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Explicit representations of changeable degree spline basis functions

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ABSTRACT

Changeable degree spline (*CD*-spline for short) basis functions, defined by an iterative integral method, are extensions of *B*-spline basis functions. A *CD*-spline basis function is a piecewise function made up of polynomials of different degrees. In this paper, we will give the explicit representations of *CD*-spline basis functions, from which the spanned linear space can be seen clearly. Our method is also feasible for the explicit representations of the other basis functions given in analogous integral ways.

1. Introduction

The famous *B*-spline has many good properties for modeling free form shapes, so it has an important role in computer aided geometric design [1]. Each *B*-spline basis function is a kind of spline function defined on a knot sequence $\boldsymbol{T}$ with a natural number n. It is a piecewise function made up of polynomials of the same degree n on its support interval. All the *B*-spline basis functions form a *B*-basis [2], possessing the optimal shape preserving property, for the space of polynomial splines over $\boldsymbol{T}$.

The changeable degree spline (*CD*-spline for short) is a kind of variable degree polynomial spline. That is, its basis function is a piecewise function comprised of polynomials of variable degrees. Early polynomial splines of nonuniform degrees were studied for shape-preserving interpolation purposes [3–6]. Later, [7] presented some requirements for *B*-spline-like properties and constructed some multi-degree splines of degrees 1, 2, and 3. In [8], some two-degree polynomial spline basis functions possessing *B*-spline-like properties are produced. In 2010, [9] introduced some splines of arbitrary degree polynomials which are extensions of *B*-splines. However, their basis functions cannot form a *B*-basis. This drawback motivates *CD*-splines, which are direct extensions of *B*-splines. *CD*-spline basis functions possess the optimal shape preserving property. Moreover, when we use them to design curves made up of polynomial segments of different degrees, the number of control points may be decreased [10].

CD-spline basis functions are defined on a knot sequence $\boldsymbol{T}$ and a degree sequence $\boldsymbol{G}$ by an iterative integral method. Their representations are not explicit, so the space spanned by these basis functions is not clear. Does this space have a truncated-power-function-like basis? If it has, how do we use this basis to represent the *CD*-spline basis functions? Similar problems have been thoroughly studied as important parts of the theories of *B*-splines. *B*-spline basis functions can be uniquely presented as linear combinations of truncated power functions [11,12]. These representations are used for calculating curves/surfaces, transforming models between different systems, studying the spline spaces, and so on [13–15]. But these problems concerning explicit representations have not been settled for *CD*-splines. In this paper, we study them in order to develop similar theories with *B*-splines.

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We firstly define some truncated polynomial functions $F_{i,D},\ i \in \mathbb{Z}$. Secondly, instead of showing $F_{i,D},\ i \in \mathbb{Z}$, to span the same space with *CD*-spline basis functions, we prove that if a function has a local support interval and fulfills some continuous conditions on the interval, it is unique up to a constant factor. Thirdly, for each *CD*-spline basis function, we use a determinant relating to $F_{i,D},\ i \in \mathbb{Z}$, to get a function which has the same support interval and the same continuity as the *CD*-spline basis function. So this *CD*-spline basis function can be expressed as a product of the determinant and some constant. Lastly, this constant will be obtained from the normalized property of *CD*-spline basis functions.

This method for explicit representation uses only a few properties of *CD*-spline basis functions. It is also feasible for many other basis functions defined in an analogous integral way [16]. They are Bernstein basis functions, *B*-spline basis functions, C-Bézier and AT *B*-spline basis functions for algebraic trigonometric polynomial space [17–19], AH Bézier and AH *B*-spline basis functions for algebraic hyperbolic polynomial space [20,21], AHT Bézier and NUAHT *B*-spline basis functions for algebraic hyperbolic trigonometric polynomial space [22], ω Bézier and ω *B*-spline basis functions [23,24], and multi-degree spline basis functions [9].

The rest of the paper is divided into four parts. We review the definitions and some properties of *CD*-spline basis functions in the next section. In Sections 3 and 4, we give the explicit representations of *CD*-spline basis functions for simple and multiple knots, respectively. The last section includes some results.

2. Review

In this section, we review the definitions and some properties of *CD*-spline basis functions in [10] which will be used for explicit representations.

