Characterizations of Nonemptiness and Compactness of the Set of Weakly Efficient Solutions for Convex Vector Optimization and Applications

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Submitted by Koichi Mizukami

Received March 9, 2000

In this paper, we give characterizations for the nonemptiness and compactness of the set of weakly efficient solutions of an unconstrained/constrained convex vector optimization problem with extended vector-valued functions in terms of the 0-coercivity of some scalar functions. Finally, we apply these results to discuss solution characterizations of a constrained convex vector optimization problem in terms of solutions of a sequence of unconstrained vector optimization problems which are constructed with a general nonlinear Lagrangian.

Key Words: vector optimization; weakly efficient solution; convexity; coercivity; nonlinear Lagrangian.

1. INTRODUCTION

It is important to characterize the nonemptiness and compactness of solution sets for optimization problems. Sufficient conditions for the

1 This work is partially supported by a small grant from the Australian Research Council and the Research Grants Council of Hong Kong (Grant PolyU B-Q 359).
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nonemptiness and compactness of the optimal set of a general scalar optimization problem were given in [8, Chap. 1, C]. These conditions were stated in terms of some level-boundedness properties of the objective functions. Furthermore, the level boundedness properties are equivalent to some coercivity properties when the data in the optimization problems are convex. As a result, the nonemptiness and compactness of the optimal set of an unconstrained scalar convex optimization problem were characterized in terms of the $0$-coercivity of the objective function (see, e.g., [4]). Recently, several characterizations of the nonemptiness and compactness of the set of weakly efficient solutions for an unconstrained optimization problem with a finite vector-valued objective function were given by Deng [3]. This implicitly assumed that the vector-valued function is locally Lipschitz and hence continuous. In this paper, we discuss characterizations of the nonemptiness and compactness of the set of weakly efficient solutions for an unconstrained/constrained convex vector optimization problem with extended vector-valued objective function, each of whose component functions is lower semicontinuous (l.s.c. for short).

Throughout the paper, we assume that the objective space is $R_\infty^l = R^l \cup \{+\infty\}$, where $+\infty$ is an imaginary vector in $R_\infty^l$ such that each of its components is $+\infty$ in the extended real value space. Without confusion, we shall not differentiate the vector $+\infty$ in $R_\infty^l$ and the $+\infty$ in the extended real value space. We define the following orderings: for any $z^1 = (z^1_1, \ldots, z^1_l), z^2 = (z^2_1, \ldots, z^2_l) \in R_\infty^l$,

- $z^1 \leq z^2 \iff z^1_i \leq z^2_i$, $i = 1, \ldots, l$,
- $z^1 \leq z^2 \iff z^1_i = z^2_i, i = 1, \ldots, l$ with at least one $i$ such that $z^1_i < z^2_i$,
- $z^1 < z^2 \iff z^1_i < z^2_i, i = 1, \ldots, l$.

Thus, by $z^2 \geq z^1$ we naturally mean that $z^1 \leq z^2$ and by $z^1 \not\leq z^2$ we mean that $z^1 \leq z^2$ does not hold. For the orderings $\leq$, $<$, analogous relations should be understood in the same way as for $\geq$.

**Definition 1.1.** Let $f: X \to R$ be a real-valued function. $f$ is said to be l.s.c. relative to $X$ at a point $x \in X$ if for every sequence $\{x_k\} \subseteq X$ such that $x_k \to x$ as $k \to +\infty$, we have

$$\liminf_{k \to +\infty} f(x_k) \geq f(x).$$

If $f$ is l.s.c. relative to $X$ at each point $x \in X$, we say that $f$ is l.s.c. relative to $X$.

**Definition 1.2.** Let $\bar{f}: X \to R \cup \{+\infty\}$ be an extended real-valued function. $\bar{f}$ is said to be l.s.c. relative to $X$ at a point $x \in X$ if for every sequence $\{x_k\} \subseteq X$ such that $x_k \to x$ as $k \to +\infty$, we have

$$\liminf_{k \to +\infty} \bar{f}(x_k) \geq F(x).$$
If \( \bar{f} \) is \( l.s.c. \) \textit{relative to} \( X \) at each point \( x \in X \), we say that \( \bar{f} \) is \( l.s.c. \) \textit{relative to} \( X \).

Consider the vector optimization problem (\( \text{VP} \))

\[
\min \ \bar{f}(x) \\
\text{s.t.} \ x \in X,
\]

where \( X \subset R^n \) is a nonempty and closed set, and \( \bar{f}: X \to R^l_\infty \) is an extended vector-valued function such that each component function \( \bar{f}_i \) is a l.s.c. extended real-valued function relative to \( X \).

Let \( C \subset R^n \) be a nonempty and closed set and let \( f: C \to R^l \) be a vector-valued function such that each component function \( f_i \) is l.s.c. relative to \( C \). Let

\[
\bar{f}(x) = \begin{cases} 
  f(x), & \text{if } x \in C; \\
  +\infty, & \text{else.}
\end{cases}
\]

Thus, the vector optimization problem (\( VP \)) with a set constraint

\[
\min_{x \in C} f(x)
\]

is equivalent to the vector optimization problem with an extended vector-valued objective function

\[
\min_{x \in R^n} \bar{f}(x),
\]

which is a special case of (\( \text{VP} \)).

It is well known that this approach provides a unified theory and solution methods for solving many optimization problems (see [8] and the references therein). It follows from Definitions 1.1 and 1.2 that each component \( \bar{f}_i \) of \( \bar{f} \) is l.s.c. relative to \( X \) if and only if each component function \( f_i \) of \( f \) is l.s.c. relative to \( X \). It is clear that each component \( \bar{f}_i \) of \( \bar{f} \) is proper, l.s.c., and convex if and only if each component function \( f_i \) of \( f \) is l.s.c. and convex. It was showed in [8, Example 2.38] that a proper, l.s.c., convex, and positively homogeneous function fails to be continuous relative to a compact convex subset of its domain. Their example also shows that a finite, l.s.c., and convex function on a nonempty, compact convex set is not continuous and is even not bounded above on the compact convex set. It is worth mentioning that the problem (\( VP \)) was just the problem studied by Deng [1] where \( f: R^n \to R^l \) is assumed to be finite and continuous.

