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On the spectral radius and the spectral norm of Hadamard products of nonnegative matrices

Zejun Huang [1]

Department of Mathematics, East China Normal University, Shanghai 200241, China

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ABSTRACT

We prove the spectral radius inequality $\rho(A_1 \circ A_2 \circ \cdots \circ A_k) \leqslant \rho(A_1A_2\cdots A_k)$ for nonnegative matrices using the ideas of Horn and Zhang. We obtain the inequality $\|A \circ B\| \leqslant \rho(A^TB)$ for nonnegative matrices, which improves Schur's classical inequality $\|A \circ B\| \leqslant \|A\|\,\|B\|$, where $\|\cdot\|$ denotes the spectral norm. We also give counterexamples to two conjectures about the Hadamard product.

1. Introduction

Let M_n denote the set of complex matrices of order n. For matrices $A = (a_{ij}), B = (b_{ij}) \in M_n$, we denote by $\rho(A)$ the spectral radius of A, by $A \circ B = (a_{ij}b_{ij})$ the Hadamard product of A and B, and by $A \otimes B$ the Kronecker product of A and B. The notation $A \leqslant B$ means that $B - A$ is entrywise nonnegative, and $\|A\|$ denotes the spectral norm (largest singular value) of A.

Zhan [7] conjectured that $\rho(A \circ B) \leqslant \rho(AB)$ for nonnegative matrices $A, B \in M_n$, which was proved by Audenaert [1], and by Horn and Zhang [4], respectively. The aim of this paper is to generalize

E-mail address: huangzejun@yahoo.cn
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this inequality to an arbitrary finite number of nonnegative matrices by using the ideas of Horn and Zhang, and to prove the inequality $\|A \circ B\| \leqslant \rho(A^T B)$ for nonnegative matrices, which improves Schur's inequality $\|A \circ B\| \leqslant \|A\| \|B\|$ [3, Theorem 5.5.1]. In the last section, we give counterexamples to two conjectures proposed in [2,5].

2. An inequality for the spectral radius

In this section, we generalize the spectral radius inequality $\rho(A \circ B) \leqslant \rho(AB)$ to an arbitrary finite number of nonnegative matrices. For $A \in M_n$ and $\alpha \subset \{1, 2, \ldots, n\}$, $A[\alpha]$ denotes the principal submatrix of A indexed by α. One version of the following lemma can be found in [6, Lemma 2.2]. Here the statement is more explicit and we give a new proof.

Lemma 1. *Let $A_1, A_2, \ldots, A_k \in M_n$. Then*

$$A_1 \circ A_2 \circ \cdots \circ A_k = (A_1 \otimes A_2 \otimes \cdots \otimes A_k)[\alpha],$$

where

$$\alpha = \{1, (n^{k-1} + n^{k-2} + \cdots + n) + 2, 2(n^{k-1} + n^{k-2} + \cdots + n) + 3, \\ 3(n^{k-1} + n^{k-2} + \cdots + n) + 4, \ldots, n^k\}.$$

Proof. Let $e_i \in \mathbb{R}^n$ be the vector whose only nonzero component is the ith component, which equals 1 for $i = 1, \ldots, n$. Set

$$E = (\otimes^k e_1 \otimes^k e_2 \cdots \otimes^k e_n),$$

where we denote by $\otimes^k e_i = e_i \otimes \cdots \otimes e_i$, the k-fold Kronecker product of e_i. One verifies that $E^T A E = A[\alpha]$ for any $A \in M_{n^k}$.

Let $A_t = (a_{ij}^{(t)})$ for $1 \leqslant t \leqslant k$. Then

$$\begin{aligned} a_{ij}^{(1)} a_{ij}^{(2)} \cdots a_{ij}^{(k)} &= (e_i^T A_1 e_j) \otimes (e_i^T A_2 e_j) \otimes \cdots \otimes (e_i^T A_k e_j) \\ &= (\otimes^k e_i)^T (A_1 \otimes A_2 \otimes \cdots \otimes A_k)(\otimes^k e_j) \\ &= e_i^T [E^T (A_1 \otimes A_2 \otimes \cdots \otimes A_k) E] e_j. \end{aligned}$$