Unlike for *B*-spline basis functions, for *CD*-spline basis functions one needs to know not only a knot sequence but also the degree of each knot interval. Let $\boldsymbol{T} = \{t_i\}_{i\in\mathbb{Z}}$ be a nondecreasing real number sequence and $\boldsymbol{G} = \{d_i\}_{i\in\mathbb{Z}}$ be a bounded positive integer sequence satisfying the following condition:

(C) If $t_{i-1} < t_i = t_{i+1} = \cdots = t_{i+m-1} < t_{i+m}$, then $d_i = d_{i+1} = \cdots = d_{i+m-1}$ and $\max\{1, d_i - d_{i-1} + 1\} \le m \le d_i$.

Then, with definitions like those for *B*-splines, $\boldsymbol{T}$ is called a *knot sequence* and $\boldsymbol{G}$ is called a *degree sequence* of $\boldsymbol{T}$.

For each knot interval $[t_i, t_{i+1})$, its corresponding degree is d_i. For simplicity, the interval with degree n is called an *n-interval*. If $t_{i-1} < t_i = t_{i+1} = \cdots = t_{i+m-1} < t_{i+m}$, then the knots $t_j,\ j = i, \ldots, i+m-1$, all have multiplicity m. For simplicity, a knot of multiplicity m is called an *m-knot*.

In the rest of this paper, the sequence $\boldsymbol{T}$ will always be a knot sequence and the sequence $\boldsymbol{G}$ will always be a degree sequence of $\boldsymbol{T}$ satisfying Condition (C), and $D := \max_i\{d_i\}$.

For $n = 0, 1, \ldots, D$, functions $N_{i,n} = N_{i,n}(t),\ i \in \mathbb{Z}$, over $\boldsymbol{T}$ and $\boldsymbol{G}$ are generated by the following iterative method. The finally obtained functions $N_{i,D},\ i \in \mathbb{Z}$ are the *CD-spline basis functions* over $\boldsymbol{T}$ and $\boldsymbol{G}$.

$$N_{i,n}(t) := \begin{cases} 0, & d_i < D-n, \\ \begin{cases} 1, & t \in [t_i, t_{i+1}), \\ 0, & \text{otherwise}, \end{cases} & d_i = D-n, \\ \displaystyle\int_{-\infty}^{t} \left(\delta_{i,n-1}N_{i,n-1}(s) - \delta_{i+1,n-1}N_{i+1,n-1}(s)\right) ds, & d_i > D-n, \end{cases} \tag{1}$$

where

$$\delta_{i,n} := \left(\int_{-\infty}^{+\infty} N_{i,n}(t)dt\right)^{-1}. \tag{2}$$

If $N_{i,n} = 0$, then we set

$$\int_{-\infty}^{t} \delta_{i,n}N_{i,n}(s)ds = \begin{cases} 0, & t < t_i, \\ 1, & t \ge t_i. \end{cases} \tag{3}$$

Actually, in this case, $\delta_{i,n}N_{i,n}$ is the Dirac function.

If $n = 0$, then any $d_i \le D - n$. So the definitions of the initial functions $N_{i,0},\ i \in \mathbb{Z}$, are included in Formula (2).

Proposition 2.1. *CD-spline basis functions $N_{i,D},\ i \in \mathbb{Z}$, possess the following B-spline-like properties.*

(1) *(Normalized property.)*

$$\sum_{i=-\infty}^{+\infty} N_{i,D}(t) \equiv 1.$$

(2) *(Local support property.) The function $N_{i,n}$ is supported on $[t_i, t_{i+k})$, where $k := k_{i,D}$ and the sequence $\{k_{i,n}\}_{i\in\mathbb{Z}}$ is recursively defined by*

$$k_{i,n} := \begin{cases} 0, & d_i < D - n, \\ 1, & d_i = D - n, \\ \begin{cases} k_{i,n-1}, & d_{i+1} < D - n + 1, \\ k_{i+1,n-1} + 1, & d_{i+1} \geq D - n + 1, \end{cases} & d_i > D - n. \end{cases} \tag{4}$$

(3) *(Basis property) On each interval $[t_i, t_{i+1})$, only functions $N_{j,D},\ i - d_i \leq j \leq i$, are nonzero. They are linearly independent polynomials of degree d_i.*

For an explicit expression for $N_{i,D}$, we also need its continuity property. However, this property is complex since it is related to the multiplicities of knots. So we will give the property for simple knots and then the multiple case when using it.

