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Characterizations of Nonemptiness and Compactness of the Set of Weakly Efficient Solutions for Convex Vector Optimization and Applications[1]

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In this paper, we give characterizations for the nonemptiness and compactness of the set of weakly efficient solutions of an unconstrained/constrained convex vector optimization problem with extended vector-valued functions in terms of the 0-coercivity of some scalar functions. Finally, we apply these results to discuss solution characterizations of a constrained convex vector optimization problem in terms of solutions of a sequence of unconstrained vector optimization problems which are constructed with a general nonlinear Lagrangian.

Key Words: vector optimization; weakly efficient solution; convexity; coercivity; nonlinear Lagrangian.

1. INTRODUCTION

It is important to characterize the nonemptiness and compactness of solution sets for optimization problems. Sufficient conditions for the

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nonemptiness and compactness of the optimal set of a general scalar optimization problem were given in [8, Chap. 1, C]. These conditions were stated in terms of some level-boundedness properties of the objective functions. Furthermore, the level boundedness properties are equivalent to some coercivity properties when the data in the optimization problems are convex. As a result, the nonemptiness and compactness of the optimal set of an unconstrained scalar convex optimization problem were characterized in terms of the 0-coercivity of the objective function (see, e.g., [4]). Recently, several characterizations of the nonemptiness and compactness of the set of weakly efficient solutions for an unconstrained optimization problem with a finite vector-valued objective function were given by Deng [3]. This implicitly assumed that the vector-valued function is locally Lipschitz and hence continuous. In this paper, we discuss characterizations of the nonemptiness and compactness of the set of weakly efficient solutions for an unconstrained/constrained convex vector optimization problem with extended vector-valued objective function, each of whose component functions is lower semicontinuous (l.s.c. for short).

Throughout the paper, we assume that the objective space is $R^l_\infty = R^l \cup \{+\infty\}$, where $+\infty$ is an imaginary vector in R^l_∞ such that each of its components is $+\infty$ in the extended real value space. Without confusion, we shall not differentiate the vector $+\infty$ in R^l_∞ and the $+\infty$ in the extended real value space. We define the following orderings: for any $z^1 = (z^1_1, \dots, z^1_l)$, $z^2 = (z^2_1, \dots, z^2_l) \in R^l_\infty$,

$$\begin{aligned} z^1 &\leqq z^2 \Leftrightarrow z^1_i \leqq z^2_i, && i = 1, \dots, l, \\ z^1 &\leq z^2 \Leftrightarrow z^1_i \leqq z^2_i, && i = 1, \dots, l \text{ with at least one } i \text{ such that } z^1_i < z^2_i, \\ z^1 &< z^2 \Leftrightarrow z^1_i < z^2_i, && i = 1, \dots, l. \end{aligned}$$

Thus, by $z^2 \geqq z^1$ we naturally mean that $z^1 \leqq z^2$ and by $z^1 \nleqq z^2$ we mean that $z^1 \leqq z^2$ does not hold. For the orderings "$\leq$," "$<$," analogous relations should be understood in the same way as for $\geqq$.

DEFINITION 1.1. Let f: $X \to R$ be a real-valued function. f is said to be *l.s.c. relative to* X at a point $x \in X$ if for every sequence $\{x_k\} \subseteq X$ such that $x_k \to x$ as $k \to +\infty$, we have

$$\liminf_{k\to+\infty} f(x_k) \geqq f(x).$$

If f is *l.s.c. relative to* X at each point $x \in X$, we say that f is *l.s.c. relative to* X.

DEFINITION 1.2. Let $\bar{f}$: $X \to R \cup \{+\infty\}$ be an extended real-valued function. $\bar{f}$ is said to be *l.s.c. relative to* X at a point $x \in X$ if for every sequence $\{x_k\} \subseteq X$ such that $x_k \to x$ as $k \to +\infty$, we have

$$\liminf_{k\to+\infty} \bar{f}(x_k) \geqq F(x).$$

If $\bar{f}$ is *l.s.c. relative to* X at each point $x \in X$, we say that $\bar{f}$ is *l.s.c. relative to* X.

Consider the vector optimization problem $(\overline{VP})$

$$\begin{aligned} &\min \quad \bar{f}(x) \\ &\text{s.t.} \;\; x \in X, \end{aligned}$$

where $X \subset R^n$ is a nonempty and closed set, and $\bar{f}: X \to R^l_\infty$ is an extended vector-valued function such that each component function $\bar{f}_i$ is a l.s.c. extended real-valued function relative to X.

Let $C \subset R^n$ be a nonempty and closed set and let $f: C \to R^l$ be a vector-valued function such that each component function f_i is l.s.c. relative to C. Let

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \in C; \\ +\infty, & \text{else.} \end{cases}$$

Thus, the vector optimization problem (VP) with a set constraint

$$\min_{x \in C} f(x)$$

is equivalent to the vector optimization problem with an extended vector-valued objective function

$$\min_{x \in R^n} \bar{f}(x),$$

which is a special case of $(\overline{VP})$.

It is well known that this approach provides a unified theory and solution methods for solving many optimization problems (see [8] and the references therein). It follows from Definitions 1.1 and 1.2 that each component $\bar{f}_i$ of $\bar{f}$ is l.s.c. relative to X if and only if each component function f_i of f is l.s.c. relative to X. It is clear that each component $\bar{f}_i$ of $\bar{f}$ is proper, l.s.c., and convex if and only if each component function f_i of f is l.s.c. and convex. It was showed in [8, Example 2.38] that a proper, l.s.c., convex, and positively homogeneous function fails to be continuous relative to a compact convex subset of its domain. Their example also shows that a finite, l.s.c., and convex function on a nonempty, compact convex set is not continuous and is even not bounded above on the compact convex set. It is worth mentioning that the problem (VP) was just the problem studied by Deng [1] where $f: R^n \to R^l$ is assumed to be finite and continuous.

