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Near-best quasi-interpolants associated with H-splines on a three-direction mesh☆

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Abstract

Spline quasi-interpolants with optimal approximation orders and small norms are useful in several applications. In this paper, we construct the so-called near-best discrete and integral quasi-interpolants based on H-splines, i.e., B-splines with regular hexagonal supports on the uniform three-directional mesh of the plane. These quasi-interpolants are obtained so as to be exact on some space of polynomials and to minimize an upper bound of their infinite norms which depend on a finite number of free parameters. We show that this problem has always a solution, which is not unique in general. Concrete examples of these types of quasi-interpolants are given in the two last sections.

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Let τ be the uniform triangulation of the plane, whose set of vertices is $\mathbb{Z}^2$, and whose edges are parallel to the three directions $e_1=(1,0)$, $e_2=(0,1)$ and $e_3=(1,1)$. Let $\mathbb{P}_n$ be the space of bivariate polynomials of total degree at most n, and let $\mathbb{P}_n^k(\tau)$, $k\in\mathbb{N}$, be the space of piecewise polynomial functions of degree n

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and class C^k defined on τ. In this paper, we consider only H-splines, i.e., B-splines with regular hexagonal supports (whose sides are composed of the same number of edges of τ). The family of H-splines contains the classical box-splines in $\mathbb{P}^{2k}_{3k+1}(\tau)$ for $k \geqslant 0$, together with new families of B-splines introduced in [8,11,14].

For a given H-spline φ, $\mathscr{S}(\varphi)$ denotes the space of splines $\{\sum c(\alpha)\varphi(. - \alpha), \alpha \in \mathbb{Z}^2$ and $c(\alpha) \in \mathbb{R}\}$ generated by the family of translates $\mathscr{B}(\varphi) = \{\varphi(. - \alpha), \alpha \in \mathbb{Z}^2\}$.

All the families $\mathscr{B}(\varphi)$ that we use are globally linearly independent, namely,

$$\sum_{\alpha \in \mathbb{Z}^2} c(\alpha)\varphi(. - \alpha) = 0 \quad \text{implies } c(\alpha) = 0 \quad \text{for all } \alpha \in \mathbb{Z}^2.$$

We denote by $\mathbb{P}(\varphi)$ the space of polynomials of maximal total degree included in $\mathscr{S}(\varphi)$. We construct new families of discrete or integral quasi-interpolants from $C^{k+1}(\mathbb{R}^2)$ into $\mathscr{S}(\varphi)$ which are exact on $\mathbb{P}(\varphi)$, and minimize a simple upper bound of their uniform norm. These quasi-interpolants can be considered as extensions to the bivariate case of those introduced in [2] and [3]. They have the form $Qf = \sum_{\alpha \in \mathbb{Z}^2} \lambda_\alpha(f)\varphi(. - \alpha)$, where $\lambda_\alpha(f)$ is a finite combination of values $f(\beta)$ or mean values $\langle f, \varphi(. - \beta)\rangle = \int f(x)\varphi(x - \beta)\mathrm{d}x$, with $\beta \in \mathbb{Z}^2$ lying in some hexagon centered at $\alpha \in \mathbb{Z}^2$. Such operators have already been considered by many authors (see [5,4]), but the ones presented here seem to be new and interesting.

The paper is organized as follows. In Section 2, we recall some results on H-splines and hexagonal sequences. Then, in Section 3, we introduce discrete and integral quasi-interpolants (QIs) based on some H-spline φ and which are exact on $\mathbb{P}(\varphi)$. Starting from these QIs, we study in Section 4 new families of QIs. They are obtained by solving a minimization problem that admits always a solution. Finally, in Sections 4 and 5, we give two examples of each type of these operators. In particular, we show that they are not unique in general.

1. H-splines, symmetrical hexagonal sequences and difference operators

1.1. H-splines

For $p \geqslant 0$, we denote by H_p the hexagon in τ centered at the origin, with sides of length p. For $p = 0$, we define $H_0 = \{0\}$.

Let π_r, $r \geqslant 0$, be a H-spline supported on H_1 of class C^r and of minimal degree $d(r)$ for which $\mathscr{B}(\pi_r)$ is a partition of unity. It is proved in [9] that π_r is unique with $d(r) = 3r + 1$ for r even and $3r + 2$ for r odd. If we put $\pi = \pi_0$ the classical piecewise affine pyramid, then $\pi^k = \pi * \cdots * \pi$ (k times) is the box-spline in $\mathbb{P}^{2k}_{3k+1}(\tau)$. For $k = 0$, we define $\pi^0_r = \pi_r$ and for $k \geqslant 1$, $\pi^k_r = \pi_r * \pi^{k-1}$. Note that the power is the convolution power.

Using classical results on the convolution product of piecewise polynomial functions and the Strang–Fix theory (see [15]), the following result has been established in [14] (see also [13]).

Theorem 1. (i) *The support of* π^k_r *is the hexagon* H_{k+1}.

(ii) π^k_r *is a positive B-spline of class* C^{r+2k}, *of degree* $3(r+k)+1$ *for* r *even and of degree* $3(r+k)+2$ *for* r *odd*.

(iii) *For $k \geqslant 1$ we have*

$$\mathbb{P}(\pi_r^k) = \begin{cases} \mathbb{P}_{2k+1} & \textit{when } r = 0, \\ \mathbb{P}_{2k} & \textit{when } r \geqslant 1. \end{cases}$$

(iv) *The family $\mathscr{B}(\pi_r^k)$ is globally linearly independent.*

From Property (iii), we deduce immediately that the approximation order of a smooth function in the space $\mathscr{S}(\pi_r^k)$ is $2k+2$ for $r=0$ and $2k+1$ for $r \geqslant 1$. In the literature, there exist different methods to construct spline operators giving this order of approximation. For instance, in [4] and [5] are described quasi-interpolants using Appell sequences, Neumann series or Fourier transform. In [10] and [12], discrete and integral quasi-interpolants are defined from the values of an H-spline on a three direction mesh by exploiting the relation between hexagonal sequences and central difference operators. It seems that this later method is best adapted for the study proposed here. So, in the following subsections we recall some properties of hexagonal sequences and of the associated algebra of difference operators. For more details see e.g. [10].

1.2. Hexagonal sequences

Let $\mathscr{H}_p$ be the vector space of real sequences $\{c(\alpha), \alpha \in \mathbb{Z}^2\}$ having their support in H_p, i.e., satisfying $c(\alpha) = 0$ for all $\alpha \notin H_p^* = H_p \cap \mathbb{Z}^2$, and which are invariant by the group of symmetries and rotations of the hexagon H_p. It is easy to prove the following result.

Theorem 2.

$$\dim \mathscr{H}_p = \begin{cases} (q+1)^2 & \textit{when } p = 2q, \\ (q+1)(q+2) & \textit{when } p = 2q+1. \end{cases}$$

Then, with any sequence $c \in \mathscr{H}_p$, we associate a list $\tilde{c} = [c_{\alpha_1}, \ldots, c_{\alpha_n}]$, where $n = \dim \mathscr{H}_p$. The correspondence between the list and the actual sequence is described in Fig. 1 for $p = 2$, $n = 4$.

Let $\mathsf{d}_1 \in \mathscr{H}_1$ and $\mathsf{d}_2 \in \mathscr{H}_2$ be two hexagonal sequences associated respectively with the lists $\tilde{\mathsf{d}}_1 = [-6, 1]$ and $\tilde{\mathsf{d}}_2 = [-6, 0, 1, 0]$. We denote by $I \in \mathscr{H}_0$, the sequence associated with the list reduced to [1]. For $p \geqslant 0$, let T_p be the subset of $(m, n) \in \mathbb{N}^2$ such that $0 \leqslant m + 2n \leqslant p$ and $\mathscr{B}_p = \{\mathsf{d}_1^m \mathsf{d}_2^n, (m, n) \in T_p\}$, where the products are convolution products, i.e., the elements d_1^m, d_1^n and $\mathsf{d}_1^m \mathsf{d}_2^n$ of the spaces $\mathscr{H}_m$, $\mathscr{H}_n$ and

$$\begin{array}{ccccc}
 & & c_{20} & c_{21} & c_{20} \\
 & c_{21} & c_{10} & c_{10} & \underline{c_{21}} \\
c_{20} & c_{10} & \underline{c_{00}} & \underline{c_{10}} & \underline{c_{20}} \\
c_{21} & c_{10} & c_{10} & c_{21} & \\
c_{20} & c_{21} & c_{20} & &
\end{array} \qquad \tilde{c} = [c_{00}, c_{10}, c_{20}, c_{21}]$$

Fig. 1. A sequence c and its corresponding $\tilde{c}$.