3. Explicit representations for simple knots

Starting from the most basic case and progressing onwards to the more complicated ones, we focus on simple knots in this section. If all the knots in $\boldsymbol{T}$ are simple, then the degree sequence $\boldsymbol{G}$ should be decreasing for Condition (C). That is, $d_{i-1} \geq d_i$ for any i.

In this case, for each $N_{i,D}$, consider the continuity at the knots on its support interval $[t_i, t_{i+k}]$. Since each integration increases the order of continuity of a function by 1, it is easy to get from (1) the continuous order p_j^i of $N_{i,D}$ at t_j.

$$p_j^i := \begin{cases} d_j - 1, & i \leq j \leq i + k - 1, \\ c, & j = i + k, \end{cases}$$

where $c := c_{i,D}$ is iteratively obtained by the following recursion for $n = 0, 1, 2, \ldots, D$:

$$c_{i,n} := \begin{cases} -2, & d_i < D - n, \\ -1, & d_i = D - n, \\ \begin{cases} c_{i,n-1}, & d_{i+1} < D - n + 1, \\ c_{i+1,n-1} + 1, & d_{i+1} \geq D - n + 1, \end{cases} & d_i > D - n. \end{cases} \tag{5}$$

We denote the linear space spanned by all the *CD*-spline basis functions over $\boldsymbol{T}$ and $\boldsymbol{G}$ as $\Omega_{\boldsymbol{G}}[\boldsymbol{T}]$. The space $\Omega_{\boldsymbol{G}}[\boldsymbol{T}]$ is called the *changeable degree spline space* (*CDS-space* for short) in this paper. Then, we define a subspace of $\Omega_{\boldsymbol{G}}[\boldsymbol{T}]$ as follows:

$$\begin{aligned} \Gamma_{\boldsymbol{G}}^p[t_i, t_{i+k}] := \{u(t) \in \Omega_{\boldsymbol{G}}[\boldsymbol{T}] | u(t) = 0 \text{ if } t \notin [t_i, t_{i+k}],\ \text{and for } j \in [i, i+k],\ \text{the continuous order of } u(t) \\ \text{at the knot } t_j \text{ is greater than or equal to } p_j^i\}. \end{aligned}$$

We call $\Gamma_{\boldsymbol{G}}^p[t_i, t_{i+k}]$ the *local support continuity changeable degree spline subspace (LSCCDS-subspace for short).*

3.1. A class of truncated functions

Truncated power functions are usually used for explicit representations of *B*-spline basis functions. Each of them is a piecewise function made up of zero and a power function. The power function possesses the same degree on each knot interval.

Here, we will use some truncated power-like functions. They may have different degrees on the knot intervals. These functions, denoted as $F_i = F_i(t),\ i \in \mathbb{Z}$, are defined as follows:

$$F_i(t) := \begin{cases} 0, & t < t_i, \\ f_j^i, & t_j \leq t < t_{j+1},\ j = i, i+1, \ldots, \end{cases} \tag{6}$$

where $f_j^i := f_j^i(t)$ is a function on $[t_j, t_{j+1})$ defined by

$$f_j^i(t) := \begin{cases} (t - t_i)^{d_i}, & j = i, \\ \displaystyle\sum_{w=0}^{d_j} \frac{1}{w!} {f_{j-1}^i}^{(w)}(t_j)(t - t_j)^w, & j > i. \end{cases} \tag{7}$$

In fact, f_j^i is the Taylor polynomial [25] of order d_j of f_{j-1}^i at t_j if $j > i$. From (6) and (7), it is easy to see the following properties of F_i.

Proposition 3.1. (1) *On the interval $[t_j, t_{j+1})$, each F_i is a polynomial of degree d_j if $j \geq i$.*
(2) *Each $F_i \in \Omega_{\boldsymbol{G}}[\boldsymbol{T}]$.*
(3) *The functions $F_i,\ i \in \mathbb{Z}$, are linearly independent.*

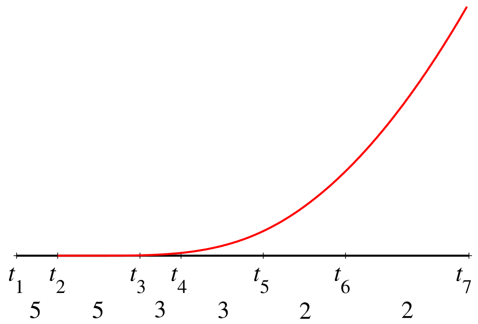

Fig. 1. Example of a truncated function for simple knots.