Hence

$$A_1 \circ A_2 \circ \cdots \circ A_k = E^T (A_1 \otimes A_2 \otimes \cdots \otimes A_k) E = (A_1 \otimes A_2 \otimes \cdots \otimes A_k)[\alpha]. \quad \square$$

Lemma 2. *Let $A, B, C \in M_n$ be nonnegative and let $\beta \subset \{1, \ldots, n\}$ be nonempty.*

(1) *If $A \leqslant B$, then $\rho(A) \leqslant \rho(B)$.*
(2) *$\rho(A[\beta]) \leqslant \rho(A)$.*
(3) *$A[\beta]B[\beta] \leqslant (AB)[\beta]$.*
(4) *If $A \leqslant B$, then $AC \leqslant BC$.*

Proof. Lemma 2.1 of [4] contains (1), (2), and (3). One verifies (4) with a computation. $\square$

Theorem 3. *Let $A_1, A_2, \ldots, A_k \in M_n$ be nonnegative matrices. Then*

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_k) \leqslant \rho(A_1 A_2 \cdots A_k). \tag{1}$$

Proof. For nonnegative matrices $A_1, A_2, \ldots, A_k \in M_n$, Lemmas 1 and 2 ensure that

$$\begin{aligned} &(A_1 \circ A_2 \circ \cdots \circ A_k)^k \\ &\quad = (A_1 \circ A_2 \circ \cdots \circ A_k)(A_2 \circ \cdots \circ A_k \circ A_1) \cdots (A_k \circ A_1 \circ \cdots \circ A_{k-1}) \end{aligned}$$

$$= (A_1 \otimes A_2 \otimes \cdots \otimes A_k)[\alpha](A_2 \otimes \cdots \otimes A_k \otimes A_1)[\alpha] \cdots (A_k \otimes A_1 \otimes \cdots \otimes A_{k-1})[\alpha]$$
$$\leqslant ((A_1 \otimes A_2 \otimes \cdots \otimes A_k)(A_2 \otimes \cdots \otimes A_k \otimes A_1))[\alpha](A_3 \otimes \cdots \otimes A_k \otimes A_1 \otimes A_2)[\alpha]$$
$$\cdots (A_k \otimes A_1 \otimes \cdots \otimes A_{k-1})[\alpha] \text{ (by (3) and (4) of Lemma 2)}$$
$$\cdots$$
$$\leqslant ((A_1 \otimes A_2 \otimes \cdots \otimes A_k)(A_2 \otimes \cdots \otimes A_k \otimes A_1) \cdots (A_k \otimes A_1 \otimes \cdots \otimes A_{k-1}))[\alpha]$$
$$= ((A_1A_2 \cdots A_k) \otimes (A_2 \cdots A_kA_1) \otimes \cdots \otimes (A_kA_1 \cdots A_{k-1}))[\alpha].$$

Therefore,

$$\rho^k(A_1 \circ A_2 \circ \cdots \circ A_k)$$
$$= \rho((A_1 \circ A_2 \circ \cdots \circ A_k)^k)$$
$$\leqslant \rho(((A_1A_2 \cdots A_k) \otimes (A_2 \cdots A_kA_1) \otimes \cdots \otimes (A_kA_1 \cdots A_{k-1}))[\alpha]) \text{ (by (1) of Lemma 2)}$$
$$\leqslant \rho((A_1A_2 \cdots A_k) \otimes (A_2 \cdots A_kA_1) \otimes \cdots \otimes (A_kA_1 \cdots A_{k-1})) \text{ (by (2) of Lemma 2)}$$
$$= \rho(A_1A_2 \cdots A_k)\rho(A_2 \cdots A_kA_1) \cdots \rho(A_kA_1 \cdots A_{k-1})$$
$$= \rho^k(A_1A_2 \cdots A_k). \qquad \square$$

Since the Hadamard product is commutative, it follows from (1) that

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_k) \leqslant \min_p \rho(A_{p(1)}A_{p(2)} \cdots A_{p(k)}),$$

where p is any permutation of $1, 2, \ldots, k$.