Thus, it is a natural question whether results in [3] can be extended to a vector optimization problem with a proper extended vector-valued convex objective function, each of whose component function is l.s.c. and convex. This paper gives an affirmative answer to this question. It is worth noting that Lemma 2.2 in [3] is not valid if not all of the component functions of the objective vector-valued function are finite in the whole space R^n. To establish solution characterizations for a vector optimization problem with a proper extended vector-valued convex objective function, a variant of Lemma 2.2 in [3] must be provided. Furthermore, we shall utilize solution characterizations of related scalar optimization problems in [1, 4].

The outline of the paper is as follows. In Section 2, we shall give a characterization for the nonemptiness and compactness of the weakly efficient solutions of a convex vector optimization problem with no explicit constraints in terms of the nonemptiness and compactness of the optimal sets of scalar optimization problems. This chracterization is analogous to one of the characterizations given by Deng [3]. We shall also give a characterization in terms of the 0-coercivity of each component function $\bar{f}_i$ $(i = 1, \ldots, l)$ of $\bar{f}$ in this section. In Section 3, we shall provide a characterization of the nonemptiness and compactness of the optimal set of an inequality constrained convex scalar optimization problem in terms of the 0-coercivity of the max-composite function of the objective and constraint functions. Then we shall characterize the nonemptiness and compactness of the set of weakly efficient solutions for an inequality constrained convex vector optimization problem in terms of the 0-coercivity of several scalar convex functions. Finally, in Sections 4 and 5, using the characterization results obtained in Section 3, we shall discuss the solution of an inequality constrained convex vector optimization problem by solving a series of unconstrained vector optimization problems.

To conclude this section, we recall some basic concepts in [1, 6, 8] which will be needed in the sequel.

DEFINITION 1.3. A function f: $X \to R^l_\infty$ is said to be *proper* if the domain of f, defined by $\mathrm{dom}(f) = \{x \in X : f(x) \neq +\infty\}$, is not empty.

DEFINITION 1.4. Let f: $X \to R^l$.

(i) A point $x^* \in X$ is said to be an *efficient solution* of f over X if there exists no $x \in X$ such that $f(x) \leq f(x^*)$.

(ii) A point $x^* \in X$ is said to be a *weakly efficient solution* of f over X if there exists no $x \in X$ such that $f(x) < f(x^*)$.

DEFINITION 1.5. Let F: $X \to R^1 \cup \{+\infty\}$ be an extended real-valued function. F is said to be 0-*coercive* if

$$\lim_{\|x\| \to +\infty,\, x \in X} F(x) = +\infty.$$

DEFINITION 1.6. (i) Let C be a nonempty and convex subset of R^n. The *recession cone* of C is defined by

$$C^\infty = \left\{ y \in R^n : \exists\, t_k \to +\infty,\, x_k \in C \text{ such that } y = \lim_{k \to +\infty} x_k / t_k \right\}.$$

(ii) Let $F: R^n \to R^1 \cup \{+\infty\}$ be an extended real-valued convex function. The *recession* function F^∞ of F is defined by $\text{epi}(F^\infty) = [\text{epi}(F)]^\infty$, where the epigraph of F, $\text{epi}(F) = \{(x, r) \in R^n \times R^1 : F(x) \leqq r\}$.

As a straightforward consequence, we get

$$F^\infty(y) = \inf\left\{\liminf_{k\to+\infty} F(t_k x_k)/t_k \colon t_k \to +\infty, x_k \to y\right\},$$

where $\{t_k\}$ and $\{x_k\}$ are sequences in R^1 and R^n, respectively (see [1]).

2. CHARACTERIZATION OF SOLUTION SETS FOR CONVEX VECTOR OPTIMIZATION WITH A SET CONSTRAINT

In this section, we consider the vector optimization problem $(\overline{VP})$

$$\begin{array}{ll} \min & \bar{f}(x) \\ \text{s.t.} & x \in X, \end{array}$$

where $X \subset R^n$ is a nonempty, closed, and convex set, and $\bar{f} = (\bar{f}_1, \ldots, \bar{f}_l)\colon X \to R^l_\infty$ is a proper extended vector-valued function such that its component function $\bar{f}_i$ $(i = 1, \ldots, l)$ is l.s.c. relative to X and convex on X.

We also consider the following related scalar optimization problems $(\overline{P}_i)$ $(i = 1, \ldots, l)$:

$$\begin{array}{ll} \min & \bar{f}_i(x) \\ \text{s.t.} & x \in X. \end{array}$$

Let WE_1 denote the set of weakly efficient solutions of $(\overline{VP})$ and let S_i denote the set of optimal solutions of $(\overline{P}_i)$ $(i = 1, \ldots, l)$.

We need the following lemma.

LEMMA 2.1. *Let X be a nonempty and closed subset of R^n. Let $\varphi\colon X \to R^l_\infty$ be a proper extended vector-valued function such that each component φ_i of φ is l.s.c. relative to X. Suppose that $x_0 \in \text{mbox}(\varphi)$ such that $X_1 = \{x \in X : \varphi(x) \leqq \varphi(x_0)\}$ is a compact set. Then there exists $x^* \in X_1$ such that $\varphi(x) - \varphi(x^*) \nleq 0, \forall x \in X$.*

Proof. It is easy to see that each component φ_i of φ is finite on X_1 and l.s.c. relative to X_1. Applying Corollary 3.2.1 in [6], we obtain a point $x^* \in X_1$ such that it is an efficient solution of φ on X_1, namely,

$$\varphi(x) \nleq \varphi(x^*), \qquad \forall x \in X_1. \tag{1}$$

Since $x^* \in X_1$, showing by contradiction, we obtain that

$$\varphi(x) \nleq \varphi(x^*), \qquad \forall x \in X \backslash X_1. \tag{2}$$

The conclusion follows from (1) and (2). ∎

We have the following result.