Thus, it is a natural question whether results in [3] can be extended to a vector optimization problem with a proper extended vector-valued convex objective function, each of whose component function is l.s.c. and convex. This paper gives an affirmative answer to this question. It is worth noting that Lemma 2.2 in [3] is not valid if not all of the component functions of
the objective vector-valued function are finite in the whole space $R^n$. To establish solution characterizations for a vector optimization problem with a proper extended vector-valued convex objective function, a variant of Lemma 2.2 in [3] must be provided. Furthermore, we shall utilize solution characterizations of related scalar optimization problems in [1, 4].

The outline of the paper is as follows. In Section 2, we shall give a characterization for the nonemptiness and compactness of the weakly efficient solutions of a convex vector optimization problem with no explicit constraints in terms of the nonemptiness and compactness of the optimal sets of scalar optimization problems. This characterization is analogous to one of the characterizations given by Deng [3]. We shall also give a characterization in terms of the 0-coercivity of each component function $\bar{f}_i$ ($i = 1, \ldots, l$) of $\bar{f}$ in this section. In Section 3, we shall provide a characterization of the nonemptiness and compactness of the optimal set of an inequality constrained convex scalar optimization problem in terms of the 0-coercivity of the max-composite function of the objective and constraint functions. Then we shall characterize the nonemptiness and compactness of the set of weakly efficient solutions for an inequality constrained convex vector optimization problem in terms of the 0-coercivity of several scalar convex functions. Finally, in Sections 4 and 5, using the characterization results obtained in Section 3, we shall discuss the solution of an inequality constrained convex vector optimization problem by solving a series of unconstrained vector optimization problems.

To conclude this section, we recall some basic concepts in [1, 6, 8] which will be needed in the sequel.

**Definition 1.3.** A function $f: X \rightarrow R^l_\infty$ is said to be proper if the domain of $f$, defined by $\text{dom}(f) = \{x \in X : f(x) \neq +\infty\}$, is not empty.

**Definition 1.4.** Let $f: X \rightarrow R^l$.

(i) A point $x^* \in X$ is said to be an efficient solution of $f$ over $X$ if there exists no $x \in X$ such that $f(x) \leq f(x^*)$.

(ii) A point $x^* \in X$ is said to be a weakly efficient solution of $f$ over $X$ if there exists no $x \in X$ such that $f(x) < f(x^*)$.

**Definition 1.5.** Let $F: X \rightarrow R^1 \cup \{+\infty\}$ be an extended real-valued function. $F$ is said to be 0-coercive if

$$\lim_{\|x\| \to +\infty, x \in X} F(x) = +\infty.$$  

**Definition 1.6.** (i) Let $C$ be a nonempty and convex subset of $R^n$. The recession cone of $C$ is defined by

$$C^\infty = \left\{ y \in R^n : \exists t_k \rightarrow +\infty, x_k \in C \text{ such that } y = \lim_{k \to +\infty} x_k/t_k \right\}.$$
Let $F : R^n \to R^1 \cup \{+\infty\}$ be an extended real-valued convex function. The recession function $F^\infty$ of $F$ is defined by $\text{epi}(F^\infty) = [\text{epi}(F)]^\infty$, where the epigraph of $F$, $\text{epi}(F) = \{(x, r) \in R^n \times R^1 : F(x) \leq r\}$.

As a straightforward consequence, we get

$$F^\infty(y) = \inf \left\{ \liminf_{k \to +\infty} F(t_k x_k)/t_k : t_k \to +\infty, x_k \to y \right\},$$

where $\{t_k\}$ and $\{x_k\}$ are sequences in $R^1$ and $R^n$, respectively (see [1]).

2. CHARACTERIZATION OF SOLUTION SETS FOR CONVEX VECTOR OPTIMIZATION WITH A SET CONSTRAINT

In this section, we consider the vector optimization problem $(VP)$

$$\min \quad \tilde{f}(x)$$

s.t. $x \in X,$

where $X \subset R^n$ is a nonempty, closed, and convex set, and $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_l) : X \to R^l_\infty$ is a proper extended vector-valued function such that its component function $\tilde{f}_i$ $(i = 1, \ldots, l)$ is l.s.c. relative to $X$ and convex on $X$.

We also consider the following related scalar optimization problems $(P_i)$ $(i = 1, \ldots, l)$:

$$\min \quad \tilde{f}_i(x)$$

s.t. $x \in X.$

Let $WE_1$ denote the set of weakly efficient solutions of $(VP)$ and let $S_i$ denote the set of optimal solutions of $(P_i)$ $(i = 1, \ldots, l)$.

We need the following lemma.

**Lemma 2.1.** Let $X$ be a nonempty and closed subset of $R^n$. Let $\varphi : X \to R_\infty^l$ be a proper extended vector-valued function such that each component $\varphi_i$ of $\varphi$ is l.s.c. relative to $X$. Suppose that $x_0 \in \text{mbox}{\varphi}$ such that $X_1 = \{x \in X : \varphi(x) \leq \varphi(x_0)\}$ is a compact set. Then there exists $x^* \in X_1$ such that $\varphi(x) - \varphi(x^*) \not\leq 0, \forall x \in X.$

**Proof.** It is easy to see that each component $\varphi_i$ of $\varphi$ is finite on $X_1$ and l.s.c. relative to $X_1$. Applying Corollary 3.2.1 in [6], we obtain a point $x^* \in X_1$ such that it is an efficient solution of $\varphi$ on $X_1$, namely,

$$\varphi(x) \not\leq \varphi(x^*), \quad \forall x \in X_1.$$  \hfill (1)

Since $x^* \in X_1$, showing by contradiction, we obtain that

$$\varphi(x) \not\leq \varphi(x^*), \quad \forall x \in X \setminus X_1.$$  \hfill (2)

The conclusion follows from (1) and (2). \hfill \blacksquare
We have the following result.

**Lemma 2.2.** Consider problem \((\bar{VP})\) and \((\bar{Pi})\) \((i = 1, \ldots, l)\). If \(WE_1\) is nonempty and compact, then, for each \(i \in \{1, \ldots, l\}\), \(S_i\) is nonempty and compact.