3. Inequalities for the spectral norm

It is known [3, Theorem 5.5.1] that for matrices $A, B \in M_n$, $\|A \circ B\| \leqslant \|A\|\,\|B\|$. A natural question is whether $\|A \circ B\| \leqslant \|AB\|$. This is not true even for two nonnegative matrices A and B. Consider

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

for which

$$\|A \circ B\| = 1 > 0 = \|AB\|.$$

Now we give some inequalities for the spectral norm of the Hadamard product of nonnegative matrices, one of which improves the inequality $\|A \circ B\| \leqslant \|A\|\,\|B\|$.

Denote by A^T the transpose of a matrix A. For an arbitrary finite number of nonnegative matrices, we have

Theorem 4. *Let $A_1, A_2, \ldots, A_k \in M_n$ be nonnegative matrices. Then*

$$\|A_1 \circ A_2 \circ \cdots \circ A_k\| \leqslant \rho^{1/2}(A_1A_1^TA_2A_2^T \cdots A_kA_k^T).$$

Proof. For nonnegative matrices $A_1, A_2, \ldots, A_k \in M_n$, Lemmas 1 and 2 ensure that

$$((A_1 \circ A_2 \circ \cdots \circ A_k)(A_1 \circ A_2 \circ \cdots \circ A_k)^T)^k$$
$$= (A_1 \circ A_2 \circ \cdots \circ A_k)(A_1 \circ A_2 \circ \cdots \circ A_k)^T(A_2 \circ \cdots \circ A_k \circ A_1)(A_2 \circ \cdots \circ A_k \circ A_1)^T$$
$$\cdots (A_k \circ A_1 \circ \cdots \circ A_{k-1})(A_k \circ A_1 \circ \cdots \circ A_{k-1})^T$$
$$= (A_1 \circ A_2 \circ \cdots \circ A_k)(A_1^T \circ A_2^T \circ \cdots \circ A_k^T)(A_2 \circ \cdots \circ A_k \circ A_1)(A_2^T \circ \cdots \circ A_k^T \circ A_1^T)$$
$$\cdots (A_k \circ A_1 \circ \cdots \circ A_{k-1})(A_k^T \circ A_1^T \circ \cdots \circ A_{k-1}^T)$$
$$= (A_1 \otimes A_2 \otimes \cdots \otimes A_k)[\alpha](A_1^T \otimes A_2^T \otimes \cdots \otimes A_k^T)[\alpha](A_2 \otimes \cdots \otimes A_k \otimes A_1)[\alpha]$$
$$(A_2^T \otimes \cdots \otimes A_k^T \otimes A_1^T)[\alpha] \cdots (A_k \otimes A_1 \otimes \cdots \otimes A_{k-1})[\alpha](A_k^T \otimes A_1^T \otimes \cdots \otimes A_{k-1}^T)[\alpha]$$

$$\leqslant ((A_1 \otimes A_2 \otimes \cdots \otimes A_k)(A_1^T \otimes A_2^T \otimes \cdots \otimes A_k^T)(A_2 \otimes \cdots \otimes A_k \otimes A_1)$$
$$(A_2^T \otimes \cdots \otimes A_k^T \otimes A_1^T) \cdots (A_k \otimes A_1 \otimes \cdots \otimes A_{k-1})(A_k^T \otimes A_1^T \otimes \cdots \otimes A_{k-1}^T))[\alpha]$$
$$= ((A_1A_1^TA_2A_2^T \cdots A_kA_k^T) \otimes (A_2A_2^T \cdots A_kA_k^TA_1A_1^T) \otimes \cdots \otimes (A_kA_k^TA_1A_1^T \cdots A_{k-1}A_{k-1}^T))[\alpha]$$