$\mathscr{H}_{m+2n}$ respectively are given by:

$$\mathsf{d}_1^m = \Big\{ \mathsf{d}_1^m(j) \text{ such that } \mathsf{d}_1^1(j) = \mathsf{d}_1(j) \text{ for } j \in H_1^*, \text{ and}$$
$$\mathsf{d}_1^m(j) = \sum_{i \in H_1^*} \mathsf{d}_1(i)\mathsf{d}_1^{m-1}(j-i) \text{ for } j \in H_m^* \Big\},$$

$$\mathsf{d}_2^n = \Big\{ \mathsf{d}_2^n(j) \text{ such that } \mathsf{d}_2^1(j) = \mathsf{d}_2(j) \text{ for } j \in H_2^*, \text{ and}$$
$$\mathsf{d}_2^n(j) = \sum_{i \in H_2^*} \mathsf{d}_2(i)\mathsf{d}_2^{n-1}(j-i) \text{ for } j \in H_{2n}^* \Big\},$$

and

$$\mathsf{d}_1^m\mathsf{d}_2^n = \Big\{ \mathsf{d}^{m,n}(j) \text{ such that } \mathsf{d}^{m,n}(j) = \sum_{i \in H_m^*} \mathsf{d}_1^m(i)\mathsf{d}_2^n(j-i) \text{ for } j \in H_{2n}^* \Big\}.$$

Then, it is easy to check that $\dim \mathscr{H}_p = \operatorname{card} \mathscr{B}_p$ and, by induction on p, one can prove that $\mathscr{B}_p$ is a basis for the space $\mathscr{H}_p$.

1.3. The algebra of difference operators

To the above hexagonal sequences d_1 and d_2 of the spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, we associate the following difference operators Δ_1 and Δ_2 defined, for $k = 1$ or 2, by

$$(\Delta_k f)(x) = f(x + ke_1) + f(x + ke_2) + f(x + ke_3) - 6f(x) + f(x - ke_1) + f(x - ke_2) + f(x - ke_3),$$

which stand for the discrete schemes of the Laplacian operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s y^2}$.

Then, the relation between hexagonal sequences and these difference operators is given by the following identity:

$$(\Delta_k f)(\alpha) = (\mathsf{d}_k * f)(\alpha),$$

where f denotes here the sequence $\{f(\alpha), \alpha \in \mathbb{Z}^2\}$.

Moreover, if we denote by $Ł_p$, $p \geqslant 0$, the space with basis $\{\Delta_1^m \Delta_2^n, (m, n) \in T_p\}$, then it is clear that the two spaces $Ł_p$ and $Ł_p$ are isomorphic. On the other hand, it is simple to see that each element D of $Ł_p$, $p \geqslant 0$, has an hexagonal support. Then, its inverse D^{-1} in the convolution algebra $l^1(\mathbb{Z}^2)$ has a non-bounded support. However, we show in the following result that D^{-1} is finite when restricted to some spaces of polynomials.

Lemma 3. *Let $k \in \mathbb{N}^*$ and $D = \sum_{(m,n) \in T_p} \alpha(m, n)\Delta_1^m \Delta_2^n \in Ł_p$. Then the inverse D^{-1} of D restricted to the space $\mathbb{P}_{2k+1}$ is an element of $Ł_{2p}$ and it is given by*

$$D^{-1} = \sum_{r+s \leqslant k} \beta(r, s)\Delta_1^r \Delta_2^s,$$

where $\beta(r,s)$ are solutions of the following linear system:

$$\sum_{r+m\leqslant u,\ s+n\leqslant v} \alpha(m,n)\beta(r,s)=\begin{cases}1 & \textit{for } (u,v)=(0,0),\\ 0 & \textit{for } (u,v)\neq(0,0).\end{cases}$$

Proof. It derives from the fact that $\Delta_1^m\Delta_2^n p=0$ for all $p\in\mathbb{P}_{2r-1}$ such that $m+n=r\geqslant 1$, and the degree $2r-1$ is maximal. □

2. Quasi-interpolants based on H-splines

As indicated in the introduction, our aim is to study new families of discrete and integral quasi-interpolants based on some H-spline φ. They are obtained by solving minimization problems under some linear constraints. In order to give the explicit formulae of these linear constraints, it is necessary to express all the monomials of $\mathbb{P}(\varphi)$ as linear combinations of integer translates of φ. To do this, we need some results concerning differential quasi-interpolants (see [6]).

2.1. Differential quasi-interpolants (DQIs)

Let φ be a H-spline of support H_{k+1}, $k\geqslant 0$, and let $\hat{\varphi}$ be its Fourier transform. As $\hat{\varphi}(0)=1$, we have in some neighbourhood of the origin

$$\frac{1}{\hat{\varphi}(y)}=\sum_{\alpha\in\mathbb{N}^2} a_\alpha y^\alpha.$$

Let d be the integer such that $\mathbb{P}_d=\mathbb{P}(\varphi)$ and $m_\alpha(x)=x^\alpha$ the monomials of $\mathbb{P}(\varphi)$. We denote by d the following differential operator

$$\mathrm{d}f=\sum_{|\alpha|\leqslant d}(-\mathbf{i})^{|\alpha|}a_\alpha D^\alpha f,\qquad \text{where } \mathbf{i} \text{ is the complex such that } \mathbf{i}^2=-1,$$

and by $Sf=\sum_{i\in\mathbb{Z}^2} f(i)\varphi(.-i)$ denotes the classical Schoenberg operator. Then it is well known , see e.g., [8,12], that S is an automorphism on $\mathbb{P}(\varphi)$ and satisfies

$$Sm_\alpha=\sum_{\beta\leqslant\alpha}\frac{\alpha!}{\beta!}(-\mathbf{i}D)^\beta\hat{\varphi}(0)D^\beta m_\alpha,\quad \text{and}\quad S^{-1}m_\alpha=g_\alpha\quad \text{for all } \alpha\in\Gamma_\varphi,$$

where $\Gamma_\varphi=\{\alpha\in\mathbb{N}^2,\ m_\alpha\in\mathbb{P}(\varphi)\}$ and g_α is a recursive family of polynomials defined by

$$g_0=m_0,$$
$$g_\alpha=m_\alpha-\sum_{j\in\mathbb{Z}^2}\varphi(j)\sum_{\beta\leqslant\alpha,\beta\neq\alpha}\frac{(-j)^{\alpha-\beta}\alpha!}{(\alpha-\beta)!}g_\beta. \tag{1}$$

Moreover, we have the following result.

Lemma 4. *The operator* d *coincides on* $\mathbb{P}(\varphi)$ *with* S^{-1}. *Therefore* d *is also an automorphism on* $\mathbb{P}(\varphi)$.