LEMMA 2.2. *Consider problem* $(\overline{VP})$ *and* $(\overline{P}_i)$ $(i = 1, \ldots, l)$. *If* WE_1 *is nonempty and compact, then, for each* $i \in \{1, \ldots, l\}, S_i$ *is nonempty and compact.*

Proof. Let

$$X_2 = \{x \in X : \bar{f}(x) < +\infty\},$$
$$f(x) = \bar{f}(x), \qquad x \in X_2.$$

Then $(\overline{VP})$ is equivalent to the vector optimization problem (VP)

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in X_2, \end{array}$$

and $(\overline{P}_i)$ is equivalent to the scalar optimization problem (P_i)

$$\begin{array}{ll} \min & f_i(x) \\ \text{s.t.} & x \in X_2, \end{array}$$

where f_i is the ith component function of f.

It is clear that each component function f_i is finite and convex on the nonempty, closed, and convex subset X_2. It is also l.s.c. relative to X_2. As WE_1 is nonempty and compact, the set of weakly efficient solutions of (VP) is nonempty and compact. It follows from the proof of [3, Lemma 2.1] that

$$X_2^\infty \cap \left(\bigcap_{1 \leqq i \leqq l} \{u : f_i^\infty(u) \leqq 0\} \right) = \{0\}. \tag{3}$$

In what follows, we prove that $S_i \neq \emptyset, \forall\, i \in \{1, \ldots, l\}$. Suppose to the contrary that there exists $k \in \{1, \ldots, l\}$ such that $S_k = \emptyset$. Let

$$A = \{x \in WE_1 : f_k(x) \leqq f_k(y), \forall\, y \in WE_1\}.$$

Since f_k is l.s.c. relative to X_2 and WE_1 is nonempty and compact, it follows that $A \neq \emptyset$. Suppose that $x^* \in A$. Let us prove that $x^* \in S_k$. Suppose to the contrary that there exists $z \in X_2$ such that

$$f_k(z) < f_k(x^*). \tag{4}$$

Then the set

$$D = \{x \in X_2 : f_i(x) \leqq f_i(z),\ i = 1, \ldots, l\}$$

is nonempty and closed since $z \in D$ and f_i is l.s.c. relative to X_2. We show that D is bounded. Otherwise, there exists a sequence $\{x_k\} \subset X_2$ such that $\|x_k\| \to +\infty$ as $k \to +\infty$ and

$$f_i(x_k) \leqq f_i(z), \qquad i = 1, \ldots, l.$$

Without loss of generality, we assume that $x_k/\|x_k\| \to u$. Clearly, $\|u\| = 1$ and $u \in X_2^\infty$ and $f_i^\infty(u) \leqq 0$, $i = 1, \ldots, l$. This contradicts (3). Hence D is nonempty and compact. By Lemma 2.1, there exists $x_0 \in D$ such that

$$f(x) \nleqq f(x_0), \qquad \forall\, x \in X_2;$$

thus, $x_0 \in WE_1$. This, combined with $x^* \in A$, yields

$$f_k(x^*) \leqq f_k(x_0). \tag{5}$$

However, $x_0 \in D$ implies that

$$f_k(x_0) \leqq f_k(z). \tag{6}$$

Inequalities (4)–(6) jointly yield a contradiction. The proof is complete. ∎

LEMMA 2.3. *Let $F\colon X \to R^1 \cup \{+\infty\}$ be a proper, l.s.c. relative to X, and convex function. Then the optimal set of the problem $\min_{x\in X} F(x)$ is nonempty and compact if and only if F is 0-coercive on X.*

Proof. Let

$$\overline{F}(x) = \begin{cases} F(x), & \text{if } x \in X; \\ +\infty, & x \in R^n \setminus X. \end{cases}$$

Then it is easy to see that $\overline{F}$ is a proper, l.s.c., and convex function on R^n, and the optimal set of the problem $\min_{x\in X} \overline{F}(x)$ is the same as that of $\min_{x\in X} F(x)$, which is nonempty and compact. By [4, Proposition 3.2.5 and Definition 3.2.6], we see that this holds if and only if $\overline{F}$ is 0-coercive on R^n. Hence, we deduce that F is 0-coercive on X. ∎

LEMMA 2.4. *Consider $(\overline{VP})$ and $(\overline{P}_i)$ $(i = 1, \ldots, l)$. If for each $i = 1, \ldots, l$, S_i is nonempty and compact, then WE_1 is nonempty and compact.*

Proof. Since $S_i \subset WE_1, \forall\, i \in \{1, \ldots, l\}$, it follows that $WE_1 \neq \emptyset$. As each $\bar{f}_i$ is l.s.c. relative to X, we see that WE_1 is closed. Finally, we show that WE_1 is bounded. Since each S_i is nonempty and compact, by Lemma 2.3, we deduce that each $\bar{f}_i$ is 0-coercive on X. Let $x^* \in \operatorname{dom}(\bar{f})$. Suppose to the contrary that WE_1 is unbounded. Then there exists $\{x_k\} \subset \mathrm{WE}_1$ such that $\|x_k\| \to +\infty$ as $k \to +\infty$. Thus, $\bar{f}_i(x_k) \to +\infty$ as $k \to +\infty$. Hence, when k is sufficiently large,

$$\bar{f}_i(x_k) > \bar{f}_i(x^*), \qquad i = 1, \ldots, l,$$

which contradicts the fact that $x_k \in WE_1$. Thus, WE_1 is bounded. ∎

THEOREM 2.1. *Consider $(\overline{VP})$ and $(\overline{P}_i)$. WE_1 is nonempty and compact if and only if each S_i $(i = 1, \ldots, l)$ is nonempty and compact if and only if each component function $\bar{f}_i$ of $\bar{f}$ $(i = 1, \ldots, l)$ is 0-coercive on X.*

Proof. The conclusion follows from Lemmas 2.2 and 2.4. ∎

3. CHARACTERIZATION OF SOLUTION SETS FOR CONSTRAINED CONVEX VECTOR OPTIMIZATION

In this section, we consider the inequality constrained convex vector optimization problem $(\overline{CVP})$

$$\begin{aligned} &\min\ \bar{f}(x) \\ &\text{s.t.}\ \ x \in X, \\ &\qquad g_j(x) \leqq 0, \qquad j = 1, \ldots, m, \end{aligned}$$

where $X \subset R^n$ is a nonempty, closed, and convex set; $\bar{f} = (\bar{f}_1, \ldots, \bar{f}_l)$: $X \to R^l_\infty$ is a proper vector-valued function such that its component function $\bar{f}_i$ $(i = 1, \ldots, l)$ is l.s.c. relative to X and convex on X; and each g_j: $X \to R^1 \cup \{+\infty\}$ is a proper, l.s.c. relative to X, and convex function.