and hence

$$\begin{aligned}
&\|A_1 \circ A_2 \circ \cdots \circ A_k\|^{2k} \\
&= \rho^k((A_1 \circ A_2 \circ \cdots \circ A_k)(A_1 \circ A_2 \circ \cdots \circ A_k)^T) \\
&= \rho(((A_1 \circ A_2 \circ \cdots \circ A_k)(A_1 \circ A_2 \circ \cdots \circ A_k)^T)^k) \\
&\leqslant \rho(((A_1A_1^TA_2A_2^T \cdots A_kA_k^T) \otimes (A_2A_2^T \cdots A_kA_k^TA_1A_1^T) \otimes \cdots \otimes (A_kA_k^TA_1A_1^T \cdots A_{k-1}A_{k-1}^T))[\alpha]) \\
&\leqslant \rho((A_1A_1^TA_2A_2^T \cdots A_kA_k^T) \otimes (A_2A_2^T \cdots A_kA_k^TA_1A_1^T) \otimes \cdots \otimes (A_kA_k^TA_1A_1^T \cdots A_{k-1}A_{k-1}^T)) \\
&= \rho(A_1A_1^TA_2A_2^T \cdots A_kA_k^T)\rho(A_2A_2^T \cdots A_kA_k^TA_1A_1^T) \cdots \rho(A_kA_k^TA_1A_1^T \cdots A_{k-1}A_{k-1}^T)) \\
&= \rho^k(A_1A_1^TA_2A_2^T \cdots A_kA_k^T). \qquad \square
\end{aligned}$$

Theorem 5. *Let $A_1, A_2, \ldots, A_k \in M_n$ be nonnegative matrices. If k is even then*

$$\|A_1 \circ A_2 \circ \cdots \circ A_k\| \leqslant \rho^{1/2}(A_1^TA_2A_3^TA_4 \cdots A_{k-1}^TA_k)\rho^{1/2}(A_1A_2^TA_3A_4^T \cdots A_{k-1}A_k^T);$$

if k is odd then

$$\|A_1 \circ A_2 \circ \cdots \circ A_k\| \leqslant \rho^{1/2}(A_1A_2^TA_3A_4^T \cdots A_{k-1}^TA_kA_1^TA_2A_3^TA_4 \cdots A_{k-1}A_k^T).$$

Proof. If k is even, Lemmas 1 and 2 ensure that

$$\begin{aligned}
&((A_1 \circ A_2 \circ \cdots \circ A_k)^T(A_1 \circ A_2 \circ \cdots \circ A_k))^{k/2} \\
&= (A_1^T \circ A_2^T \circ \cdots \circ A_k^T)(A_2 \circ \cdots \circ A_k \circ A_1)(A_3^T \circ \cdots \circ A_k^T \circ A_1^T \circ A_2^T) \\
&\quad (A_4 \circ \cdots \circ A_k \circ A_1 \circ A_2 \circ A_3) \cdots (A_{k-1}^T \circ A_k^T \circ A_1^T \circ \cdots \circ A_{k-2}^T)(A_k \circ A_1 \circ \cdots \circ A_{k-1}) \\
&= (A_1^T \otimes A_2^T \otimes \cdots \otimes A_k^T)[\alpha](A_2 \otimes \cdots \otimes A_k \otimes A_1)[\alpha](A_3^T \otimes \cdots \otimes A_k^T \otimes A_1^T \otimes A_2^T)[\alpha] \\
&\quad \cdots (A_{k-1}^T \otimes A_k^T \otimes A_1^T \otimes \cdots \otimes A_{k-2}^T)[\alpha](A_k \otimes A_1 \otimes \cdots \otimes A_{k-1})[\alpha] \\
&\leqslant ((A_1^T \otimes A_2^T \otimes \cdots \otimes A_k^T)(A_2 \otimes \cdots \otimes A_k \otimes A_1)(A_3^T \otimes \cdots \otimes A_k^T \otimes A_1^T \otimes A_2^T) \\
&\quad \cdots (A_{k-1}^T \otimes A_k^T \otimes A_1^T \otimes \cdots \otimes A_{k-2}^T)(A_k \otimes A_1 \otimes \cdots \otimes A_{k-1}))[\alpha] \\
&= ((A_1^TA_2A_3^TA_4 \cdots A_{k-1}^TA_k) \otimes (A_2^TA_3 \cdots A_{k-1}A_k^TA_1) \otimes \\
&\quad \cdots \otimes (A_{k-1}^TA_kA_1^TA_2 \cdots A_{k-3}^TA_{k-2}) \otimes (A_k^TA_1A_2^T \cdots A_{k-2}^TA_{k-1}))[\alpha].
\end{aligned}$$