Proof. Consider the power series expansion $\hat{\varphi}(y)=\sum_{\beta\in\mathbb{N}^2}\frac{1}{\beta!}D^\beta\hat{\varphi}(0)y^\beta$. Hence, $\hat{\varphi}\hat{\varphi}^{-1}=1$ implies that

$$\sum_{\alpha+\beta=\gamma}\frac{a_\alpha}{\beta!}D^\beta\hat{\varphi}(0)=\delta_{0\gamma}=\begin{cases}1 & \text{when } \gamma=0,\\ 0 & \text{when } \gamma\neq 0.\end{cases}$$

On the other hand, for all $\alpha\in\Gamma_\varphi$ we have

$$\begin{aligned}m_\alpha &= \sum_{\gamma\leqslant\alpha}(-\mathbf{i}D)^\gamma m_\alpha\delta_{0\gamma}=\sum_{\gamma\leqslant\alpha}(-\mathbf{i}D)^\gamma m_\alpha\sum_{\beta+\theta=\gamma}\frac{a_\beta}{\theta!}D^\theta\hat{\varphi}(0)\\ &= \sum_{\beta,\theta\leqslant\alpha}(-\mathbf{i}D)^{\beta+\theta}m_\alpha\frac{a_\beta}{\theta!}D^\theta\hat{\varphi}(0)=\sum_{\theta\leqslant\alpha}\left(\sum_{\beta\in\Gamma_\varphi}a_\beta(-\mathbf{i}D)^\beta(D^\theta m_\alpha)\right)\frac{(-\mathbf{i}D)^\theta\hat{\varphi}(0)}{\theta!}\\ &= \sum_{\theta\leqslant\alpha}\mathrm{d}(D^\theta m_\alpha)\frac{(-\mathbf{i}D)^\theta\hat{\varphi}(0)}{\theta!}=\mathrm{d}\left(\sum_{\theta\leqslant\alpha}D^\theta m_\alpha\frac{(-\mathbf{i}D)^\theta\hat{\varphi}(0)}{\theta!}\right)\\ &= \mathrm{d}Sm_\alpha.\end{aligned}$$

Then, we deduce that $\mathrm{d}=S^{-1}$ and consequently d is an automorphism on $\mathbb{P}(\varphi)$. $\square$

Now, using the operator d, we define the following differential quasi-interpolant:

$$\mathrm{d}f=S\mathrm{d}f=\sum_{j\in\mathbb{Z}^2}\left(\sum_{|\alpha|\leqslant d}(-\mathbf{i})^{|\alpha|}a_\alpha D^\alpha f(j)\right)\varphi(.-j).$$

Thus, it is clear that d is exact on $\mathbb{P}_d$.

According to Section 2, the space $\mathbb{P}_d$ coincides with $\mathbb{P}_{2k+1}$ when φ is a box-spline in $\mathbb{P}^{2k}_{3k+1}(\tau)$. In this case, the Fourier transform $\hat{\varphi}$ is well known and the computation of the coefficients a_α can be done directly. Therefore, as

$$\mathrm{d}m_\alpha=m_\alpha,\quad \text{for all } \alpha\in\mathbb{P}_{2k+1},$$

we easily deduce the needed expressions of m_α.

For a H-spline φ which is not a box-spline, we have not in general the explicit formula of its Fourier transform. However, as shown in the following result, the associated coefficients a_α are determined only in terms of the values $\varphi(j)$, $j\in\mathrm{supp}(\varphi)\cap\mathbb{Z}^2$, which can be computed by standard convolution algorithms (see e.g., [8]).

Lemma 5. *For any $\alpha\in\Gamma_\varphi$, we have*

$$a_\alpha=\mathbf{i}^{|\alpha|}g_\alpha(0).$$

Proof. It derives from the fact that $g_\alpha=S^{-1}m_\alpha=\mathrm{d}m_\alpha$, for all $\alpha\in\Gamma_\varphi$. $\square$

2.2. Discrete quasi-interpolants (dQIs)

Let $\Phi=\{\varphi(\alpha),\ \alpha \in H_k^*=H_k \cap \mathbb{Z}^2\}$ be the hexagonal sequence of h_k associated with the H-spline φ, and $D \in Ł_k$ its corresponding difference operator. As the above Schoenberg operator S is an automorphism on $\mathbb{P}(\varphi)$, there exists for each $p \in \mathbb{P}(\varphi)$ a unique $q \in \mathbb{P}(\varphi)$ such that $p = Sq$. Then, according to the definition of S, we obtain

$$Sp = \sum_{i\in\mathbb{Z}^2} Sq(i)\varphi(.-i) = \sum_{i\in\mathbb{Z}^2}\left(\sum_{\alpha\in\mathbb{Z}^2} q(\alpha)\varphi(i-\alpha)\right)\varphi(.-i)$$
$$= \sum_{i\in\mathbb{Z}^2}\left(\sum_{\alpha\in H_k^*}\varphi(\alpha)q(i+\alpha)\right)\varphi(.-i) = \sum_{i\in\mathbb{Z}^2} Dq(i)\varphi(.-i).$$

On the other hand, using the fact that

$$\sum_{i\in\mathbb{Z}^2}\Delta_r q(i)\varphi(.-i) = \sum_{i\in\mathbb{Z}^2} q(i)\Delta_r\varphi(.-i), \quad r = 1 \text{ or } 2,$$

we deduce that

$$Sq = \sum_{i\in\mathbb{Z}^2} Dq(i)\varphi(.-i) = \sum_{i\in\mathbb{Z}^2} q(i)D\varphi(.-i) = DSq = Dp.$$

Hence, S coincides with D on $\mathbb{P}(\varphi)$.

Now, if we set D^{-1} the inverse of D on $\mathbb{P}(\varphi)$, then the discrete quasi-interpolant defined by

$$Qf = SD^{-1}f = \sum_{i\in\mathbb{Z}^2} D^{-1}f(i)\varphi(.-i) = \sum_{i\in\mathbb{Z}^2} f(i)(D^{-1}\varphi)(.-i) = D^{-1}Sf$$

is exact on $\mathbb{P}(\varphi)$.

According to Lemma 2.1, the operator D^{-1} is finite on $\mathbb{P}(\varphi)$, and it can be written in the form

$$D^{-1}f = \sum_{\alpha\in H_k^*} c_\alpha f(.+\alpha).$$

Therefore, the above expression of Qf becomes

$$Qf = \sum_{i\in\mathbb{Z}^2}\left(\sum_{\alpha\in H_k^*} c_\alpha f(i+\alpha)\right)\varphi(.-i),$$

which is equivalent to

$$Qf = \sum_{i\in\mathbb{Z}^2} f(i)L(.-i),$$

where L denotes the fundamental function defined by

$$L = \sum_{\alpha\in H_k^*} c_\alpha\varphi(.-\alpha).$$

It is simple to verify that

$$\|Q\|_\infty \leqslant v(c) = \sum_{\alpha\in H_k^*} |c_\alpha|.$$

2.3. Integral quasi-interpolants (iQIs)

It was shown in [8] and [14], that each H-spline φ considered in this paper satisfies $\int \varphi(x)\,\mathrm{d}x = 1$. Then, we can introduce the following integral form of the Schoenberg operator:

$$\tilde{S} f = \sum_{i\in\mathbb{Z}^2} \langle f(.+i), \varphi\rangle \varphi(.-i),$$

where $\langle f, \varphi\rangle = \int f(x)\varphi(x)\,\mathrm{d}x$.

As S, the operator $\tilde{S}$ is also an automorphism on $\mathbb{P}(\varphi)$ and coincides with a difference operator. Indeed, according to Section 3.2, for any $p \in \mathbb{P}(\varphi)$ there exists a unique $q \in \mathbb{P}(\varphi)$ such that $\tilde{S}q = p$. Then,

$$\tilde{S} p = \sum_{i\in\mathbb{Z}^2} \langle Sq(.+i), \varphi\rangle \varphi(.-i) = \sum_{i\in\mathbb{Z}^2} \left(\sum_{\alpha\in\mathbb{Z}^2} v_\alpha q(\alpha+i) \right) \varphi(.-i),$$

where $v_\alpha = \int \varphi(x)\varphi(x-\alpha)\,\mathrm{d}x$. It is simple to see that $v_\alpha = 0$ for all $\alpha \notin H_k^*$. Then, if we put $\tilde{D}q(x) = \sum_{\alpha\in H_k^*} v_\alpha q(x+\alpha)$, we verify easily that

$$\tilde{S} p = \sum_{i\in\mathbb{Z}^2} \tilde{D}q(i)\varphi(.-i) = \sum_{i\in\mathbb{Z}^2} q(i)\tilde{D}\varphi(.-i) = \tilde{D}Sq = \tilde{D}p.$$

Consequently, $\tilde{S}$ coincides on $\mathbb{P}(\varphi)$ with $\tilde{D}$, and $\tilde{D}^{-1}$ has a finite expression on $\mathbb{P}(\varphi)$.