(4) *If $j \geq i$, then the order of continuity of F_i at t_j is equal to $d_j - 1$ if $j = i$, and greater than or equal to d_j otherwise.*

We give an example of such functions $F_i,\ i \in \mathbb{Z}$, in Fig. 1. Its knot sequence is

$$\{\cdots < t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < t_7 < \cdots\}$$

and its corresponding degree sequence is

$$\{\ldots, 5, 5, 3, 3, 2, 2, \ldots\}.$$

The piecewise function F_2 is shown by a solid curve. On each d_j-interval $[t_j, t_{j+1})$, F_2 is a polynomial of degree d_j, where $j = 2, 3, 4, \ldots$.

3.2. The dimension of an LSCCDS-subspace

In this subsection, we consider the dimension of the LSCCDS-subspace.

Lemma 3.1. *For any function $h = h(t) \in \Omega_{\boldsymbol{G}}[\boldsymbol{T}]$, if each right derivative $h^{(j)}(t_i+) = 0$ for $j = 0, 1, \ldots, d_i - 1$, then there exists one real number a_i such that*

$$h(t) = a_i F_i(t), \quad t \in [t_i, t_{i+1}).$$

Proof. Since h is a polynomial function of degree less than or equal to d_i on $[t_i, t_{i+1})$, then it can be written as

$$h = \sum_{l=0}^{d_i} b_l (t - t_i)^l$$

for some real numbers b_l. Taking into account that $h^{(j)}(t_i+) = 0$ for $j = 0, 1, \ldots, d_i - 1$, we have $b_l = 0$ for $l = 0, 1, \ldots, d_i - 1$. Therefore,

$$h = b_{d_i} (t - t_i)^{d_i}.$$

According to the definition of F_i, we have $F_i(t) = (t - t_i)^{d_i},\ t \in [t_i, t_{i+1})$, and the claim follows with $a_i = b_{d_i}$. □

From (4) and (5), we deduce the following results:

Lemma 3.2. $k = c + 2$.

Lemma 3.3. *Given integers i, q and l such that $l \geq 1$ and $i \leq q \leq i + l$, the determinant*

$$\begin{vmatrix} F_i(t_q) & F_{i+1}(t_q) & \cdots & F_{i+l}(t_q) \\ F_i'(t_q) & F_{i+1}'(t_q) & \cdots & F_{i+l}'(t_q) \\ \vdots & \vdots & \vdots & \vdots \\ F_i^{(l)}(t_q) & F_{i+1}^{(l)}(t_q) & \cdots & F_{i+l}^{(l)}(t_q) \end{vmatrix} \neq 0.$$

Proof. Use reduction to absurdity. Assume that the determinant is equal to zero. Then, there are $l+1$ numbers $\{b_j\}_{j=i}^{i+l}$, which are not all equal to zero, satisfying

$$\sum_{j=i}^{i+l} b_j \begin{pmatrix} F_j(t_q) \\ F_j'(t_q) \\ \vdots \\ F_j^{(l)}(t_q) \end{pmatrix} = \mathbf{0}.$$

Let $Y(t) = \sum_{j=i}^{i+l} b_j F_j(t)$. From Proposition 3.1(1) and (4), we see that, on the interval $[t_{q-1}, t_q)$, Y is a polynomial function whose degree is less than or equal to its order of continuity at t_q. Since $Y(t_q) = 0$, it follows that $Y(t) \equiv 0$, which means that $b_j = 0$ for all $i \le j \le i+l$. This conflicts with the assumption. So the lemma is proved. $\square$

From the above lemmas, we get the dimension of $\Gamma_{\boldsymbol{G}}^{p}[t_i, t_{i+k}]$.

Theorem 3.1. *The dimension of the linear space $\Gamma_{\boldsymbol{G}}^{p}[t_i, t_{i+k}]$ is* 1.

Proof. Let $u = u(t)$ be an arbitrary function in $\Gamma_{\boldsymbol{G}}^{r}[t_i, t_{i+k}]$. Thus, we have

$$u^{(l)}(t_j-) = u^{(l)}(t_j) = u^{(l)}(t_j+),$$

where $l = 0, 1, \ldots, d_j - 1$ and $j = i,\ i+1, \ldots, i+k-1$.