So

$$\begin{aligned}
&\|A_1 \circ A_2 \circ \cdots \circ A_k\|^k \\
&= \rho^{k/2}((A_1 \circ A_2 \circ \cdots \circ A_k)^T(A_1 \circ A_2 \circ \cdots \circ A_k)) \\
&= \rho(((A_1 \circ A_2 \circ \cdots \circ A_k)^T(A_1 \circ A_2 \circ \cdots \circ A_k))^{k/2}) \\
&\leqslant \rho(((A_1^TA_2A_3^TA_4 \cdots A_{k-1}^TA_k) \otimes (A_2^TA_3 \cdots A_{k-1}A_k^TA_1) \otimes \\
&\quad \cdots \otimes (A_{k-1}^TA_kA_1^TA_2 \cdots A_{k-3}^TA_{k-2}) \otimes (A_k^TA_1A_2^T \cdots A_{k-2}^TA_{k-1}))[\alpha]) \\
&\leqslant \rho((A_1^TA_2A_3^TA_4 \cdots A_{k-1}^TA_k) \otimes (A_2^TA_3 \cdots A_{k-1}A_k^TA_1) \otimes \\
&\quad \cdots \otimes (A_{k-1}^TA_kA_1^TA_2 \cdots A_{k-3}^TA_{k-2}) \otimes (A_k^TA_1A_2^T \cdots A_{k-2}^TA_{k-1}))
\end{aligned}$$

$$= \rho(A_1^TA_2A_3^TA_4\cdots A_{k-1}^TA_k)\rho(A_2^TA_3\cdots A_{k-1}A_k^TA_1)$$
$$\cdots\rho(A_{k-1}^TA_kA_1^TA_2\cdots A_{k-3}^TA_{k-2})\rho(A_k^TA_1A_2^T\cdots A_{k-2}^TA_{k-1})$$
$$= \rho^{k/2}(A_1^TA_2A_3^TA_4\cdots A_{k-1}^TA_k)\rho^{k/2}(A_1A_2^TA_3\cdots A_{k-1}A_k^T),$$

in which the final equality follows from the two inequalities

$$\rho(A_1^TA_2A_3^TA_4\cdots A_{k-1}^TA_k) = \rho(A_3^TA_4\cdots A_{k-1}^TA_kA_1^TA_2) = \cdots = \rho(A_{k-1}^TA_kA_1^TA_2\cdots A_{k-3}^TA_{k-2})$$

and

$$\rho(A_1A_2^TA_3\cdots A_{k-1}A_k^T) = \rho(A_2^TA_3\cdots A_{k-1}A_k^TA_1) = \cdots = (A_k^TA_1A_2^TA_3\cdots A_{k-2}^TA_{k-1}).$$

If k is odd, consider

$$\|A_1\circ A_2\circ\cdots\circ A_k\|^{2k} = \rho^k((A_1\circ A_2\circ\cdots\circ A_k)^T(A_1\circ A_2\circ\cdots\circ A_k))$$

and

$$((A_1\circ A_2\circ\cdots\circ A_k)^T(A_1\circ A_2\circ\cdots\circ A_k))^k$$
$$= (A_1^T\circ A_2^T\circ\cdots\circ A_k^T)(A_2\circ\cdots\circ A_k\circ A_1)(A_3^T\circ\cdots\circ A_k^T\circ A_1^T\circ A_2^T)$$
$$(A_4\circ\cdots\circ A_k\circ A_1\circ A_2\circ A_3)\cdots(A_{k-1}\circ A_k\circ A_1\circ\cdots\circ A_{k-2})(A_k^T\circ A_1^T\circ\cdots\circ A_{k-1}^T)$$
$$(A_1\circ A_2\circ\cdots\circ A_k)(A_2^T\circ\cdots\circ A_k^T\circ A_1^T)(A_3\circ\cdots\circ A_k\circ A_1\circ A_2)\cdots$$
$$(A_{k-1}^TA_k^T\circ A_1^T\circ\cdots\circ A_{k-2}^T)(A_k\circ A_1\circ\cdots\circ A_{k-1}).$$

It follows that

$$\|A_1\circ A_2\circ\cdots\circ A_k\|^{2k}$$
$$\leqslant \rho^{(k+1)/2}(A_1^TA_2A_3^TA_4\cdots A_{k-1}A_k^TA_1A_2^TA_3A_4^T\cdots A_{k-1}^TA_k)$$
$$\rho^{(k-1)/2}(A_1A_2^TA_3A_4^T\cdots A_{k-1}^TA_kA_1^TA_2A_3^TA_4\cdots A_{k-1}A_k^T)$$
$$= \rho^k(A_1A_2^TA_3A_4^T\cdots A_{k-1}^TA_kA_1^TA_2A_3^TA_4\cdots A_{k-1}A_k^T). \quad \square$$