We denote by WE_2 the set of weakly efficient solutions of $(\overline{CVP})$. Throughout this section, we assume that

$$(\mathrm{H}_1) \qquad \mathrm{dom}(\bar{f}) \cap X_0 \neq \emptyset,$$

$$\text{where } X_0 = \{x \in X\colon g_j(x) \leqq 0, j = 1, \ldots, m\}.$$

Remark 3.1. If $\bar{f}$ is finite on X, then (H_1) is equivalent to $X_0 \neq \emptyset$.

Consider the following scalar optimization problems $(\overline{CP}_i)$ $(i = 1, \ldots, l)$:

$$\begin{aligned} &\min\ \bar{f}_i(x) \\ &\text{s.t.}\ \ x \in X, \\ &\qquad g_j(x) \leqq 0, \qquad j = 1, \ldots, m. \end{aligned}$$

Since (H_1) holds, it follows that

$$\mathrm{dom}(\bar{f}_i) \cap X_0 \neq \emptyset, \qquad \forall\, i \in \{1, \ldots, l\}.$$

Let us still denote by S_i the optimal set of $(\overline{CP}_i)$ $(i = 1, \ldots, l)$.

LEMMA 3.1. *Let $i \in \{1, \ldots, l\}$ and $h_i(x) = \max\{\bar{f}_i(x), g_1(x), \ldots, g_m(x)\}$. Then S_i is nonempty and compact if and only if h_i is 0-coercive on X.*

Proof. "Sufficiency." Let h_i be 0-coercive on X. Let $x_0 \in \mathrm{dom}(\bar{f}_i) \cap X_0$. Then

$$X_3 = \{x \in X_0\colon \bar{f}_i(x) \leqq \bar{f}_i(x_0)\}$$

is nonempty and compact. Indeed, it is clear from the l.s.c. of $\bar{f}_i$ relative to X that X_3 is nonempty and closed. We show that X_3 is bounded. Otherwise, there exists $\{x_k\} \subset X_0$ such that $\|x_k\| \to +\infty$ as $k \to +\infty$ and $\bar{f}_i(x_k) \leqq \bar{f}_i(x_0)$. It follows from the 0-coercivity of h_i that $h_i(x_k) \to +\infty$ as

$k \to +\infty$. Thus, $\max\{g_1(x_k), \ldots, g_m(x_k)\} \to +\infty$, a contradiction of the fact that $\{x_k\} \subset X_0$; thus X_3 is bounded.

Thus we deduce that $S_i \neq \emptyset$. Noting that $S_i \subseteq X_3$, S_i is bounded. In addition, because $\bar{f}_i$ is l.s.c. relative to $X \supseteq X_0$ and X_0 is closed, S_i is closed. Thus we have proved that S_i is nonempty and compact.

"Necessity." Let

$$\bar{f}_i'(x) = \begin{cases} \bar{f}_i(x), & \text{if } x \in X; \\ +\infty, & \text{if } x \in R^n \backslash X, \end{cases}$$

and for $j = 1, \ldots, m$,

$$g_j'(x) = \begin{cases} g_j(x), & \text{if } x \in X; \\ +\infty, & \text{if } x \in R^n \backslash X. \end{cases}$$

Then $(\overline{CP}_i)$ is equivalent to the optimization problem $(\overline{CP}_i')$

$$\begin{aligned} &\min\ \bar{f}_i'(x) \\ &\text{s.t.}\ \ x \in X, \\ &\qquad g_j'(x) \leqq 0, \qquad j = 1, \ldots, m, \end{aligned}$$

which is a problem considered in [1]. It follows from [1] that the optimal set S_i of $(\overline{CP}_i')$ is nonempty and compact if and only if

$$(\bar{f}_i')^{\infty}(u) \leqq 0, \quad (g_j')^{\infty}(u) \leqq 0, \qquad j = 1, \ldots, m \Rightarrow u = 0. \tag{7}$$

Now we prove by contradiction that h_i is 0-coercive on X. Suppose that there exists a sequence $\{x_k\} \subset X$ and a real number $M > 0$ such that

$$h_i(x_k) \leqq M, \qquad \forall\, k.$$

It follows that

$$\bar{f}_i(x_k) \leqq M, \quad g_j(x_k) \leqq M, \qquad \forall\, k,\ j = 1, \ldots, m.$$

Without loss of generality, we assume that

$$\lim_{k \to +\infty} x_k / \|x_k\| = u.$$

Thus

$$\|u\| = 1. \tag{8}$$

Then

$$(\bar{f}_i')^{\infty}(u) \leqq \liminf_{k \to +\infty} \bar{f}_i(x_k)/\|x_k\| \leqq 0, \tag{9}$$

$$(\bar{g}_j')^{\infty}(u) \leqq \liminf_{k \to +\infty} g_j(x_k)/\|x_k\| \leqq 0, \qquad j = 1, \ldots, m. \tag{10}$$

The combination of (8), (9), and (10) contradicts (7). The proof is complete. ∎

THEOREM 3.1. *Consider* $(\overline{CVP})$. *Then the set* WE_2 *is nonempty and compact if and only if each* $h_i(x) = \max\{f_i(x), g_1(x), \ldots, g_m(x)\}$ $(i = 1, \ldots, l)$ *is* 0*-coercive on* X.