**Proof.** Let

\[
X_2 = \{x \in X : \bar{f}(x) < +\infty\},
\]

\[
f(x) = \bar{f}(x), \quad x \in X_2.
\]

Then \((\bar{VP})\) is equivalent to the vector optimization problem \((VP)\)

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad x \in X_2,
\end{align*}
\]

and \((\bar{Pi})\) is equivalent to the scalar optimization problem \((Pi)\)

\[
\begin{align*}
\min & \quad f_i(x) \\
\text{s.t.} & \quad x \in X_2,
\end{align*}
\]

where \(f_i\) is the \(i\)th component function of \(f\).

It is clear that each component function \(f_i\) is finite and convex on the nonempty, closed, and convex subset \(X_2\). It is also l.s.c. relative to \(X_2\). As \(WE_1\) is nonempty and compact, the set of weakly efficient solutions of \((VP)\) is nonempty and compact. It follows from the proof of [3, Lemma 2.1] that

\[
X_2^\infty \cap \left( \bigcap_{1 \leq i \leq l} \{u : f_i^\infty(u) \leq 0\} \right) = \{0\}. \tag{3}
\]

In what follows, we prove that \(S_i \neq \emptyset, \forall i \in \{1, \ldots, l\}\). Suppose to the contrary that there exists \(k \in \{1, \ldots, l\}\) such that \(S_k = \emptyset\). Let

\[
A = \{x \in WE_1 : f_k(x) \leq f_k(y), \forall y \in WE_1\}.
\]

Since \(f_k\) is l.s.c. relative to \(X_2\) and \(WE_1\) is nonempty and compact, it follows that \(A \neq \emptyset\). Suppose that \(x^* \in A\). Let us prove that \(x^* \in S_k\). Suppose to the contrary that there exists \(z \in X_2\) such that

\[
f_k(z) < f_k(x^*). \tag{4}
\]

Then the set

\[
D = \{x \in X_2 : f_i(x) \leq f_i(z), \quad i = 1, \ldots, l\}
\]

is nonempty and closed since \(z \in D\) and \(f_i\) is l.s.c. relative to \(X_2\). We show that \(D\) is bounded. Otherwise, there exists a sequence \(\{x_k\} \subset X_2\) such that \(\|x_k\| \to +\infty\) as \(k \to +\infty\) and

\[
f_i(x_k) \leq f_i(z), \quad \quad i = 1, \ldots, l.
\]
Without loss of generality, we assume that \( x_k/\|x_k\| \to u \). Clearly, \( \|u\| = 1 \) and \( u \in X_2^\infty \) and \( f_i^\infty(u) \leq 0, \ i = 1, \ldots, l \). This contradicts (3). Hence \( D \) is nonempty and compact. By Lemma 2.1, there exists \( x_0 \in D \) such that

\[
f(x) \not\leq f(x_0), \quad \forall x \in X_2;
\]

thus, \( x_0 \in \text{WE}_1 \). This, combined with \( x^* \in A \), yields

\[
f_k(x^*) \leq f_k(x_0). \tag{5}\]

However, \( x_0 \in D \) implies that

\[
f_k(x_0) \leq f_k(z). \tag{6}\]

Inequalities (4)–(6) jointly yield a contradiction. The proof is complete.  

**Lemma 2.3.** Let \( F : X \to R^l \cup \{+\infty\} \) be a proper, l.s.c. relative to \( X \), and convex function. Then the optimal set of the problem \( \min_{x \in X} F(x) \) is nonempty and compact if and only if \( F \) is 0-coercive on \( X \).

**Proof.** Let

\[
\overline{F}(x) = \begin{cases} 
F(x), & \text{if } x \in X; \\
+\infty, & \text{if } x \in R^n \setminus X.
\end{cases}
\]

Then it is easy to see that \( \overline{F} \) is a proper, l.s.c., and convex function on \( R^n \), and the optimal set of the problem \( \min_{x \in X} \overline{F}(x) \) is the same as that of \( \min_{x \in X} F(x) \), which is nonempty and compact. By [4, Proposition 3.2.5 and Definition 3.2.6], we see that this holds if and only if \( \overline{F} \) is 0-coercive on \( R^n \). Hence, we deduce that \( F \) is 0-coercive on \( X \).

**Lemma 2.4.** Consider (\( \overline{VP} \)) and \( (\overline{P}_i) \) (\( i = 1, \ldots, l \)). If for each \( i = 1, \ldots, l, S_i \) is nonempty and compact, then \( \text{WE}_1 \) is nonempty and compact.

**Proof.** Since \( S_i \subset \text{WE}_1, \forall i \in \{1, \ldots, l\} \), it follows that \( \text{WE}_1 \neq \emptyset \). As each \( \bar{f}_i \) is l.s.c. relative to \( X \), we see that \( \text{WE}_1 \) is closed. Finally, we show that \( \text{WE}_1 \) is bounded. Since each \( S_i \) is nonempty and compact, by Lemma 2.3, we deduce that each \( \bar{f}_i \) is 0-coercive on \( X \). Let \( x^* \in \text{dom}(\bar{f}) \). Suppose to the contrary that \( \text{WE}_1 \) is unbounded. Then there exists \( \{x_k\} \subset \text{WE}_1 \) such that \( \|x_k\| \to +\infty \) as \( k \to +\infty \). Thus, \( \bar{f}_i(x_k) \to +\infty \) as \( k \to +\infty \). Hence, when \( k \) is sufficiently large,

\[
\bar{f}_i(x_k) > \bar{f}_i(x^*), \quad i = 1, \ldots, l,
\]

which contradicts the fact that \( x_k \in \text{WE}_1 \). Thus, \( \text{WE}_1 \) is bounded.

**Theorem 2.1.** Consider (\( \overline{VP} \)) and \( (\overline{P}_i) \). \( \text{WE}_1 \) is nonempty and compact if and only if each \( S_i \) (\( i = 1, \ldots, l \)) is nonempty and compact if and only if each component function \( \bar{f}_i \) of \( \bar{f} \) (\( i = 1, \ldots, l \)) is 0-coercive on \( X \).