We now consider the following integral quasi-interpolant based on $\tilde{D}^{-1}$:

$$\begin{aligned} Tf = \tilde{S}\tilde{D}^{-1} f &= \sum_{i\in\mathbb{Z}^2} \langle \tilde{D}^{-1} f(.+i), \varphi\rangle \varphi(.-i) \\ &= \sum_{i\in\mathbb{Z}^2} \left(\sum_{\alpha\in H_k^*} d_\alpha \langle f(.+i+\alpha), \varphi\rangle \right) \varphi(.-i). \end{aligned}$$

We remark that for all $p \in \mathbb{P}(\varphi)$, we have $Tp = \tilde{S}\tilde{D}^{-1}p = \tilde{D}\tilde{D}^{-1}p = p$. Thus, the iQI T is exact on $\mathbb{P}(\varphi)$.

Once again, as we obtained above for the dQI Q,

$$\|T\|_\infty \leqslant v(d) = \sum_{\alpha\in H_k^*} |d_\alpha|.$$

The study of these iQIs, illustrated by examples, is given in [8,10,14].

Let us denote by $\mathcal{Q}$ one of the above dQI Q or iQI T. It is well known that the infinite norm of $\mathcal{Q}$ appears in the approximation error of f by $\mathcal{Q}f$. More specifically, we have

$$\|f - \mathcal{Q}f\|_\infty \leqslant (1 + \|\mathcal{Q}\|_\infty)\,\mathrm{dist}(f, S(\varphi)).$$

Then, it is interesting to construct a quasi-interpolant Q with a small norm. In general, it is difficult to minimize the true norm. To remedy partially this problem, Sablonnière has proposed in [12], a method for defining discrete quasi-interpolant with minimal infinite norm. It consisted to constructing bases of the algebras of hexagonal sequences in order to get small norms for the corresponding discrete quasi-interpolants. In the next section, we present another method which seems more interesting.

3. Near-best dQIs and iQIs based on H-splines

The proposed method consists in choosing a priori a sequence c (resp. d) with a larger support than that of φ and afterwards in minimizing $v(c)$ (resp. $v(d)$) under the linear constraints consisting of reproducing all monomials in $\mathbb{P}(\varphi)$. More specifically, for $s \geqslant k$, we construct families of discrete or integral quasi-interpolants:

$$Q_{k+1,s} f = \sum_{i\in\mathbb{Z}^2} \left(\sum_{\alpha\in H_s^*} c_\alpha f(i+\alpha) \right) \varphi(.-i), \tag{2}$$

$$T_{k+1,s} f = \sum_{i\in\mathbb{Z}^2} \left(\sum_{\alpha\in H_s^*} d_\alpha \langle f(.+i+\alpha), \varphi\rangle \right) \varphi(.-i) \tag{3}$$

which satisfy the two following properties:

(i) $Q_{k+1,s}$ and $T_{k+1,s}$ are exact on $\mathbb{P}(\varphi)$.
(ii) The coefficients c_α (resp. d_α), $\alpha \in H_s^*$, are those that minimize the l_1-norm $v(c)$ (resp. $v(d)$) of c (resp. d) under the linear constraints consisting of reproducing all monomials in $\mathbb{P}(\varphi)$.

As a sequence c (resp. d) is fully determined by a list $\tilde{c}=[c_{\alpha_1},\ldots,c_{\alpha_n}]$ (resp. $\tilde{d}=[d_{\alpha_1},\ldots,d_{\alpha_n}]$), it is clear that the exactness of $Q_{k+1,s}$ (resp. $T_{k+1,s}$) on $\mathbb{P}(\varphi)$ implies that there exist a $p\times n$ matrix A of rank $p<n$ and a vector b_1 (resp. b_2) in $\mathbb{R}^p$ such that $A\tilde{c}=b_1$ (resp. $A\tilde{d}=b_2$). For $i=1,2$, set $V_i=\{\tilde{x}\in\mathbb{R}^n : A\tilde{x}=b_i\}$. Then the construction of $Q_{k+1,s}$ or $T_{k+1,s}$ is equivalent to solving the following minimization problem:

Solve $\mathrm{Min}\{\|x\|_1, \tilde{x}\in V_i\}, \quad i=1,2.$

Definition 6. If c (resp. d) is a solution of Problem (1) (resp. Problem (2)), then the associated dQI (resp. iQI) defined by (2) (resp. (3)) is called a near-best dQI (resp. near-best iQI).

Proposition 7. *For $i=1$ or 2, the minimization Problem* (i) *has at least one solution.*

Proof. Since the rank of A is p, the above system $A\tilde{x}=b_i$, $i=1$ or 2, can be solved and each x_{α_j}, $1\leqslant j\leqslant n$, is an affine function of $n-p$ parameters of $\tilde{x}$. Moreover, the sequence x is an element of H_k. On the other hand, by substituting the affine functions x_{α_j} in the expression of $\|x\|_1$, we obtain a $n\times(n-p)$ matrix $\tilde{A}$ and a vector $\tilde{b_i}$ such that $\|x\|_1=\|\tilde{b_i}-\tilde{A}\tilde{x}\|_1$. Thus, solving Problem (i) is equivalent to determine the best linear l_1-approximation of $\tilde{b_i}$ using the elements of $\tilde{A}\tilde{x}$, and the existence of at least one solution is guaranteed. $\square$

Before giving examples of such quasi-interpolants, note that the exactness equations of $T_{k+1,s}$ on $\mathbb{P}(\varphi)$ need the moments $\mu_\alpha(\varphi) = \int m_\alpha(x)\varphi(x)\,\mathrm{d}x,\ \alpha \in \Gamma_\varphi$, of φ. It was shown in [14] that $\mu_\alpha(\varphi) = (\mathbf{i})^{|\alpha|}\mathbf{D}^\alpha\hat{\varphi}(\mathbf{0})$, $|\alpha| = \alpha_1 + \alpha_2$. Then, when φ is a box-spline, we know explicitly its Fourier transform $\hat{\varphi}$ and therefore the computation of $\mu_\alpha(\varphi)$ can be done easily. But, for φ which is not a box-spline, we can determine its corresponding moments by using only the values $\varphi(j),\ \ j \in H_k \cap \mathbb{Z}^2$. Indeed, if we put $t_\alpha = \sum_{j\in\mathbb{Z}^2} m_\alpha(j)\varphi(j)$, then we have the following result.

Lemma 8. *For any $\alpha \in \Gamma_\varphi$ we have*

$$\mu_\alpha(\varphi) = \begin{cases} t_\alpha & \textit{when } |\alpha| \textit{ is even}, \\ 0 & \textit{when } |\alpha| \textit{ is odd}. \end{cases}$$

Proof. According to expression (1), we get the following connection between t_α and g_α.

$$g_\alpha = m_\alpha - \sum_{\beta\leqslant\alpha,\beta\neq\alpha} \frac{(-1)^{|\alpha-\beta|\alpha!}}{(\alpha-\beta)}\, t_{\alpha-\beta}g_\beta. \tag{4}$$

On the other hand, see e.g. [5], the sequence $(g_\alpha)_{\alpha\in\mathbb{N}^2}$ may be written in the form

$$g_0 = m_0,$$
$$g_\alpha = m_\alpha - \sum_{j\in\mathbb{Z}^2}\ \sum_{\beta\leqslant\alpha,\beta\neq\alpha} \frac{\alpha!}{(\alpha-\beta)}\,(-\mathbf{i}D)^{\alpha-\beta}\hat{\varphi}(0)g_\beta. \tag{5}$$

Hence, by comparing (4) and (5), we obtain

$$t_\alpha = (-\mathbf{i}D)^{\alpha-\beta}\hat{\varphi}(0) = \mu_\alpha(\varphi).$$

Using the symmetries of φ, we easily verify that $t_\alpha = (-1)^{|\alpha|}t_\alpha$, i.e., $t_\alpha = 0$ for all α such that $|\alpha|$ is odd. Then, the announced result yields. □

4. Examples of near-best dQIs

4.1. Near-best dQI based on the quartic box-spline π_0^2

The differential quasi-interpolant based on the C^2 quartic box-spline π_0^2 ($k=1$) is given by

$$Df = \sum_{j\in\mathbb{Z}^2} \left(f(i) - \frac{1}{6}\,(D^{(2,0)}f(i) + D^{(1,1)}f(i) + D^{(0,2)}f(i)) \right) \pi_0^2(.-i).$$

As D is exact on $\mathbb{P}_3$, we get the following expressions:

$$m_{0,0} = \sum_{j\in\mathbb{Z}^2} \pi_0^2(.-i), \quad m_{1,0} = \sum_{j\in\mathbb{Z}^2} i_1\pi_0^2(.-i),$$

$$m_{2,0} = \sum_{j\in\mathbb{Z}^2} \left(i_1^2 - \frac{1}{3}\right) \pi_0^2(. - i), \quad m_{1,1} = \sum_{j\in\mathbb{Z}^2} \left(i_1 i_2 - \frac{1}{6}\right) \pi_0^2(. - i),$$

$$m_{3,0} = \sum_{j\in\mathbb{Z}^2} (i_1^3 - i_1)\pi_0^2(. - i), \quad m_{2,1} = \sum_{j\in\mathbb{Z}^2} \left(i_1^2 i_2 - \frac{1}{3} i_1 - \frac{1}{3} i_2\right) \pi_0^2(. - i),$$

and by symmetry we deduce the expressions of $m_{0,1}, m_{0,2}, m_{1,2}$ and $m_{0,3}$.