Proof. Applying Theorem 2.1 with X replaced by X_0, we see that WE_2 is nonempty and compact if and only if the optimal set S_i of $(\overline{CP}_i)$ is nonempty and compact for each $i \in \{1, \ldots, l\}$. By lemma 3.1, the latter is true if and only if h_i is 0-coercive on X for each $i \in \{1, \ldots, l\}$. ∎

4. APPLICATION I: EXACT SOLUTIONS

In this section, we apply the results of the previous sections to propose a scheme to solve a class of constrained convex vector optimization problems by means of unconstrained vector optimization via a nonlinear Lagrangian.

Consider the inequality constrained vector optimization problem (CVP)

$$\begin{aligned} &\min\ f(x) \\ &\text{s.t.}\ \ x \in X, \\ &\qquad g_j(x) \leqq 0, \qquad j = 1, \ldots, m, \end{aligned}$$

where $X \subset R^n$ is a nonempty and closed set, $f = (f_1, \ldots, f_l)$: $X \to R^l$ is a vector-valued function such that its component function f_i $(i = 1, \ldots, l)$ is l.s.c. relative to X, and each g_j: $X \to R^1$ is l.s.c. relative to X.

Assume throughout this section that $f_i(x) \geqq 0$, $i = 1, \ldots, l$, $\forall x \in X$. This assumption is not restrictive. If it does not hold, we can replace each component function f_i of f with $\exp(f_i)$, and the resulting vector optimization problem has the same sets of efficient solutions and weakly efficient solutions as that of (CVP) and satisfies this assumption. We also assume throughout this section that the feasible set $X_0 = \{x \in X\colon g_j(x) \leqq 0,\ j = 1, \ldots, m\}$ is nonempty.

Now we recall the nonlinear Lagrangian for a constrained vector optimization problem (see [5] for details). Let p: $R^l_+ \times R^m \to R^l$ be a vector-valued function such that each component function p_i $(i = 1, \ldots, l)$ of p is l.s.c. and p enjoys the following properties:

(A) p is increasing: $\forall (z^1, y^1), (z^2, y^2) \in R^l_+ \times R^m$, if $(z^2, y^2) - (z^1, y^1) \in R^l_+ \times R^m_+$, then $p(z^1, y^1) \leqq p(z^2, y^2)$;

(B) there exist positive real numbers $a_1, \ldots, a_m$ such that $p(z, y) \geqq z$ and $p(z, y) \geqq (\max_{1 \leqq j \leqq m}\{a_i y_i\})e_l$, $\forall z \in R^l_+$, $y = (y_1, \ldots, y_m) \in R^m$, where $e_l = (1, \ldots, 1) \in R^l$;

(C) $p(z, 0) = z$, $\forall z \in R^l_+$.

Then

$$L(x, d) = p(f(x), d_1 g_1(x), \ldots, d_m g_m(x)),$$
$$x \in X, \ d = (d_1, \ldots, d_m) \in R^m_+$$

is called a *nonlinear Lagrangian function* corresponding to p.

We attempt to solve (CVP) by solving a sequence of unconstrained vector optimization problems (VQ_d),

$$\begin{array}{l} \min \ L(x, d) \\ \text{s.t.} \ \ x \in X, \end{array}$$

where $d \in R^m_+$.

Let E, E_d, WE_3, and WE_d denote the sets of efficient solutions and weakly efficient solutions of (CVP) and (VQ_d), respectively.

We need the following lemmas.

LEMMA 4.1 [5]. *For any $x \in X_0$ and $d \in R^m_+$, we have $L(x, d) = f(x)$.*

LEMMA 4.2. *Each component function L_i of L is l.s.c. relative to X.*

Proof. This is a direct consequence of [5, Lemma 2.3]. ∎

We have the following result.

THEOREM 4.1. *Consider (CVP) and (VQ_d). Let $h_i(x) = \max\{f_i(x), g_1(x), \ldots, g_m(x)\}$ be 0-coercive on X, $i = 1, \ldots, l$. Let $e_m = (1, \ldots, 1) \in R^m$. Then*

(i) *for each $d \in R^m_+$ with $d - e_m \in R^m_+$, WE_d is nonempty and compact; E_d is nonempty and bounded;*

(ii) *for each selection $x^*_d \in WE_d$, $\{x^*_d\}$ is bounded and all of its limit points belong to WE_3;*

(iii) *$E \neq \emptyset$ and E is bounded; for each $x^* \in E$, there exists a selection $x^*_d \in E_d$ such that $f(x^*) = \lim_{d\to+\infty} f(x^*_d)$ (where $d \to +\infty$ means that each component d_j of d tends to $+\infty$); and WE_3 is nonempty and compact.*

Proof. (i) First we prove that $E_d \neq \emptyset$ for each $d \in R^m_+$ with $d - e_m \in R^m_+$. By the definition of L, we deduce that for each $i \in \{1, \ldots, l\}$,

$$\begin{aligned} L_i(x, d) &\geqq \max\{f_i(x), a_1 d_1 g_1(x), \ldots, a_m d_m g_m(x)\} \\ &\geqq \max\{f_i(x), a_1 g_1(x), \ldots, a_m g_m(x)\}, \end{aligned}$$

where L_i is the ith component function of L and $d - e_m \in R^m_+$. As each h_i is 0-coercive, we deduce that each L_i is 0-coercive. Arbitrarily fix an $x_0 \in C$. It follows from Lemmas 4.1 and 4.2 that $\{x \in X: L(x, d) \leqq L(x_0, d) = f(x_0)\}$ is nonempty and compact. By Lemma 2.1, E_d is nonempty. Hence WE_d is nonempty.