**Proof.** The conclusion follows from Lemmas 2.2 and 2.4.
3. CHARACTERIZATION OF SOLUTION SETS FOR CONSTRAINED CONVEX VECTOR OPTIMIZATION

In this section, we consider the inequality constrained convex vector optimization problem \((\text{CVP})\)

\[
\begin{align*}
\min & \quad \bar{f}(x) \\
\text{s.t.} & \quad x \in X, \\
& \quad g_j(x) \leq 0, \quad j = 1, \ldots, m,
\end{align*}
\]

where \(X \subset \mathbb{R}^n\) is a nonempty, closed, and convex set; \(\bar{f} = (\bar{f}_1, \ldots, \bar{f}_l): X \to \mathbb{R}^l\) is a proper vector-valued function such that its component function \(\bar{f}_i (i = 1, \ldots, l)\) is l.s.c. relative to \(X\) and convex on \(X\); and each \(g_j: X \to \mathbb{R}^1 \cup \{+\infty\}\) is a proper, l.s.c. relative to \(X\), and convex function.

We denote by \(\text{WE}_2\) the set of weakly efficient solutions of \((\text{CVP})\).

Throughout this section, we assume that \((\text{H}_1)\)

\[
\text{dom}(\bar{f}) \cap X_0 \neq \emptyset,
\]

where \(X_0 = \{x \in X: g_j(x) \leq 0, j = 1, \ldots, m\}\).

Remark 3.1. If \(\bar{f}\) is finite on \(X\), then \((\text{H}_1)\) is equivalent to \(X_0 \neq \emptyset\).

Consider the following scalar optimization problems \((\text{CP}_i)\) \((i = 1, \ldots, l)\):

\[
\begin{align*}
\min & \quad \bar{f}_i(x) \\
\text{s.t.} & \quad x \in X, \\
& \quad g_j(x) \leq 0, \quad j = 1, \ldots, m.
\end{align*}
\]

Since \((\text{H}_1)\) holds, it follows that

\[
\text{dom}(\bar{f}_i) \cap X_0 \neq \emptyset, \quad \forall i \in \{1, \ldots, l\}.
\]

Let us still denote by \(S_i\) the optimal set of \((\text{CP}_i)\) \((i = 1, \ldots, l)\).

**Lemma 3.1.** Let \(i \in \{1, \ldots, l\}\) and \(h_i(x) = \max\{\bar{f}_i(x), g_1(x), \ldots, g_m(x)\}\). Then \(S_i\) is nonempty and compact if and only if \(h_i\) is 0-coercive on \(X\).

**Proof.** “Sufficiency.” Let \(h_i\) be 0-coercive on \(X\). Let \(x_0 \in \text{dom}(\bar{f}_i) \cap X_0\). Then

\[
X_3 = \{x \in X_0: \bar{f}_i(x) \leq \bar{f}_i(x_0)\}
\]

is nonempty and compact. Indeed, it is clear from the l.s.c. of \(\bar{f}_i\) relative to \(X\) that \(X_3\) is nonempty and closed. We show that \(X_3\) is bounded. Otherwise, there exists \(\{x_k\} \subset X_0\) such that \(\|x_k\| \to +\infty\) as \(k \to +\infty\) and \(\bar{f}_i(x_k) \leq \bar{f}_i(x_0)\). It follows from the 0-coercivity of \(h_i\) that \(h_i(x_k) \to +\infty\) as
Thus we deduce that $S_i \neq \emptyset$. Noting that $S_i \subseteq X_1$, $S_i$ is bounded. In addition, because $\bar{f}_i$ is l.s.c. relative to $X \supseteq X_0$ and $X_0$ is closed, $S_i$ is closed. Thus we have proved that $S_i$ is nonempty and compact.

“Necessity.” Let

$$\bar{f}_i(x) = \begin{cases} \tilde{f}_i(x), & \text{if } x \in X; \\ +\infty, & \text{if } x \in \mathbb{R}^n \setminus X, \end{cases}$$

and for $j = 1, \ldots, m$,

$$g'_j(x) = \begin{cases} g_j(x), & \text{if } x \in X; \\ +\infty, & \text{if } x \in \mathbb{R}^n \setminus X. \end{cases}$$

Then $(\overline{CP}_i)$ is equivalent to the optimization problem $(\overline{CP}'_i)$

$$\min \bar{f}_i(x) \quad \text{s.t. } x \in X, \quad g'_j(x) \leq 0, \quad j = 1, \ldots, m,$$

which is a problem considered in [1]. It follows from [1] that the optimal set $S_i$ of $(\overline{CP}_i)$ is nonempty and compact if and only if

$$(\bar{f}_i)^\infty(u) \leq 0, \quad (g'_j)^\infty(u) \leq 0, \quad j = 1, \ldots, m \Rightarrow u = 0. \quad (7)$$

Now we prove by contradiction that $h_i$ is 0-coercive on $X$. Suppose that there exists a sequence $\{x_k\} \subset X$ and a real number $M > 0$ such that

$$h_i(x_k) \leq M, \quad \forall k.$$ 

It follows that

$$\tilde{f}_i(x_k) \leq M, \quad g_j(x_k) \leq M, \quad \forall k, \ j = 1, \ldots, m.$$ 

Without loss of generality, we assume that

$$\lim_{k \to +\infty} x_k/\|x_k\| = u.$$

Thus

$$\|u\| = 1. \quad (8)$$

Then

$$(\bar{f}_i)^\infty(u) \leq \liminf_{k \to +\infty} \frac{\tilde{f}_i(x_k)}{\|x_k\|} \leq 0, \quad \quad (9)$$

$$(g'_j)^\infty(u) \leq \liminf_{k \to +\infty} \frac{g_j(x_k)}{\|x_k\|} \leq 0, \quad j = 1, \ldots, m. \quad (10)$$

The combination of (8), (9), and (10) contradicts (7). The proof is complete. $\blacksquare$
THEOREM 3.1. Consider (CVP). Then the set $WE_2$ is nonempty and compact if and only if each $h_i(x) = \max\{f_i(x), g_1(x), \ldots, g_m(x)\} \ (i = 1, \ldots, l)$ is 0-coercive on $X$.