Now, by using the properties of the hexagonal sequences $(c_\alpha)_{\alpha\in H_s^*}$, it is simple to verify that the quasi-interpolant

$$Q_{2,s} f = \sum_{i\in\mathbb{Z}^2} \left(\sum_{\alpha\in H_s^*} c_\alpha f(i+\alpha) \right) \pi_0^2(. - i), \quad s \geqslant 1,$$

is exact on $\mathbb{P}_3$ if and only if the coefficients c_α satisfy the following equations:

$$\sum_{\alpha\in H_s^*} c_\alpha = 1 \quad \text{and} \quad \sum_{\alpha\in H_s^*} \alpha_1^2 c_\alpha = -\frac{1}{3}.$$

Remark 9. For $s = 1$, the dimension of H_1 coincides with the number of the exactness conditions of $Q_{2,1}$ on $\mathbb{P}_3$. Therefore, $Q_{2,1}$ is unique and it is given by

$$Q_{2,1} f = \sum_{i\in\mathbb{Z}^2} \left(\frac{3}{2} f(i) - \frac{1}{12} \sum_{l=1}^{3} f(i \pm e_l) \right) \pi_0^2(. - i).$$

Thus, in order to have parameters in the minimization problem, it is necessary to take $s > 1$.

Proposition 10. *Let* $c_{0,0}^* = 1 + \frac{1}{2(2t)^2}$ *and* $c_{2t,0}^* = -\frac{1}{12(2t)^2}$. *Then*

$$(c_{0,0}^*, \underbrace{0, \ldots, 0}_{t^2+t-1}, c_{2t,0}^*, \underbrace{0, \ldots, 0}_{t})^{\mathrm{T}} \in \mathbb{R}^{(t+1)^2}$$

is a solution of Problem (1) *for* $k = 1$ *and* $s = 2t$, $t \geqslant 1$.

Proof. For $k = 1$ and $s = 2t$, $t \geqslant 1$, the expression of $\|c\|_1$ is

$$\|c\|_1 = |c_{0,0}| + 6 \sum_{j=1}^{t} (|c_{2j,j}| + |c_{2j,0}|) + 12 \sum_{j=2}^{t} \sum_{l=1}^{j-1} |c_{2j,l}| + 6 \sum_{j=1}^{t-1} |c_{2j+1,0}| + 12 \sum_{j=1}^{t} \sum_{l=1}^{j-1} |c_{2j-1,l}|,$$

and the associated linear constraints in Problem (1) are

$$
\begin{aligned}
1 &= c_{0,0} + 6\sum_{j=1}^{t} c_{2j,j} + c_{2j,0} + 12\sum_{j=2}^{t}\sum_{l=1}^{j-1} c_{2j,l} + 6\sum_{j=1}^{t-1} c_{2j+1,0} + 12\sum_{j=1}^{t}\sum_{l=1}^{j-1} c_{2j-1,l} \\
-\frac{1}{3} &= \sum_{j=1}^{t}\{4(2j)^2 c_{2j,0} + (2(2j)^2 + 4j^2)c_{2j,j}\} + \sum_{j=2}^{t}\sum_{l=1}^{j-1} 4\{(2j)^2 + l^2 + (2j-l)^2\}c_{2j,l} \\
&+ \sum_{j=2}^{t}\sum_{l=1}^{j-1} 4\{(2j-1)^2 + l^2 + (2j-1-l)^2\}c_{2j-1,l}.
\end{aligned} \tag{6}
$$

If we put

$$
\begin{aligned}
\|c\|_1 = \omega(&c_{0,0}, c_{1,0}, c_{2,0}, c_{2,1}, c_{3,0}, c_{3,1}, \ldots, c_{2t-1,0}, c_{2t-1,1}, \ldots, \\
&c_{2t-1,t-1}, c_{2t,0}, c_{2t,1}, \ldots, c_{2t,t-1}, c_{2t,t}),
\end{aligned}
$$

then, by using Eqs. (6), we can express $c_{0,0}$ and $c_{2t,0}$ in terms of the other coefficients of the hexagonal sequence c. Therefore, minimizing $\|c\|_1$ under the linear constraints given in (6) becomes equivalent to minimizing in $\mathbb{R}^{(t+1)^2-2}$ the polyhedral convex function ω of the following variables

$$
c_{1,0}, c_{2,0}, c_{2,1}, c_{3,0}, c_{3,1}, \ldots, c_{2t-1,0}, c_{2t-1,1}, \ldots, c_{2t-1,t-1}, c_{2t,1}, \ldots, c_{2t,t-1}, c_{2t,t}. \tag{7}
$$

Let $c_{i,j}$ be any variable in (7). Denote by $\bar{\omega}(c_{i,j})$ the restriction of ω obtained by replacing its variables by zero except $c_{i,j}$. We will prove that this univariate function $\bar{\omega}(c_{i,j})$ admits a minimum at $0 \in \mathbb{R}$. Indeed, assume for example $c_{i,j} = c_{1,0}$. Then, by annulling the other variables in Eqs. (6), we get the expressions of $c_{0,0}$ and $c_{2t,0}$ in terms of $c_{1,0}$. More precisely, we obtain

$$
\begin{aligned}
c_{0,0} &= c^*_{0,0} - \frac{6}{(2t)^2}((2t)^2 - 1)c_{1,0}, \\
c_{2t,0} &= c^*_{2t,0} - \frac{1}{(2t)^2}c_{1,0}.
\end{aligned}
$$

Thus, $\bar{\omega}(c_{1,0})$ takes the following expression

$$
\begin{aligned}
\bar{\omega}(c_{1,0}) &= |c_{0,0}| + 6|c_{2t,0}| + 6|c_{1,0}| \\
&= \left|c^*_{0,0} - \frac{6}{(2t)^2}((2t)^2 - 1)c_{1,0}\right| + 6\left|c^*_{2t,0} - \frac{1}{(2t)^2}c_{1,0}\right| + 6|c_{1,0}|.
\end{aligned}
$$

It is simple to see that for small values of $c_{1,0}$, $\bar{\omega}(c_{1,0})$ becomes

$$
\begin{aligned}
\bar{\omega}(c_{1,0}) &= c^*_{0,0} - \frac{6}{(2t)^2}\,((2t)^2 - 1)c_{1,0} - 6\left(c^*_{2t,0} - \frac{1}{(2t)^2}\,c_{1,0}\right) + 6|c_{1,0}| \\
&= (c^*_{0,0} - 6c^*_{2t,0}) - \frac{6}{(2t)^2}\,((2t)^2 - 1)c_{1,0} - \frac{6}{(2t)^2}\,c_{1,0} + 6|c_{1,0}| \\
&= \omega^* + \frac{6}{(2t)^2}\,(2 - (2t)^2)c_{1,0} + 6|c_{1,0}|.
\end{aligned}
$$

Therefore, in both cases $c_{1,0} > 0$ and $c_{1,0} < 0$, we easily verify that

$$
\bar{\omega}(c_{1,0}) > \omega^* = \bar{\omega}(0).
$$

A similar technique can be applied for each of the other variables in (7).