To prove that E_d is bounded, it is sufficient to prove that WE_d is bounded. Suppose to the contrary that $\exists\{x_d^k\} \subset WE_d$ such that $\|x_d^k\| \to +\infty$ as $k \to +\infty$. As each L_i is 0-coercive, we deduce that $L_i(x_d^k, d) \to +\infty$ as $k \to +\infty$. It follows that $L_i(x_d^k, d) > L_i(x_0, d) = f_i(x_0)$, $i = 1, \ldots, l$, when k is sufficiently large, which contradicts the fact that $x_d^k \in WE_d$, $\forall\, k$. Thus we have proved that WE_d is nonempty and bounded. In addition, WE_d is closed by Lemma 4.2. So, WE_d is nonempty and compact.

(ii) Now we show that for each selection $x_d^* \in WE_d$, $\{x_d^*\}$ is bounded. Suppose to the contrary that $\exists\, d^k \to +\infty$ and $x_{d^k}^* \in WE_{d^k}$ such that $\|x_{d^k}^*\| \to +\infty$. As

$$L_i(x_{d^k}^*, d^k) \geqq \max\{f_i(x_{d^k}^*), a_1 g_1(x_{d^k}^*), \ldots, a_m g_m(x_{d^k}^*)\}$$

and

$$\max\{f_i(x_{d^k}^*), a_1 g_1(x_{d^k}^*), \ldots, a_m g_m(x_{d^k}^*)\} \to +\infty$$

by the 0-coercivity of h_i, $i = 1, \ldots, l$, we deduce that

$$L_i(x_{d^k}^*, d^k) \to +\infty, \qquad i = 1, \ldots, l.$$

Hence,

$$L_i(x_{d^k}^*, d^k) > L_i(x_0, d^k) = f_i(x_0), \qquad i = 1, \ldots, l,$$

when k is sufficiently large, which contradicts the fact that $x_{d^k}^* \in WE_{d^k}$, $\forall\, k$. This proves that $\{x_d^*\}$ is bounded.

Suppose that x^* is a limit point of $\{x_d^*\}$, i.e., $\exists\, d^k \to +\infty$ and $x_{d^k}^* \in WE_{d^k}$ such that $x_{d^k}^* \to x^*$ as $k \to +\infty$. We show that $x^* \in WE_3$. First, we show that $x^* \in X_0$. Otherwise,

$$\max\{g_1(x^*), \ldots, g_m(x^*)\} \geqq m_0 > 0$$

for some real number m_0. It follows that

$$\max\{g_1(x_{d^k}^*), \ldots, g_m(x_{d^k}^*)\} \geqq m_0/2$$

when $k \geqq k_0$ for some $k_0 > 0$. So

$$L_i(x_{d^k}^*, d^k) \geqq \frac{m_0}{2} \cdot \min_{1 \leqq j \leqq m} a_j \cdot \min_{1 \leqq j} d_j^k, \qquad i = 1, \ldots, l,$$

when $k \geqq k_0$. Hence,

$$L_i(x_{d^k}^*, d^k) \to +\infty, \qquad i = 1, \ldots, l,$$

as $k \to +\infty$. Thus, for sufficiently large k,

$$L_i(x_{d^k}^*, d^k) > L(x_0, d^k) = f_i(x_0), \qquad i = 1, \ldots, l,$$

a contradiction of the fact that $x_{d^k}^* \in WE_{d^k}$, $\forall\, k$.

Now we show that $x^* \in WE_3$. Suppose to the contrary that there exists $x \in X_0$ and a positive real number m such that

$$f(x) - f(x^*) \leqq -me_l.$$

It follows from the l.s.c. of f_i and the fact that $x^*_{d^k} \to x^*$ that there exists $k_1 > 0$ such that, for $k \geqq k_1$,

$$f(x) - f(x^*_{d^k}) \leqq -m/2e_l.$$

By Lemma 4.1, we have, for $k \geqq k_1$,

$$L(x, d^k) - L(x^*_{d^k}, d^k) \leqq -m/2e_l,$$

a contradiction of the fact that $x^*_{d^k} \in WE_{d^k}$, $\forall k$. Hence $x^* \in WE_3$.

(iii) Let $x_0 \in X_0$ and $X_4 = \{x \in X_0: f(x) \leqq f(x_0)\}$. Then

$$\begin{aligned} X_4 &= \bigcap_{1 \leqq i \leqq l} \{x \in X_0: f_i(x) \leqq f_i(x_0)\} \\ &= \bigcap_{1 \leqq i \leqq l} \{x \in X_0: \max\{f_i(x), g_1(x), \ldots, g_m(x)\} \leqq f_i(x_0)\}, \end{aligned}$$

and X_3 is nonempty and compact since h_i is l.s.c. relative to X and 0-coercive. By Lemma 2.1, $E \neq \emptyset$.

Let $x^* \in E$. For each $d \in R^m_+$ with $d - e_m \in R^m_+$, let

$$X^d = \{x \in X: L(x, d) \leqq L(x^*, d) = f(x^*)\}.$$

Then

$$\begin{aligned} X^d &= \bigcap_{1 \leqq i \leqq l} \{x \in X: L_i(x, d) \leqq f_i(x^*)\} \\ &\subset \bigcap_{1 \leqq i \leqq l} \{x \in X: \max\{f_i(x), a_1 d_1 g_1(x), \ldots, a_m d_m g_m(x)\} \leqq f_i(x^*)\} \\ &\subset \bigcap_{1 \leqq i \leqq l} \{x \in X: \max\{f_i(x), a_1 g_1(x), \ldots, a_m g_m(x)\} \leqq f_i(x^*)\}. \end{aligned}$$

Then X^d is nonempty and compact. By Lemma 2.1, there exists $x^*_d \in E_d \cap X^d$. As $x^*_d \in WE_d$, we deduce from the above proof that $\{x^*_d\}$ is bounded and all of the limit points of $\{x^*_d\}$ belong to X_0. Arbitrarily take a limit point $\bar{x}$ of $\{x^*_d\}$. Then $\exists\, d^k \to +\infty$ and $x^*_{d^k} \in E_{d^k}$ such that

$$\lim_{k \to +\infty} x^*_{d^k} = \bar{x}.$$

Since $x^*_{d^k} \in X^{d^k}$, we have

$$f(x^*_{d^k}) \leqq L(x^*_{d^k}, d^k) \leqq f(x^*).$$

That is,

$$f_i(x^*_{d^k}) \leqq f_i(x^*), \qquad i = 1, \dots, l.$$

It follows from the l.s.c. of f_i relative to X that

$$f_i(\bar{x}) \leqq \liminf_{k\to+\infty} f_i(x^*_{d^k}) \leqq f_i(x^*), \qquad i = 1, \dots, l. \tag{11}$$