Consider u on $[t_i, t_{i+1})$ firstly. From Lemma 3.1, there is a real number a_i such that

$$u(t) = a_i F_i(t), \quad t \in [t_i, t_{i+1}).$$

Secondly, consider the function $u - a_i F_i$ on the interval $[t_i, t_{i+2})$. According to the continuity of u, there exists an a_{i+1} such that

$$u(t) - a_i F_i(t) = a_{i+1} F_{i+1}(t).$$

That is,

$$u(t) = a_i F_i(t) + a_{i+1} F_{i+1}(t), \quad t \in [t_i, t_{i+2}).$$

Recursively, we deduce that there exist k real numbers $a_l,\ i \le l \le i+k-1$, such that

$$u(t) = \sum_{l=i}^{i+k-1} a_l F_l(t), \quad t \in [t_i, t_{i+k}).$$

Then consider the continuity of u at t_{i+k}. We have $u^{(l)}(t_{i+k}) = 0$, for $l = 0,\ 1, \ldots, c$. That is,

$$\begin{cases} \displaystyle\sum_{l=i}^{i+k-1} a_l F_l(t_{i+k}) = 0, \\ \displaystyle\sum_{l=i}^{i+k-1} a_l F_l'(t_{i+k}) = 0, \\ \vdots \\ \displaystyle\sum_{l=i}^{i+k-1} a_l F_l^{(c)}(t_{i+k}) = 0. \end{cases} \tag{8}$$

Since $k = c+2$, (8) can also be represented as

$$\begin{pmatrix} F_i(t_{i+k}) & F_{i+1}(t_{i+k}) & \cdots & F_{i+c+1}(t_{i+k}) \\ F_i'(t_{i+k}) & F_{i+1}'(t_{i+k}) & \cdots & F_{i+c+1}'(t_{i+k}) \\ \vdots & \vdots & \vdots & \vdots \\ F_i^{(c)}(t_{i+k}) & F_{i+1}^{(c)}(t_{i+k}) & \cdots & F_{i+c+1}^{(c)}(t_{i+k}) \end{pmatrix} \begin{pmatrix} a_i \\ a_{i+1} \\ \vdots \\ a_{i+c+1} \end{pmatrix} = \mathbf{0}. \tag{9}$$

The dimension of $\Gamma_{\boldsymbol{G}}^{p}[t_i, t_{i+k}]$ is equal to the dimension of the solution space of the system of linear equations (9). From Lemma 3.3, we see that the coefficient matrix of (9) has full rank $c+1$. And since the number of variables of (9) is $c+2$, the dimension of its solution space is 1 [26]. Therefore, $\Gamma_{\boldsymbol{G}}^{p}[t_i, t_{i+k}]$ is a linear space of dimension 1. $\square$

From the proof of Theorem 3.1, we see that each *CD*-spline basis function $N_{i,D}$ can be represented as a linear combination of some functions F_j. So, the functions $F_i,\ i \in \mathbb{Z}$, also form a basis for the CDS-space.

3.3. The determinantal representation

In this subsection, we will use the basis $\{F_j\}_{i\in\mathbb{Z}}$ to explicitly represent $N_{i,D}$.

According to Theorem 3.1 and the definition of $\Gamma^r_{\boldsymbol{G}}[t_i, t_{i+k}]$, we have $N_{i,D} \in \Gamma^r_{\boldsymbol{G}}[t_i, t_{i+k}]$, and the dimension of $\Gamma^r_{\boldsymbol{G}}[t_i, t_{i+k}]$ is 1. If we find a function $H \in \Gamma^r_{\boldsymbol{G}}[t_i, t_{i+k}]$, then there must be $N_{i,D} = \alpha H$ for some real number α. Thus we give the following theorem.

Theorem 3.2. *Let*

$$\varphi_i = \varphi_i(t) := \begin{vmatrix} F_i(t) & F_{i+1}(t) & \cdots & F_{i+c+1}(t) \\ F_i(t_{i+k}) & F_{i+1}(t_{i+k}) & \cdots & F_{i+c+1}(t_{i+k}) \\ F_i'(t_{i+k}) & F_{i+1}'(t_{i+k}) & \cdots & F_{i+c+1}'(t_{i+k}) \\ \vdots & \vdots & \vdots & \vdots \\ F_i^{(c)}(t_{i+k}) & F_{i+1}^{(c)}(t_{i+k}) & \cdots & F_{i+c+1}^{(c)}(t_{i+k}) \end{vmatrix}. \tag{10}$$