Consequently, we conclude that the convex function ω without constraints attains its global minimum at $0 \in \mathbb{R}^{(t+1)^2-2}$. In other words, we have

$$\omega^* = \omega(c^*_{0,0}, \underbrace{0, \ldots, 0}_{t^2+t-1}, c^*_{2t,0}, \underbrace{0, \ldots, 0}_{t}) = \min\{\|c\|_1, \tilde{c} \in V_1\}. \qquad \square$$

Remark 11. A similar result can be obtained when s is odd, i.e., $s = 2t + 1$, $t \geqslant 1$. In this case we have

$$\begin{aligned}\|c\|_1 = |c_{0,0}| &+ 6 \sum_{j=1}^{t} (|c_{2j,j}| + |c_{2j,0}|) + 12 \sum_{j=2}^{t} \sum_{l=1}^{j-1} |c_{2j,l}| \\ &+ 6 \sum_{j=0}^{t} |c_{2j+1,0}| + 12 \sum_{j=1}^{t+1} \sum_{l=1}^{j-1} |c_{2j-1,l}|.\end{aligned}$$

Moreover, if we put $c^*_{0,0} = 1 + \frac{1}{2(2t+1)^2}$ and $c^*_{2t+1,0} = -\frac{1}{12(2t+1)^2}$, then the vector

$$\tilde{c}^* = (c^*_{0,0}, 0, \ldots, 0, c^*_{2t+1,0}, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{R}^{(t+1)(t+2)}$$

is a solution of Problem (1) for $k = 1$ and $s = 2t + 1$, $t \geqslant 1$.

According to Proposition 5.1 and Remark 5.2, the near minimally normed dQIs associated with H_s, $s \geqslant 2$, and exact on $\mathbb{P}_3$ are given by

$$Q_{2,s} f = \sum_{i \in \mathbb{Z}^2} \left(\left(1 + \frac{1}{2s^2}\right) f(i) - \frac{1}{12s^2} \sum_{l=1}^{3} f(i \pm se_l) \right) \pi_0^2(. - i). \tag{8}$$

Proposition 12. *For all $s \geqslant 1$ we have*

$$\|Q_{2,s}\|_\infty \leqslant 1 + \frac{1}{s^2}.$$

Moreover, the sequence $(Q_{2,s})_{s \geqslant 1}$ converges in the infinite norm to the Schoenberg's operator S.

Proof. Let $f \in C(\mathbb{R}^2)$ such that $\|f\|_\infty \leqslant 1$. Then, from (8) we obtain

$$\begin{aligned}|Q_{2,s} f| &\leqslant \sum_{i \in \mathbb{Z}^2} \left(\left(1 + \frac{1}{2s^2}\right) |f(i)| + \frac{1}{12s^2} \sum_{l=1}^{3} |f(i \pm se_l)| \right) \pi_0^2(. - i) \\ &\leqslant \|f\|_\infty \sum_{i \in \mathbb{Z}^2} \left(\left(1 + \frac{1}{2s^2}\right) + \frac{6}{12s^2} \right) \pi_0^2(. - i) \\ &\leqslant 1 + \frac{1}{s^2}.\end{aligned}$$

Hence, $\|Q_{2,s}\|_\infty \leqslant 1 + \frac{1}{s^2}$.

On the other hand, by using the expression of S given in Section 3.1, we get

$$Q_{2,s} f - Sf = \frac{1}{2s^2} \left(f(i) - \frac{1}{6} \sum_{l=1}^{3} f(i \pm se_l) \right) \pi_0^2(. - i).$$

Therefore

$$|Q_{2,s}f - Sf| \leqslant \frac{1}{2s^2} \sum_{i\in\mathbb{Z}^2} (2\|f\|_\infty)\pi_0^2(\cdot - i) \leqslant \frac{1}{s^2}.$$

Then, we conclude that $\|Q_{2,s} - S\|_\infty \leqslant \frac{1}{s^2}$, i.e., $Q_{2,s}$ converges to S when $s \longrightarrow +\infty$. $\square$

Remark 13. Using the Bernstein–Bézier form of π_0^2, we can easily compute the infinite norm of $Q_{2,s}$ for the first values of s. For instance, if $s = 1, 2, 3$, we get

$$\|Q_{2,1}\|_\infty = \tfrac{193}{144} \simeq 1.34028,$$
$$\|Q_{2,2}\|_\infty = \tfrac{59}{48} \simeq 1.22917,$$
$$\|Q_{2,3}\|_\infty = \tfrac{119}{108} \simeq 1.10185.$$

On the other hand, it is simple to check that $\|Q_{2,1}\|_\infty \leqslant 2$, and from Proposition 5.2, we have

$$\|Q_{2,2}\|_\infty \leqslant \tfrac{5}{4} = 1.25 \text{ and } \|Q_{2,3}\|_\infty \leqslant \tfrac{10}{9} \simeq 1.1111.$$

Therefore, the bounds of $\|Q_{2,s}\|_\infty,\ s = 2, 3$, are small in comparison with that of $\|Q_{2,1}\|_\infty$. Moreover, these bounds are close to the exact values of the infinite norm of these new dQIs. In Fig. 2, we give the graphs of fundamental functions corresponding respectively to $Q_{2,1}$, $Q_{2,2}$ and $Q_{2,3}$.

4.2. Near-best dQI based on the box-spline π_0^3

The interest in the study of this example is to show that Problem (1) can have an infinite set of solutions. Indeed, according to Section 2, the box-spline π_0^3 is of class C^4, degree 7 and support H_3. The differential quasi-interpolant based on π_0^3 which is exact on $\mathbb{P}_5$ is defined by

$$Df = \sum_{i\in\mathbb{Z}^2} \lambda_i(f)\pi_0^3(\cdot - i),$$

where

$$\begin{aligned}\lambda_i(f) = f(i) &- \tfrac{1}{4}(D^{(2,0)}f(i) + D^{(1,1)}f(i) + D^{(0,2)}f(i)) \\ &+ \tfrac{1}{30}(D^{(4,0)}f(i) + 2D^{(3,1)}f(i) + 3D^{(2,2)}f(i) + 2D^{(1,3)}f(i) + D^{(0,4)}f(i)).\end{aligned}$$

Then, with the help of D we easily get the expressions of the monomials m_α, $|\alpha| \leqslant 4$, as linear combinations of the integer translates of π_0^3 (see e.g., [7] for more details).

Now, let us consider the dQI

$$Q_{3,s}f = \sum_{i\in\mathbb{Z}^2} \left(\sum_{\alpha\in H_s^*} c_\alpha f(i+\alpha) \right) \pi_0^3(\cdot - i).$$

Using the properties of $(c_\alpha)_{\alpha\in H_s^*}$, we verify that $Q_{3,s}$ is exact on $\mathbb{P}_5$ if and only if

$$\sum_{\alpha\in H_s^*} c_\alpha = 1, \quad \sum_{\alpha\in H_s^*} \alpha_1^2 c_\alpha = -\frac{1}{2} \quad \text{and} \quad \sum_{\alpha\in H_s^*} \alpha_1^4 c_\alpha = \frac{4}{5}.$$