This combined with $x^* \in E$ and $\bar{x} \in X_0$ implies that

$$f(\bar{x}) = f(x^*). \tag{12}$$

(11) and (12) jointly yield

$$\lim_{k\to+\infty} f(x^*_{d^k}) = f(\bar{x}) = f(x^*).$$

Since the limit point $\bar{x}$ has been arbitrarily taken, we conclude that

$$\lim_{d\to+\infty} f(x^*_d) = f(x^*).$$

Finally, we show that WE_3 is bounded, which further implies the compactness of WE_3 and the boundedness of E. Suppose to the contrary that WE_3 is unbounded. Then $\exists\, \bar{x}^k \in WE_3$ such that $\|\bar{x}^k\| \to +\infty$ as $k \to +\infty$. Arbitrarily fix an $x_0 \in X_0$. Since $h_i(\bar{x}^k) = \max\{f_i(\bar{x}^k), g_1(\bar{x}^k), \dots, g_m(\bar{x}^k)\} = f_i(\bar{x}^k) \to +\infty$ as $k \to +\infty$, $i = 1, \dots, l$, we deduce that, for sufficiently large k,

$$f_i(\bar{x}^k) > f_i(x_0), \qquad i = 1, \dots, l,$$

a contradiction of $\bar{x}^k \in WE_3$. The proof is complete. ∎

In the following, we consider the convex case of (CVP). We assume

(H_2) $X \subset R^n$ is nonempty, closed and convex; f: $X \to R^l$ is a vector-valued function such that each component function f_i of f is l.s.c. and convex; and each g_j: $X \to R^1$ is l.s.c. and convex.

THEOREM 4.2. *Let assumption* (H_2) *hold. Suppose that* WE_3 *is nonempty and compact. Then*

(i) *for each* $d \in R^m_+$ *with* $d - e_m \in R^m_+$, WE_d *is nonempty and compact*; E_d *is nonempty and bounded*;

(ii) *for each selection* $x^*_d \in WE_d$, $\{x^*_d\}$ *is bounded, and all of its limit points belong to* WE_3;

(iii) $E \neq \emptyset$ *and* E *is bounded*; *for each* $x^* \in E$, *there exists a selection* $x^*_d \in E_d$ *such that* $f(x^*) = \lim_{d\to+\infty} f(x^*_d)$.

Proof. Under assumption (H_2), by Theorem 3.1, that WE_3 is nonempty and compact implies that each $h_i(x) = \max\{f_i(x),\ g_1(x), \ldots, g_m(x)\}$ is 0-coercive, $i = 1, \ldots, l$. Applying Theorem 4.1, the conclusions follow. ∎

Remark 4.1. Each (VQ_d) may not be, in general, a convex vector optimization problem. But each component function p_i of the function p defining the nonlinear Lagrangian L is convex and (H_2) holds; then (VQ_d) is a convex vector optimization problem.

5. APPLICATION II: APPROXIMATE SOLUTIONS

For many vector optimization problems, there do not exist efficient solutions (or even weakly efficient solutions). Then we have to resort to approximate efficient solutions. In this section, we shall consider the acquisition of so-called ϵ-quasi-efficient solutions (or ϵ-quasi-weakly efficient solutions) of a vector optimization problem by using the results in Section 4.

We consider (CVP) and (VQ_d). We assume throughout this section that $f(x) \geqq 0$, $\forall\, x \in X$ and the feasible set $X_0 \neq \emptyset$.

Our consideration is based on the following Ekeland variational principle for vector-valued functions.

PROPOSITION 5.1. *Consider* (CVP). *For any* $\epsilon > 0$, *there exists* $x^* \in X_0$ *such that*

(i) $f(x) - f(x^*) + \epsilon e_l \nleqq 0$, $\forall\, x \in X_0$;

(ii) $f(x) + \epsilon \|x - x^*\| e_l - f(x^*) \nleqq 0$, $\forall\, x \in X_0$.

Proof. It follows from Lemma 1.1 and Corollary 2.1 in [6] that the conclusion holds. ∎

DEFINITION 5.1. Let $x^* \in X_0$. Then

(i) The point x^* is called an *ϵ-quasi-efficient solution* of (CVP) if

$$f(x) + \epsilon \|x - x^*\| e_l - f(x^*) \nleqq 0, \quad \forall\, x \in X_0;$$

(ii) The point x^* is called an *ϵ-quasi-weakly efficient solution* of (CVP) if

$$f(x) + \epsilon \|x - x^*\| e_l \not< f(x^*), \quad \forall\, x \in X_0.$$

Denote by QWE_ϵ the set of all of the ϵ-quasi-weakly efficient solutions of (CVP). Let $x_0 \in X$ and

$$f_\epsilon(x) = f(x) + \epsilon \|x - x_0\| e_l,$$

$$L_\epsilon(x,\ d) = p(f_\epsilon(x),\ d_1 g_1(x),\ \ldots,\ d_m g_m(x)),$$

$$x \in X\ ,\ d = (d_1, \ldots, d_m) \in R^m_+,$$

where p is defined as in Section 4.