Proof. Applying Theorem 2.1 with $X$ replaced by $X_0$, we see that $WE_2$ is nonempty and compact if and only if the optimal set $S_i$ of (CP$_i$) is nonempty and compact for each $i \in \{1, \ldots, l\}$. By lemma 3.1, the latter is true if and only if $h_i$ is 0-coercive on $X$ for each $i \in \{1, \ldots, l\}$. $lacksquare$

4. APPLICATION I: EXACT SOLUTIONS

In this section, we apply the results of the previous sections to propose a scheme to solve a class of constrained convex vector optimization problems by means of unconstrained vector optimization via a nonlinear Lagrangian.

Consider the inequality constrained vector optimization problem (CVP)

$$\min f(x)$$

s.t. $x \in X,$

$$g_j(x) \leq 0, \quad j = 1, \ldots, m,$$

where $X \subset \mathbb{R}^n$ is a nonempty and closed set, $f = (f_1, \ldots, f_l): X \rightarrow \mathbb{R}^l$ is a vector-valued function such that its component function $f_i (i = 1, \ldots, l)$ is l.s.c. relative to $X$, and each $g_j: X \rightarrow \mathbb{R}^l$ is l.s.c. relative to $X$.

Assume throughout this section that $f_i(x) \geq 0, \ i = 1, \ldots, l, \forall x \in X$. This assumption is not restrictive. If it does not hold, we can replace each component function $f_i$ of $f$ with $\exp(f_i)$, and the resulting vector optimization problem has the same sets of efficient solutions and weakly efficient solutions as that of (CVP) and satisfies this assumption. We also assume throughout this section that the feasible set $X_0 = \{x \in X: g_j(x) \leq 0, \ j = 1, \ldots, m\}$ is nonempty.

Now we recall the nonlinear Lagrangian for a constrained vector optimization problem (see [5] for details). Let $p: \mathbb{R}^l_+ \times \mathbb{R}^m \rightarrow \mathbb{R}^l$ be a vector-valued function such that each component function $p_i (i = 1, \ldots, l)$ of $p$ is l.s.c. and $p$ enjoys the following properties:

(A) $p$ is increasing: $\forall (z^1, y^1), (z^2, y^2) \in R^l_+ \times \mathbb{R}^m$, if $(z^1, y^2) - (z^1, y^1) \in R^l_+ \times \mathbb{R}^m$, then $p(z^1, y^1) \leq p(z^2, y^2)$;

(B) there exist positive real numbers $a_1, \ldots, a_m$ such that $p(z, y) \geq z$ and $p(z, y) \geq (\max_{1 \leq j \leq m} \{a_j y_j\}) e_1$, $\forall z \in \mathbb{R}^l_+, y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, where $e_1 = (1, \ldots, 1) \in \mathbb{R}^l$;

(C) $p(z, 0) = z$, $\forall z \in \mathbb{R}^l_+$.
Then

\[ L(x, d) = p(f(x), d_1g_1(x), \ldots, d_mg_m(x)), \]

\[ x \in X, \ d = (d_1, \ldots, d_m) \in R^m_+ \]

is called a nonlinear Lagrangian function corresponding to \( p \).

We attempt to solve \((CVP)\) by solving a sequence of unconstrained vector optimization problems \((VQ_d)\),

\[
\begin{align*}
\min L(x, d) \\
\text{s.t. } x \in X,
\end{align*}
\]

where \( d \in R^m_+ \).

Let \( E, E_d, WE_3, \) and \( WE_d \) denote the sets of efficient solutions and weakly efficient solutions of \((CVP)\) and \((VQ_d)\), respectively.

We need the following lemmas.

**Lemma 4.1** [5]. For any \( x \in X_0 \) and \( d \in R^m_+ \), we have \( L(x, d) = f(x) \).

**Lemma 4.2.** Each component function \( L_i \) of \( L \) is l.s.c. relative to \( X \).

**Proof.** This is a direct consequence of [5, Lemma 2.3].

We have the following result.

**Theorem 4.1.** Consider \((CVP)\) and \((VQ_d)\). Let \( h_i(x) = \max\{f_i(x), g_1(x), \ldots, g_m(x)\} \) be 0-coercive on \( X \), \( i = 1, \ldots, l \). Let \( e_m = (1, \ldots, 1) \in R^m \). Then

(i) for each \( d \in R^m_+ \) with \( d - e_m \in R^m_+ \), \( WE_d \) is nonempty and compact; \( E_d \) is nonempty and bounded;

(ii) for each selection \( x_d^* \in WE_d \), \( \{x_d^*\} \) is bounded and all of its limit points belong to \( WE_3 \);

(iii) \( E \neq \emptyset \) and \( E \) is bounded; for each \( x^* \in E \), there exists a selection \( x_d^* \in E_d \) such that \( f(x^*) = \lim_{d \to +\infty} f(x_d^*) \) (where \( d \to +\infty \) means that each component \( d_j \) of \( d \) tends to \( +\infty \)); and \( WE_3 \) is nonempty and compact.

**Proof.** (i) First we prove that \( E_d \neq \emptyset \) for each \( d \in R^m_+ \) with \( d - e_m \in R^m_+ \). By the definition of \( L \), we deduce that for each \( i \in \{1, \ldots, l\} \),

\[
L_i(x, d) \geq \max\{f_i(x), a_1d_1g_1(x), \ldots, a_md_mg_m(x)\} \geq \max\{f_i(x), a_1g_1(x), \ldots, a_mg_m(x)\},
\]

where \( L_i \) is the \( i \)th component function of \( L \) and \( d - e_m \in R^m_+ \). As each \( h_i \) is 0-coercive, we deduce that each \( L_i \) is 0-coercive. Arbitrarily fix an \( x_0 \in C \). It follows from Lemmas 4.1 and 4.2 that \( \{x \in X: L(x, d) \leq L(x_0, d) = f(x_0)\} \) is nonempty and compact. By Lemma 2.1, \( E_d \) is nonempty. Hence \( WE_d \) is nonempty.
To prove that $E_d$ is bounded, it is sufficient to prove that $WE_d$ is bounded. Suppose to the contrary that $\exists \{x^k_d\} \subset WE_d$ such that $\|x^k_d\| \to +\infty$ as $k \to +\infty$. As $L_i$ is 0-coercive, we deduce that $L_i(x^k_d, d) \to +\infty$ as $k \to +\infty$. It follows that $L_i(x^k_d, d) > L_i(x_0, d) = f_i(x_0)$, $i = 1, \ldots, l$, when $k$ is sufficiently large, which contradicts the fact that $x^k_d \in WE_d$, $\forall k$. Thus we have proved that $WE_d$ is nonempty and bounded. In addition, $WE_d$ is closed by Lemma 4.2. So, $WE_d$ is nonempty and compact.