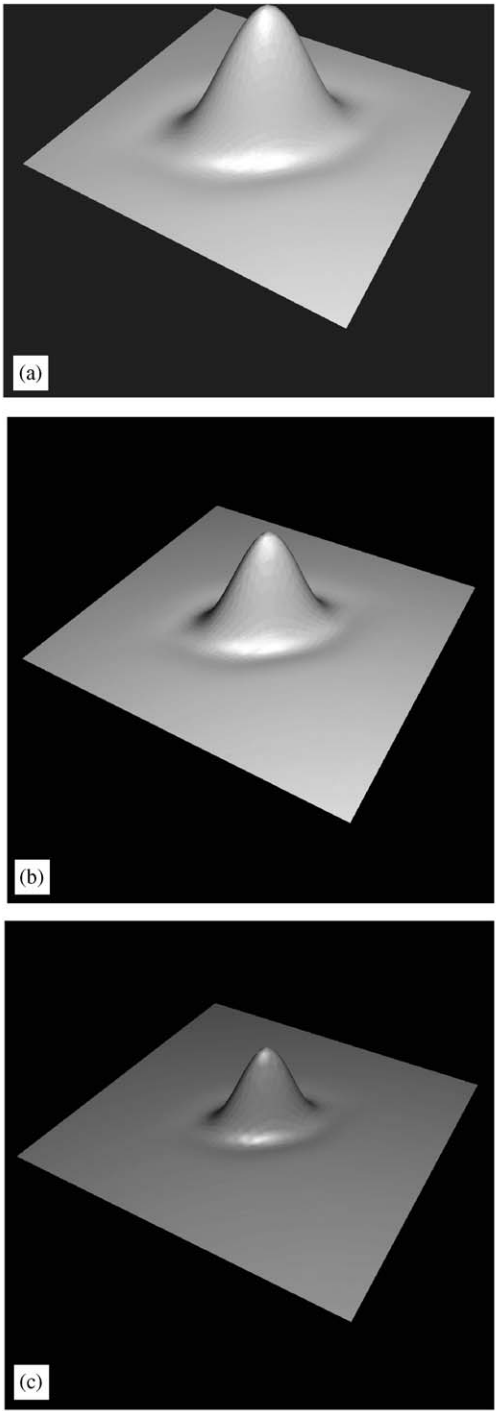

Fig. 2. Graphs of fundamental functions for $s = 1, 2, 3$, respectively.

In particular, for $s=2$, a sequence $c \in H_2$ can be determined only in terms of $c_{0,0}$, $c_{1,0}$, $c_{2,0}$, and $c_{2,1}$. Hence, the above equations of exactness become

$$\begin{aligned}
&c_{0,0}+6c_{1,0}+6c_{2,0}+6c_{2,1}=1,\\
&c_{1,0}+4c_{2,0}+3c_{2,1}=-\tfrac{1}{8},\\
&c_{1,0}+16c_{2,0}+9c_{2,1}=\tfrac{1}{5}.
\end{aligned} \tag{9}$$

Therefore, if we put $c_{2,1}=\gamma$, then the other three coefficients in (9) can be computed in terms of γ. Moreover, we have the following result.

Proposition 14. *For each $\gamma \in [-\frac{7}{30}, 0]$, $Q_{3,2}f=\sum_{i\in\mathbb{Z}^2}\lambda_i(f,\gamma)\pi_0^3(\cdot-i)$, with*

$$\begin{aligned}
\lambda_i(f,\gamma)={}&\left(\frac{179}{80}+3\gamma\right)f(i)+\left(\frac{7}{30}+\gamma\right)\sum_{l=1}^{3}f(i\pm e_l)+\left(\frac{13}{480}-\frac{1}{2}\gamma\right)\sum_{l=1}^{3}f(i\pm 2e_l)\\
&+\gamma(f(\pm(e_1+e_3))+f(\pm(e_2+e_3))+f(\pm(-e_1+e_2)))
\end{aligned}$$

is a near minimally normed dQI associated with π_0^3.

Proof. The solution of system (9) is given by

$$c_{0,0}=\tfrac{179}{80}+3\gamma,\quad c_{1,0}=\tfrac{7}{30}+\gamma,\quad c_{2,0}=\tfrac{13}{480}-\tfrac{1}{2}\gamma,\quad c_{2,1}=\gamma.$$

Then,

$$\begin{aligned}
\|c\|_1&=|c_{0,0}|+6|c_{1,0}|+6|c_{2,0}|+6|c_{2,1}|\\
&=|\tfrac{179}{80}+3\gamma|+6|\tfrac{7}{30}+\gamma|+6|\tfrac{13}{480}-\tfrac{1}{2}\gamma|+6|\gamma|.
\end{aligned}$$

It is simple to check that

$$\min_{\gamma\in\mathbb{R}}\|c\|_1=\tfrac{19}{5}\quad \text{for all } \gamma\in\left[-\tfrac{7}{30},0\right].$$

Consequently, for each $\gamma \in [-\frac{7}{30}, 0]$, we obtain a near-best dQI based on the box-spline π_0^3. In Fig. 3, we give the graph of the fundamental function corresponding to $Q_{3,2}$ for $\gamma=-7/30$. □

5. Examples of near-best iQIs

5.1. Near-best iQI based on the H-spline π_1^1

According to Section 2, the H-spline π_1^1 is supported on H_2, and it is of class C^3 and degree 8. Moreover, the space $S(\pi_1^1)$ contains $\mathbb{P}_2$. It was shown in [14] that the associated differential quasi-interpolant is

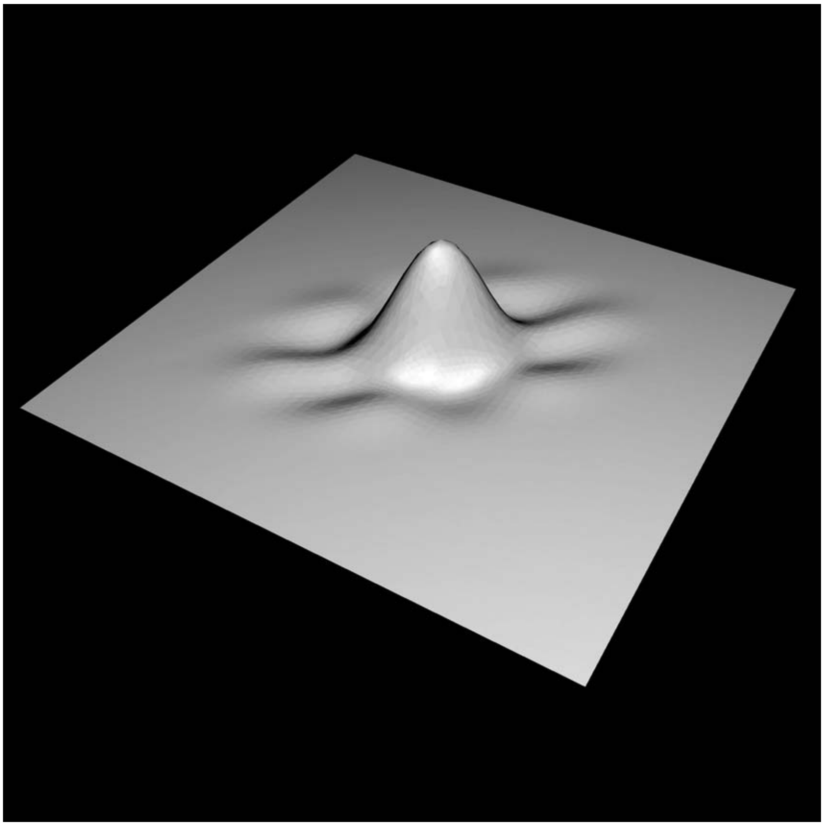

Fig. 3. Graph of the fundamental function of $Q_{3,2}$ for $\gamma = -\frac{7}{30}$.

defined by

$$Df = \sum_{i\in\mathbb{Z}^2} \left(f(i) - \frac{25}{168}\,(D^{(2,0)}f(i) + D^{(1,1)}f(i) + D^{(0,2)}f(i)) \right) \pi_1^1(. - i),$$

and it is exact on $\mathbb{P}_2$. Then we deduce the following formulae:

$$m_{0,0} = \sum_{i\in\mathbb{Z}^2} \pi_1^1(. - i), \quad m_{1,0} = \sum_{i\in\mathbb{Z}^2} i_1 \pi_1^1(. - i),$$

$$m_{2,0} = \sum_{i\in\mathbb{Z}^2} \left(i_1^2 - \frac{25}{84} \right) \pi_1^1(. - i), \quad m_{1,1} = \sum_{i\in\mathbb{Z}^2} \left(i_1 i_2 - \frac{25}{168} \right) \pi_1^1(. - i),$$

and by symmetry we get the expressions of $m_{0,1}$ and $m_{0,2}$.