Consider the approximate vector optimization problem (CVP_ϵ),

$$\begin{aligned} &\min\ f_\epsilon(x) \\ &\text{s.t.}\ \ x \in X, \\ &\qquad g_j(x) \leqq 0, \qquad j = 1, \dots, m, \end{aligned}$$

and the approximate nonlinear penalty problem (VQ_d^ϵ),

$$\begin{aligned} &\min\ L_\epsilon(x, d) \\ &\text{s.t.}\ \ x \in X. \end{aligned}$$

Let E_ϵ, E_d^ϵ, WE_ϵ, and WE_d^ϵ denote the sets of efficient solutions and the sets of weakly efficient solutions of (CVP_ϵ) and (Q_d^ϵ), respectively.

THEOREM 5.1. *We have*

(a) *for each $d \in R_+^m$ with $d - e_m \in R_+^m$, WE_d^ϵ is nonempty and compact*;

(b) *E_d^ϵ is nonempty and bounded*;

(c) *for each $x^* \in E_\epsilon$, there exists a selection $x_d^* \in E_d^\epsilon$ such that $f(x^*) = \lim_{d\to+\infty} f(x_d^*)$*;

(d) *for each selection $x_d^* \in WE_d^\epsilon$, $\{x_d^*\}$ is bounded and all of its limit points belong to WE_ϵ*;

(e) *any $x^* \in WE_\epsilon$ is an ϵ-quasi-weakly efficient solution of (CVP)*;

(f) *any $x^* \in E_\epsilon$ is an ϵ-quasi-efficient solution of (CVP).*

Proof. As each component function of f_ϵ is 0-coercive on X, all of the conditions of Theorem 4.1 are satisfied. (a), (b), (c), and (d) follow directly from Theorem 4.1. Now we prove (e) and (f). Let $x^* \in WE_\epsilon$. Then x^* is a weakly efficient solution of (CVP_ϵ), namely,

$$f_\epsilon(x) \nless f_\epsilon(x^*), \qquad \forall\, x \in X_0.$$

Thus,

$$f(x) + \epsilon\|x - x_0\|e_l \nless f(x^*) + \epsilon\|x^* - x_0\|e_l,$$

i.e.,

$$f(x) - f(x^*) + \epsilon(\|x - x_0\| - \|x^* - x^0\|)e_l \nless 0. \tag{13}$$

By the triangle inequality, we have

$$\|x - x_0\| - \|x^* - x_0\| \leqq \|x - x^*\|. \tag{14}$$

The combination of (13) and (14) yields

$$f(x) + \|x - x^*\|e_l - f(x^*) \nless 0, \qquad \forall\, x \in X_0,$$

namely, x^* is an ϵ-quasi-weakly efficient solution of (CVP). (f) can be analogously proved. ∎

Consider the scalar optimization problem (P)

$$\begin{aligned} &\inf\ f^1(x) \\ &\text{s.t. } x \in X \\ &\qquad g_j(x) \leqq 0, \qquad j = 1, \ldots, m, \end{aligned}$$

where X is a nonempty and closed subset of R^n, and f^1, g_j: $X \to R^1$ $(j = 1, \ldots, m)$ are finite and l.s.c. relative to X.

DEFINITION 5.2. Let $\epsilon > 0$. $x^* \in X_0$ is called an *ϵ-quasisolution* of (P) if

$$f^1(x^*) \leqq f^1(x) + \epsilon \|x - x^*\|, \qquad \forall\, x \in X_0.$$

Let the problem (CVP) be convex, i.e., let all of the data in the problem (CVP) be convex. It is known that $x^* \in QWE_\epsilon$ if and only if there exists $\lambda = (\lambda_1, \ldots, \lambda_l) \in R^l_+$ with $\sum_{i=1}^l \lambda_i = 1$ such that x^* is an ϵ-quasisolution for the following problem (P_λ):

$$\begin{aligned} &\inf\ \langle \lambda, f(x) \rangle \\ &\text{s.t. } x \in X \\ &\qquad g_j(x) \leqq 0, \qquad j = 1, \ldots, m. \end{aligned}$$

That is,

$$\langle \lambda, f(x^*) \rangle \leqq \langle \lambda, f(x) \rangle + \epsilon \|x - x^*\|, \qquad \forall\, x \in X_0. \tag{15}$$

On the other hand, (CVP_ϵ) is also a convex vector optimization problem. $x^{**} \in WE_\epsilon$ if and only if there exists $\lambda = (\lambda_1, \ldots, \lambda_l) \geq 0$ with $\sum_{i=1}^l \lambda_i = 1$ such that

$$\langle \lambda, f(x^{**}) \rangle + \epsilon \|x^{**} - x^0\| \leqq \langle \lambda, f(x) \rangle + \epsilon \|x - x_0\|, \qquad x \in X_0. \tag{16}$$

Furthermore, (16) implies

$$\langle \lambda, f(x^{**}) \rangle \leqq \langle \lambda, f(x) \rangle + \epsilon \|x - x^{**}\|, \qquad \forall\, x \in X_0. \tag{17}$$

Therefore, x^{**} is an ϵ-quasisolution of (P_λ). This establishes some further relationship (in addition to the relation $WE_\epsilon \subseteq QWE_\epsilon$, which is obtained in (f) of Theorem 5.1) between QWE_ϵ and WE_ϵ. That is, for any $x^* \in QWE_\epsilon$, there exists $\lambda = (\lambda_1, \ldots, \lambda_l) \in R^l_+$ with $\sum_{i=1}^l \lambda_i = 1$ such that x^* is an ϵ-quasisolution of (P_λ). Corresponding to this λ, we can find an $x^{**} \in X_0$ such that (16) and (17) hold. Hence, $x^{**} \in WE_\epsilon$, $x^{**} \in QWE_\epsilon$, and x^{**} is also an ϵ-quasisolution of (P_λ) such that $|\langle \lambda, f(x^*) \rangle - \langle \lambda, f(x^{**}) \rangle| \leqq \epsilon \|x^* - x^{**}\|$.

6. CONCLUSIONS

In this paper, characterizations of the nonemptiness and compactness of the set of weakly efficient solutions were established for a convex vector optimization problem with extended valued functions. These results were applied to discuss solution characterizations of a class of constrained convex vector optimization problems.

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