(ii) Now we show that for each selection $x^k_d \in WE_d$, $\{x^k_d\}$ is bounded. Suppose to the contrary that $\exists d^k \to +\infty$ and $x^k_{d^k} \in WE_{d^k}$ such that $\|x^k_{d^k}\| \to +\infty$. As

$$L_i(x^k_{d^k}, d^k) \geq \max\{f_i(x^k_{d^k}), a_1g_1(x^k_{d^k}), \ldots, a_mg_m(x^k_{d^k})\}$$

and

$$\max\{f_i(x^k_{d^k}), a_1g_1(x^k_{d^k}), \ldots, a_mg_m(x^k_{d^k})\} \to +\infty$$

by the 0-coercivity of $h_i$, $i = 1, \ldots, l$, we deduce that

$$L_i(x^k_{d^k}, d^k) \to +\infty, \quad i = 1, \ldots, l.$$}

Hence,

$$L_i(x^k_{d^k}, d^k) > L_i(x_0, d^k) = f_i(x_0), \quad i = 1, \ldots, l,$$

when $k$ is sufficiently large, which contradicts the fact that $x^k_{d^k} \in WE_{d^k}$, $\forall k$. This proves that $\{x^k_d\}$ is bounded.

Suppose that $x^*$ is a limit point of $\{x^k_d\}$, i.e., $\exists d^k \to +\infty$ and $x^*_{d^k} \in WE_{d^k}$ such that $x^*_{d^k} \to x^*$ as $k \to +\infty$. We show that $x^* \in WE_3$. First, we show that $x^* \in X_0$. Otherwise,

$$\max\{g_1(x^*), \ldots, g_m(x^*)\} \geq m_0 > 0$$

for some real number $m_0$. It follows that

$$\max\{g_1(x^*_{d^k}), \ldots, g_m(x^*_{d^k})\} \geq m_0/2$$

when $k \geq k_0$ for some $k_0 > 0$. So

$$L_i(x^*_{d^k}, d^k) \geq \frac{m_0}{2} \cdot \min_{1 \leq i \leq m} a_j \cdot \min_{1 \leq j \leq m} d_j^k, \quad i = 1, \ldots, l,$$

when $k \geq k_0$. Hence,

$$L_i(x^*_{d^k}, d^k) \to +\infty, \quad i = 1, \ldots, l,$$

as $k \to +\infty$. Thus, for sufficiently large $k$,

$$L_i(x^*_{d^k}, d^k) > L(x_0, d^k) = f_i(x_0), \quad i = 1, \ldots, l,$$

a contradiction of the fact that $x^*_{d^k} \in WE_{d^k}$, $\forall k$. 


Now we show that $x^* \in WE_3$. Suppose to the contrary that there exists $x \in X_0$ and a positive real number $m$ such that
\[ f(x) - f(x^*) \leq -me \]
It follows from the l.s.c. of $f_i$ and the fact that $x_{d^k}^* \to x^*$ that there exists $k_1 > 0$ such that, for $k \geq k_1$,
\[ f(x) - f(x_{d^k}^*) \leq -m/2e \]
By Lemma 4.1, we have, for $k \geq k_1$,
\[ L(x, d^k) - L(x_{d^k}^*, d^k) \leq -m/2e \]
a contradiction of the fact that $x_{d^k}^* \in WE_{d^k}, \forall k$. Hence $x^* \in WE_3$.

(iii) Let $x_0 \in X_0$ and $X_4 = \{ x \in X_0: f(x) \leq f(x_0) \}$. Then
\[ X_4 = \bigcap_{1 \leq i \leq l} \{ x \in X_0: f_i(x) \leq f_i(x_0) \} \]
\[ = \bigcap_{1 \leq i \leq l} \{ x \in X_0: \max\{ f_i(x), g_1(x), \ldots, g_m(x) \} \leq f_i(x_0) \} \]
and $X_3$ is nonempty and compact since $h_i$ is l.s.c. relative to $X$ and 0-coercive. By Lemma 2.1, $E \neq \emptyset$.
Let $x^* \in E$. For each $d \in R_+^m$ with $d - e_m \in R_+^m$, let
\[ X^d = \{ x \in X: L(x, d) \leq L(x^*, d) = f(x^*) \} \]
Then
\[ X^d = \bigcap_{1 \leq i \leq l} \{ x \in X: L_i(x, d) \leq f_i(x^*) \} \]
\[ \subset \bigcap_{1 \leq i \leq l} \{ x \in X: \max\{ f_i(x), a_1d_1g_1(x), \ldots, a_md_mg_m(x) \} \leq f_i(x^*) \} \]
\[ \subset \bigcap_{1 \leq i \leq l} \{ x \in X: \max\{ f_i(x), a_1g_1(x), \ldots, a_md_mg_m(x) \} \leq f_i(x^*) \} \]
Then $X^d$ is nonempty and compact. By Lemma 2.1, there exists $x_{d}^* \in E_d \cap X^d$. As $x_{d}^* \in WE_d$, we deduce from the above proof that $\{ x_{d}^* \}$ is bounded and all of the limit points of $\{ x_{d}^* \}$ belong to $X_0$. Arbitrarily take a limit point $\bar{x}$ of $\{ x_{d}^* \}$. Then $\exists d^k \to +\infty$ and $x_{d^k}^* \in E_{d^k}$ such that
\[ \lim_{k \to +\infty} x_{d^k}^* = \bar{x} \]
Since $x_{d^k}^* \in X_{d^k}$, we have
\[ f(x_{d^k}^*) \leq L(x_{d^k}^*, d^k) \leq f(x^*) \]
That is,

\[ f_i(x^*_k) \leq f_i(x^*), \quad i = 1, \ldots, l. \]