Corollary 6. *Let $A, B \in M_n$ be nonnegative. Then*

$$\|A\circ B\| \leqslant \rho(A^TB). \tag{3}$$

Proof. Theorem 5 ensures that

$$\|A\circ B\| \leqslant \rho^{1/2}(A^TB)\rho^{1/2}(AB^T) = \rho^{1/2}(A^TB)\rho^{1/2}(B^TA) = \rho(A^TB). \quad \square$$

Since

$$\rho(A\circ B) \leqslant \|A\circ B\| \leqslant \rho(A^TB) \leqslant \|A^TB\| \leqslant \|A\|\,\|B\|,$$

Corollary 6 improves the inequality $\|A\circ B\| \leqslant \|A\|\,\|B\|$ and implies that $\rho(A\circ B) \leqslant \rho(A^TB)$ and $\|A\circ B\| \leqslant \|A^TB\|$ for nonnegative matrices.

Remark 1. In Corollary 6 the nonnegativity condition cannot be removed and the inequality does not hold for positive semidefinite matrices. Consider the following example:

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

for which A and B are both positive semidefinite and

$$\|A\circ B\| = 2 > 0 = \rho(A^TB).$$

Remark 2. The transpose in (3) is necessary: the matrices in (2) provide a counterexample to the inequality $\|A \circ B\| \leqslant \rho(AB)$, as well as to the weaker inequality $\|A \circ B\| \leqslant \|AB\|$.

4. Counterexamples to two conjectures

Denote by $\overline{A}$ and A^* the entrywise complex conjugate and the conjugate transpose of a matrix $A \in M_n$, respectively. We write the decreasingly ordered singular values of $A \in M_n$ as $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A)$. Huang [5] and Zhan [2] made the following two conjectures, respectively: For $A, B \in M_n$,

$$\{s_j^2(A \circ B)\} \prec_w \{s_j(A \circ \overline{A})s_j(B \circ \overline{B})\}, \tag{4}$$

and

$$\|(A \circ B)(A \circ B)^*\| \leqslant \|(A \circ \overline{A})(B \circ \overline{B})^T\| \tag{5}$$

where $\prec_w$ means weak majorization and $\|\cdot\|$ is any unitarily invariant norm. Since for any $X, Y \in M_n$, $\{s_j(XY)\} \prec_w \{s_j(X)s_j(Y)\}$ [8, p. 20], (5) is stronger than (4). Du [2] proved (5) for the spectral norm, the trace norm, and the Frobenius norm. Here we give counterexamples to (4) and (5) for other unitarily invariant norms. Recall that the *Ky Fan k-norm* of a matrix $A \in M_n$ is $\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A)$.

Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

The singular values of $A \circ B$ are $\{5, 5, 2\}$, the singular values of $A \circ \overline{A} = B \circ \overline{B}$ are $\{6, 3, 3\}$, and the singular values of $(A \circ \overline{A})(B \circ \overline{B})^T$ are $\{36, 9, 9\}$. We have

$$5^2 + 5^2 = 50 > 45 = 6^2 + 3^2 = 36 + 9,$$

which contradicts (4). The contradiction

$$\|(A \circ B)(A \circ B)^*\|_{(2)} > \|(A \circ \overline{A})(B \circ \overline{B})^T\|_{(2)}$$

of (5) follows from observing that $s_j((A \circ B)(A \circ B)^*) = s_j^2(A \circ B)$ for every j.

In fact, (5) is invalid for nonnegative matrices and the Ky Fan 2-norm. One example is

$$A = \begin{pmatrix} 8 & 7 & 4 \\ 0 & 0 & 6 \\ 6 & 10 & 7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 7 & 6 & 0 \\ 3 & 1 & 6 \\ 4 & 10 & 7 \end{pmatrix}.$$

However, we have not found nonnegative matrices A, B contradicting (4).

Problem 1. Is the inequality (4) correct for nonnegative matrices A, B?

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