B
\end{pmatrix} \begin{pmatrix}
\xi \\
\eta
\end{pmatrix}.
\] (32)

Since (32) is autonomous it will have bounded solutions for all time if and only if \( \text{Re}(\lambda) = 0 \) for all characteristic roots \( \lambda \).

The characteristic equation for the matrix in (32) is

\[ \lambda^2 - \varepsilon^2 K_B^2 + \varepsilon^2 |K_A|^2 = 0, \]

so that \( \text{Re}(\lambda) = 0 \) if and only if

\[ K_B^2 \leq |K_A|^2. \]

Thus, the equations for the stability boundaries have the form,

\[ K_B(n, \delta, \varepsilon) = \pm |K_A(n, \delta, \varepsilon)|. \] (33)
If $K_A$ and $K_B$ are approximated to the first order, then (33) becomes

$$K_B^{(0)} + \varepsilon K_B^{(1)} - \pm |K_A^{(0)} + \varepsilon K_A^{(1)}|,$$

(34)

where

$$K_A^{(0)} = \frac{1}{2n} q_n, \quad K_B^{(0)} = \frac{1}{2n} \delta$$

(35)

from Eq. (27) and

$$K_A^{(1)} = \frac{1}{8n^2} \sum_{m \neq 0} \frac{1}{m} \begin{vmatrix} q_{m+n} & q_m \\ q_{-m+n} & q_{-m} \end{vmatrix},$$

$$K_B^{(1)} = \frac{1}{8n^2} \sum_{m \neq 0} \frac{1}{m} q_{m+n} \bar{q}_{m+n}$$

(36)

(37)

from Eq. (28).

In order to find the equations in the $\varepsilon - \delta$ plane of the stability boundaries, it will be convenient to write the expressions for $K_A^{(0)}$, $K_A^{(1)}$, $K_B^{(0)}$, $K_B^{(1)}$ so that their dependence on $\delta$ is explicitly shown. Doing this, Eq. (34) can be rewritten as

$$\varepsilon \delta^2 - (4n^2 + 2\varepsilon q_n) \delta + 4n^2 q_n + n\varepsilon (a_n - b_n) = 0$$

(38)

taking the plus sign in (34) and

$$\varepsilon \delta^2 - (4n^2 - 2\varepsilon q_n) \delta - 4n^2 q_n - n\varepsilon (a_n + b_n) = 0$$

(39)

if the minus sign is taken.

The symbols $a_n$ and $b_n$ appearing in (38) and (39) are defined by

$$a_n = \frac{q_{-n}}{n} (q_{2n} + q_{-2n}) + \sum_{m \neq 0} \frac{1}{m} \begin{vmatrix} q_{m+n} & q_m \\ q_{-m+n} & q_{-m} \end{vmatrix}$$

and

$$b_n = \sum_{m \neq 0} \frac{1}{m} q_{m+n} \bar{q}_{m+n}.$$  

Hill's equation will now be written again in the form (1) so that the classical notation may be used, that is

$$\ddot{x} + [\lambda + \varepsilon Q(t)]x = 0,$$

(40)

where $\lambda$ is a real positive parameter, $Q$ is a real function of period $\pi$ and further $Q_0 = 0$ where $Q(t) = \sum_{n=-\infty}^{\infty} Q_n e^{2i\pi t}$. 

The following correspondence is then apparent.

\[ \lambda = n^2 + \epsilon \delta \quad \text{and} \quad Q(t) = q(t) - \delta. \]

Let \( \delta' \) be the solution of (38)

\[ \delta' = \frac{1}{\epsilon} \left( 2n^2 + \epsilon \left| q_n \right| - \sqrt{R'} \right), \]

where

\[ R' = 4n^4 + q_n^2 \epsilon^2 - n\epsilon^2 (a_n - b_n) \]

and let \( \delta \) be the solution of (39),

\[ \delta = \frac{1}{\epsilon} \left( 2n^2 - \epsilon \left| q_n \right| - \sqrt{R} \right), \]

where

\[ R = 4n^4 + \epsilon^2 q_n^2 + n\epsilon^2 (a_n + b_n). \]

Thus

\[ \lambda_n = n^2 + \epsilon \delta, \quad \lambda_n' = n^2 + \epsilon \delta', \]

where \((\lambda_n, \lambda_n')\) is the \(n\)th interval of instability.

**APPLICATION TO MATHIEU'S EQUATION**

In this section the results of the third section, "Application to Hill's Equation," will be applied to the calculation of the stability boundaries of Mathieu's equation,

\[ \ddot{x} + (\lambda + \epsilon \cos 2t)x = 0. \quad (41) \]

This is Hill's equation in the form of Eq. (40) so that

\[ Q(t) = \cos 2t, \quad q(t) = \cos 2t + \delta, \]

and so

\[ q_1 = q_{-1} = \frac{1}{2}, \quad q_0 = \delta. \]

To approximate the stability boundaries of (41), we make use of Eq. (33) and the expressions for \( K_A^{(0)}, K_B^{(0)}, K_A^{(1)}, \) and \( K_B^{(1)} \) given by Eqs. (35), (36), and (37) for the first order approximation and, in addition, the expressions
for $K^{(2)}_I$ and $K^{(2)}_B$ given in Eqs. (30) and (31) for the second order approximations. The relation, $\lambda = n^2 + \varepsilon \delta$, will also be used so that the final results will be for the $\lambda - \varepsilon$ plane. The stability boundaries, in the $\lambda - \varepsilon$ plane, of (41) can be calculated for $n = 1$, $n = 2$, and $n = 3$. In each case, the first and second order boundaries can be calculated and compared graphically with the exact stability boundaries which are due to Ince [5, 6].

After applying the transformation $\lambda = n^2 + \varepsilon \delta$, the first and second order boundaries, together with the exact stability boundary are graphed in the $\lambda - \varepsilon$ plane in Fig. 1.

REFERENCES