It follows from the l.s.c. of \( f_i \) relative to \( X \) that

\[ f_i(\bar{x}) \leq \liminf_{k \to +\infty} f_i(x^*_k) \leq f_i(x^*), \quad i = 1, \ldots, l. \]  
(11)

This combined with \( x^* \in E \) and \( \bar{x} \in X_0 \) implies that

\[ f(\bar{x}) = f(x^*). \]  
(12)

(11) and (12) jointly yield

\[ \lim_{k \to +\infty} f(x^*_k) = f(\bar{x}) = f(x^*). \]

Since the limit point \( \bar{x} \) has been arbitrarily taken, we conclude that

\[ \lim_{d \to +\infty} f(x^*_d) = f(x^*). \]

Finally, we show that \( WE_3 \) is bounded, which further implies the compactness of \( WE_3 \) and the boundedness of \( E \). Suppose to the contrary that \( WE_3 \) is unbounded. Then \( \exists \hat{x}_k \in WE_3 \) such that \( \|\hat{x}_k\| \to +\infty \) as \( k \to +\infty \). Arbitrarily fix an \( x_0 \in X_0 \). Since \( h_i(\hat{x}_k) = \max\{f_i(\hat{x}_k), g_1(\hat{x}_k), \ldots, g_m(\hat{x}_k)\} = f_i(\hat{x}_k) \to +\infty \) as \( k \to +\infty \), \( i = 1, \ldots, l \), we deduce that, for sufficiently large \( k \),

\[ f_i(\hat{x}_k) > f_i(x_0), \quad i = 1, \ldots, l, \]
a contradiction of \( \hat{x}_k \in WE_3 \). The proof is complete.  

In the following, we consider the convex case of \( (CVP) \). We assume

(H\(_2\)) \( X \subset R^n \) is nonempty, closed and convex; \( f: X \to R^l \) is a vector-valued function such that each component function \( f_i \) of \( f \) is l.s.c. and convex; and each \( g_j: X \to R \) is l.s.c. and convex.

**Theorem 4.2.** Let assumption (H\(_2\)) hold. Suppose that \( WE_3 \) is nonempty and compact. Then

(i) for each \( d \in R^n_+ \) with \( d - e_m \in R^n_+ \), \( WE_d \) is nonempty and compact; \( E_d \) is nonempty and bounded;

(ii) for each selection \( x^*_d \in WE_d \), \( \{x^*_d\} \) is bounded, and all of its limit points belong to \( WE_3 \);

(iii) \( E \neq \varnothing \) and \( E \) is bounded; for each \( x^* \in E \), there exists a selection \( x^*_d \in E_d \) such that \( f(x^*) = \lim_{d \to +\infty} f(x^*_d) \).
Proof. Under assumption (H2), by Theorem 3.1, that $WE_3$ is nonempty and compact implies that each $h_i(x) = \max\{f_i(x), g_1(x), \ldots, g_m(x)\}$ is 0-coercive, $i = 1, \ldots, l$. Applying Theorem 4.1, the conclusions follow.

Remark 4.1. Each $(VQ_d)$ may not be, in general, a convex vector optimization problem. But each component function $p_i$ of the function $p$ defining the nonlinear Lagrangian $L$ is convex and $(H_2)$ holds; then $(VQ_d)$ is a convex vector optimization problem.

5. APPLICATION II: APPROXIMATE SOLUTIONS

For many vector optimization problems, there do not exist efficient solutions (or even weakly efficient solutions). Then we have to resort to approximate efficient solutions. In this section, we shall consider the acquisition of so-called $\varepsilon$-quasi-efficient solutions (or $\varepsilon$-quasi-weakly efficient solutions) of a vector optimization problem by using the results in Section 4.

We consider $(CVP)$ and $(VQ_d)$. We assume throughout this section that $f(x) \geq 0$, $\forall x \in X$ and the feasible set $X_0 \neq \emptyset$.

Our consideration is based on the following Ekeland variational principle for vector-valued functions.

**Proposition 5.1.** Consider $(CVP)$. For any $\varepsilon > 0$, there exists $x^* \in X_0$ such that

(i) $f(x) - f(x^*) + \varepsilon e_i \not\leq 0$, $\forall x \in X_0$;

(ii) $f(x) + \varepsilon \|x - x^*\|e_i - f(x^*) \not\leq 0$, $\forall x \in X_0$.

Proof. It follows from Lemma 1.1 and Corollary 2.1 in [6] that the conclusion holds.

**Definition 5.1.** Let $x^* \in X_0$. Then

(i) The point $x^*$ is called an $\varepsilon$-quasi-efficient solution of $(CVP)$ if

$$f(x) + \varepsilon \|x - x^*\|e_i - f(x^*) \not\leq 0, \quad \forall x \in X_0;$$

(ii) The point $x^*$ is called an $\varepsilon$-quasi-weakly efficient solution of $(CVP)$ if

$$f(x) + \varepsilon \|x - x^*\|e_i \not< f(x^*), \quad \forall x \in X_0.$$

Denote by $QWE_\varepsilon$ the set of all of the $\varepsilon$-quasi-weakly efficient solutions of $(CVP)$. Let $x_0 \in X$ and

$$f_{\varepsilon}(x) = f(x) + \varepsilon \|x - x_0\|e_i,$$

$$L_{\varepsilon}(x, d) = p(f_{\varepsilon}(x), d_1g_1(x), \ldots, d_mg_m(x)),$$  

where $p$ is defined as in Section 4.
Consider the approximate vector optimization problem \( (CVP_{\epsilon}) \),
\[
\begin{align*}
\min & \ f_{\epsilon}(x) \\
\text{s.t.} & \ x \in X, \\
& \ g_j(x) \leq 0, \quad j = 1, \ldots, m,
\end{align*}
\]
and the approximate nonlinear penalty problem \( (VQ_{d_{\epsilon}}) \),
\[
\begin{align*}
\min & \ L_{\epsilon}(x, d) \\
\text{s.t.} & \ x \in X.
\end{align*}
\]

Let \( E_{\epsilon}, E_{d_{\epsilon}}, WE_{\epsilon}, \) and \( WE_{d_{\epsilon}} \) denote the sets of efficient solutions and the sets of weakly efficient solutions of \( (CVP_{\epsilon}) \) and \( (Q_{d_{\epsilon}}) \), respectively.