The near-best iQI based on π_1^1 is given by

$$T_{2,s} f = \sum_{i\in\mathbb{Z}^2} \left(\sum_{\alpha\in H_s^*} d_\alpha \langle f(. + i + \alpha), \pi_1^1 \rangle \right) \pi_1^1(. - i).$$

From Lemma 4.1 we deduce the moments $\mu_\alpha(\pi_1^1) = \int m_\alpha(x)\pi_1^1(x)\mathrm{d}x,\ |\alpha| \leqslant 2$, of π_1^1. Their values are the following:

$$\mu_{(0,0)} = 1,\ \mu_{(1,0)} = \mu_{(1,0)} = 0,\ \mu_{(2,0)} = \mu_{(0,2)} = 2\mu_{(1,1)} = \tfrac{25}{84}.$$

Then, we easily verify that $T_{2,s}$ is exact on $\mathbb{P}_2$ if and only if the coefficients d_α satisfy

$$\sum_{\alpha \in H_s^*} d_\alpha = 1 \quad \text{and} \quad \sum_{\alpha \in H_s^*} \alpha_1^2 d_\alpha = -\frac{25}{42}.$$

In particular, for $s = 1$, these coefficients are unique and the corresponding iQI is given by

$$T_{2,1}f = \sum_{i \in \mathbb{Z}^2} \left(\frac{53}{28}\langle f, \pi_1^1\rangle - \frac{25}{168} \sum_{l=1}^{3} \langle f(. \pm e_l), \pi_1^1\rangle \right) \pi_1^1(. - i).$$

Now, assume that $s > 1$, then by using a similar technique as in Proposition 5.1, one can show the following result.

Proposition 15. *Let $c_{0,0}^* = 1 + \frac{25}{28s^2}$ and $c_{2t,0}^* = -\frac{25}{168s^2}$. Then*

$$(c_{0,0}^*, \underbrace{0, \ldots, 0}_{t^2+t-1}, c_{2t,0}^*, \underbrace{0, \ldots, 0}_{t})^{\mathrm{T}} \in \mathbb{R}^{(t+1)^2}$$

is a solution of Problem (2) *for $k = 1$ and $s > 1$.*

Hence, the near minimally normed iQI based on π_1^1 and exact on $\mathbb{P}_2$ takes the following form:

$$T_{2,s}f = \sum_{i \in \mathbb{Z}^2} \left(\left(1 + \frac{25}{28s^2}\right) \langle f, \pi_1^1\rangle - \frac{25}{168s^2} \sum_{l=1}^{3} \langle f(. \pm e_l), \pi_1^1\rangle \right) \pi_1^1(. - i).$$

It is simple to check that $\|T_{2,s}\|_\infty \leqslant 1 + \frac{25}{14s^2}$, and therefore the sequence $(T_{2,s})_{s \geqslant 1}$ converges in the infinite norm to the operator $\tilde{S}$.

5.2. *Near-best iQI based on the H-spline π_1^3*

According to Section 2, the H-spline π_1^3 is of class C^5, degree 11 and support H_3. As $S(\pi_1^3)$ contains polynomials of total degree $\leqslant 4$, one can define quasi-interpolants which are exact on $\mathbb{P}_4$. For instance, by using only the values of π_1^3 on H_2^*, see Fig. 4, we have got the following expression of its associated differential quasi-interpolant:

$$\mathrm{D}f = \sum_{i \in \mathbb{Z}^2} \lambda_i(f)\pi_1^3(. - i),$$

$$\begin{array}{ccccc} & & a_2 & a_3 & a_2 \\ & a_3 & a_1 & a_1 & a_3 \\ a_2 & a_1 & a_0 & a_1 & a_2 \\ a_3 & a_1 & a_1 & a_3 & \\ a_2 & a_3 & a_2 & & \end{array} \quad \text{with} \quad (a_0, a_1, a_2, a_3) = \left(\frac{24528}{66528}, \frac{6663}{66528}, \frac{48}{66528}, \frac{289}{66528}\right)$$

Fig. 4. The values of π_1^3 on H_2^*.

where

$$\lambda_i(f) = f(i) + \frac{13}{56}(D^{(2,0)}f(i) + D^{(1,1)}f(i) + D^{(0,2)}f(i)) + \frac{2435}{84672}(D^{(4,0)}f(i) + 2D^{(3,1)}f(i) + 3D^{(2,2)}f(i) + 2D^{(1,3)}f(i) + D^{(0,4)}f(i)).$$

Then, the exactness of D on $\mathbb{P}_4$ allows us to express the monomials m_α, $|\alpha| \leqslant 4$, in terms of the integer translates of π_1^3. On the other hand, in order to give an explicit formula of the iQI based on π_1^3, we need to compute the moments $\mu_\alpha(\pi_1^3) = \int m_\alpha(x)\pi_1^3(x)\mathrm{d}x$, $|\alpha| \leqslant 4$. Once again, these moments are determined only in terms of the values given in Fig. 4. Hence, after computation we get

$$\begin{aligned} &\mu_{(0,0)} = 1, \\ &\mu_{(1,0)} = \mu_{(0,1)} = \mu_{(1,2)} = \mu_{(2,1)} = \mu_{(3,0)} = \mu_{(0,3)} = 0, \\ &\mu_{(2,0)} = \mu_{(0,2)} = 2\mu_{(1,1)} = \tfrac{13}{28}, \\ &\mu_{(4,0)} = \mu_{(0,4)} = 2\mu_{(2,2)} = 2\mu_{(3,1)} = 2\mu_{(1,3)} = \tfrac{38}{63}. \end{aligned} \tag{10}$$

We introduce now the following iQI

$$T_{3,s}f = \sum_{i\in\mathbb{Z}^2}\left(\sum_{\alpha\in H_s^*} d_\alpha\langle f(.+i+\alpha), \pi_1^3\rangle\right)\pi_1^3(.-i).$$

Using the values given in (10) and the expressions of the monomials m_α, $|\alpha| \leqslant 4$, as linear combinations of the integer translates of π_1^3 provided by the quasi-interpolant D, we verify that the iQI $T_{3,s}$ is exact on $\mathbb{P}_4$ if and only if

$$\sum_{\alpha\in H_s^*} c_\alpha = 1, \quad \sum_{\alpha\in H_s^*} \alpha_1^2 c_\alpha = 0 \quad \text{and} \quad \sum_{\alpha\in H_s^*} \alpha_1^4 c_\alpha = \frac{307}{3528}.$$

As in Section 5.1, when $s = 2$, a sequence d of H_2 is entirely determined by its elements $d_{0,0}$, $d_{1,0}$, $d_{2,0}$, and $d_{2,1}$. In this case, the exactness equations of $T_{3,2}$ on $\mathbb{P}_4$ are

$$\begin{aligned} &d_{0,0} + 6d_{1,0} + 6d_{2,0} + 6d_{2,1} = 1, \\ &d_{1,0} + 4d_{2,0} + 3d_{2,1} = 0, \\ &d_{1,0} + 16d_{2,0} + 9d_{2,1} = \tfrac{307}{14112}. \end{aligned} \tag{11}$$

Therefore, if we put $d_{21} = \gamma$, $\gamma \in \mathbb{R}$, then the other three coefficients in (11) can be computed in terms of γ. Moreover, we have the following result.

Proposition 16. *For each* $\gamma \in [-\frac{307}{42336}, 0]$, $T_{3,2} f = \sum_{i \in \mathbb{Z}^2} \lambda_i(f, \gamma) \pi_0^3(. - i)$, *with*

$$\lambda_i(f,\gamma) = \left(\frac{29145}{28224} + 3\gamma\right) f(i) - \left(\frac{307}{42336} + \gamma\right) \sum_{l=1}^{3} f(i \pm e_l) + \left(\frac{307}{169344} - \frac{1}{2}\gamma\right) \times \sum_{l=1}^{3} f(i \pm 2e_l) \; + \gamma(f(\pm(e_1+e_3)) + f(\pm(e_2+e_3)) + f(\pm(-e_1+e_2)))$$

is a near minimally normed iQI associated to π_1^3.

Proof. The proof is similar to that of Proposition 15.3. □

Remark 17. According to Proposition 6.2, the near-minimally iQI $T_{3,2}$ is not unique. In addition, for all $\gamma \in [-\frac{307}{42336}, 0]$, we have $\|T_{3,2}\| \leqslant \frac{3835}{3528} = 1.087$. Then, we remark that this bound is close to 1, and therefore this quasi-interpolant seems very interesting.

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