**Theorem 5.1.** We have

(a) for each \( d \in R_+^n \) with \( d - e_m \in R_+^n \), \( WE_{d_{\epsilon}} \) is nonempty and compact;

(b) \( E_{d_{\epsilon}} \) is nonempty and bounded;

(c) for each \( x^* \in E_{\epsilon}, \) there exists a selection \( x^*_d \in E_{d_{\epsilon}} \) such that \( f(x^*) = \lim_{d \to +\infty} f(x^*_d); \)

(d) for each selection \( x^*_d \in WE_{d_{\epsilon}}, \) \( \{x^*_d\} \) is bounded and all of its limit points belong to \( WE_{\epsilon}; \)

(e) any \( x^* \in WE_{\epsilon} \) is an \( \epsilon \)-quasi-weakly efficient solution of \( (CVP) \);

(f) any \( x^* \in E_{\epsilon} \) is an \( \epsilon \)-quasi-efficient solution of \( (CVP). \)

**Proof.** As each component function of \( f_{\epsilon} \) is 0-coercive on \( X, \) all of the conditions of Theorem 4.1 are satisfied. (a), (b), (c), and (d) follow directly from Theorem 4.1. Now we prove (e) and (f). Let \( x^* \in WE_{\epsilon}. \) Then \( x^* \) is a weakly efficient solution of \( (CVP_{\epsilon}) \), namely,
\[
f_{\epsilon}(x) \neq f_{\epsilon}(x^*), \quad \forall x \in X_0.
\]

Thus,
\[
f(x) + \epsilon \|x - x_0\|e_l \neq f(x^*) + \epsilon \|x^* - x_0\|e_l,
\]
i.e.,
\[
f(x) - f(x^*) + \epsilon (\|x - x_0\| - \|x^* - x^0\|) e_l \neq 0. \quad (13)
\]
By the triangle inequality, we have
\[
\|x - x_0\| - \|x^* - x_0\| \leq \|x - x^*\|. \quad (14)
\]
The combination of (13) and (14) yields
\[
f(x) + \|x - x^*\|e_l - f(x^*) \neq 0, \quad \forall x \in X_0,
\]
namely, \( x^* \) is an \( \epsilon \)-quasi-weakly efficient solution of \( (CVP). \) (f) can be analogously proved. \( \blacksquare \)
Consider the scalar optimization problem \((P)\)
\[
\inf f^1(x) \\
\text{s.t. } x \in X \\
g_j(x) \leq 0, \quad j = 1, \ldots, m,
\]
where \(X\) is a nonempty and closed subset of \(\mathbb{R}^n\), and \(f^1, g_j : X \rightarrow \mathbb{R}\) \((j = 1, \ldots, m)\) are finite and l.s.c. relative to \(X\).

**Definition 5.2.** Let \(\epsilon > 0\). \(x^* \in X_0\) is called an \(\epsilon\)-quasisolution of \((P)\) if
\[
f^1(x^*) \leq f^1(x) + \epsilon \|x - x^*\|, \quad \forall x \in X_0.
\]
Let the problem \((CVP)\) be convex, i.e., let all of the data in the problem \((CVP)\) be convex. It is known that \(x^* \in QWE_\epsilon\) if and only if there exists \(\lambda = (\lambda_1, \ldots, \lambda_l) \in R_l^l\) with \(\sum_{i=1}^l \lambda_i = 1\) such that \(x^*\) is an \(\epsilon\)-quasisolution for the following problem \((P_\lambda)\):
\[
\inf \langle \lambda, f(x) \rangle \\
\text{s.t. } x \in X \\
g_j(x) \leq 0, \quad j = 1, \ldots, m.
\]
That is,
\[
\langle \lambda, f(x^*) \rangle \leq \langle \lambda, f(x) \rangle + \epsilon \|x - x^*\|, \quad \forall x \in X_0. \tag{15}
\]
On the other hand, \((CVP_\lambda)\) is also a convex vector optimization problem. \(x^{**} \in WE_\epsilon\) if and only if there exists \(\lambda = (\lambda_1, \ldots, \lambda_l) \geq 0\) with \(\sum_{i=1}^l \lambda_i = 1\) such that
\[
\langle \lambda, f(x^{**}) \rangle + \epsilon \|x^{**} - x^0\| \leq \langle \lambda, f(x) \rangle + \epsilon \|x - x_0\|, \quad x \in X_0. \tag{16}
\]
Furthermore, (16) implies
\[
\langle \lambda, f(x^{**}) \rangle \leq \langle \lambda, f(x) \rangle + \epsilon \|x - x^{**}\|, \quad \forall x \in X_0. \tag{17}
\]
Therefore, \(x^{**}\) is an \(\epsilon\)-quasisolution of \((P_\lambda)\). This establishes some further relationship (in addition to the relation \(WE_\epsilon \subseteq QWE_\epsilon\), which is obtained in (f) of Theorem 5.1) between \(QWE_\epsilon\) and \(WE_\epsilon\). That is, for any \(x^* \in QWE_\epsilon\), there exists \(\lambda = (\lambda_1, \ldots, \lambda_l) \in R_l^l\) with \(\sum_{i=1}^l \lambda_i = 1\) such that \(x^*\) is an \(\epsilon\)-quasisolution of \((P_\lambda)\). Corresponding to this \(\lambda\), we can find an \(x^{**} \in X_0\) such that (16) and (17) hold. Hence, \(x^{**} \in WE_\epsilon\), \(x^{**} \in QWE_\epsilon\), and \(x^{**}\) is also an \(\epsilon\)-quasisolution of \((P_\lambda)\) such that \(\|\langle \lambda, f(x^*) \rangle - \langle \lambda, f(x^{**}) \rangle\| \leq \epsilon \|x^* - x^{**}\|\).

6. CONCLUSIONS

In this paper, characterizations of the nonemptiness and compactness of the set of weakly efficient solutions were established for a convex vector optimization problem with extended valued functions. These results were applied to discuss solution characterizations of a class of constrained convex vector optimization problems.
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