Then we have

$$N_{i,D} = \alpha_i \varphi_i, \tag{11}$$

where

$$\alpha_i := \frac{(-1)^{d_i} \begin{vmatrix} \boldsymbol{A}^i_{i-d_i} & \cdots & \boldsymbol{A}^i_{i-1} \end{vmatrix}}{\begin{vmatrix} \varphi_{i-d_i}(t_i) & \cdots & \varphi_{i-1}(t_i) & \varphi_i(t_i) \\ \boldsymbol{A}^i_{i-d_i} & \cdots & \boldsymbol{A}^i_{i-1} & \boldsymbol{A}^i_i \end{vmatrix}} \tag{12}$$

and

$$\boldsymbol{A}^i_j := \begin{pmatrix} \varphi_j'(t_i+) \\ \varphi_j''(t_i+) \\ \vdots \\ \varphi_j^{(d_i)}(t_i+) \end{pmatrix}. \tag{13}$$

Proof. From (10), it is easy to deduce that the function φ_i is a linear combination of F_j, for $j = i,\ i+1, \ldots, i+c+1$. From the properties of F_j, it is easy for us to see that $\varphi_i \in \Gamma^p_{\boldsymbol{G}}[t_i, t_{i+k}]$. So there exists a coefficient α_i such that $N_{i,D} = \alpha_i \varphi_i$.

To get α_i, we use Proposition 2.1. Considering the nonzero *CD*-spline basis functions on the interval $[t_i, t_{i+1})$, we have $\sum_{j=i-d_i}^{i} N_{j,D}(t) \equiv 1$. Hence, there exists the following system of linear equations:

$$\begin{cases} \displaystyle\sum_{j=i-d_i}^{i} \alpha_j \varphi_j(t_i) = 1, \\ \displaystyle\sum_{j=i-d_i}^{i} \alpha_j \varphi_j'(t_i+) = 0, \\ \vdots \\ \displaystyle\sum_{j=i-d_i}^{i} \alpha_j \varphi_j^{(d_i)}(t_i+) = 0. \end{cases} \tag{14}$$

Doing the same as in the proof of Lemma 3.3, we see that the coefficient matrix of (14) is full rank. So (14) has a unique solution. Using Cramer's rule [26], we obtain that

$$\alpha_i = \frac{\begin{vmatrix} \varphi_{i-d_i}(t_i) & \cdots & \varphi_{i-1}(t_i) & 1 \\ \boldsymbol{A}^i_{i-d_i} & \cdots & \boldsymbol{A}^i_{i-1} & \boldsymbol{0} \end{vmatrix}}{\begin{vmatrix} \varphi_{i-d_i}(t_i) & \cdots & \varphi_{i-1}(t_i) & \varphi_i(t_i) \\ \boldsymbol{A}^i_{i-d_i} & \cdots & \boldsymbol{A}^i_{i-1} & \boldsymbol{A}^i_i \end{vmatrix}} = \frac{(-1)^{d_i} \begin{vmatrix} \boldsymbol{A}^i_{i-d_i} & \cdots & \boldsymbol{A}^i_{i-1} \end{vmatrix}}{\begin{vmatrix} \varphi_{i-d_i}(t_i) & \cdots & \varphi_{i-1}(t_i) & \varphi_i(t_i) \\ \boldsymbol{A}^i_{i-d_i} & \cdots & \boldsymbol{A}^i_{i-1} & \boldsymbol{A}^i_i \end{vmatrix}}.$$

Thus, the theorem is proved. □

If we focus on another interval including the knot t_i, we may get a different representation of α_i. But it must be equal to the given expression for α_i because of the linear independence of $\{F_i(t)\}_{i\in\mathbb{Z}}$ and the uniqueness of α_i.

4. Explicit representation extensions for multiple knots

We consider the case of multiple knots in this section. As it is a generalization of the previous one, some of the notation here will be defined by extension from that in the last section. We add an overline to indicate the extended notation.

In order to see the continuity of each $N_{i,D}$, we firstly give the following definition.

Definition 4.1. Let $i,\ l$ be two integers such that $i \le l$. The integer i can be $-\infty$ and l can be $+\infty$ as well. Nonnegative integral numbers $r_j^{i,l},\ j = i, i+1, \dots, l$, are defined from the knot sequence $\boldsymbol{T}$.

$$r_j^{i,l} := \text{The times that } t_j \text{ appears in the knot subsequence } \{t_v\}_{v=i}^{l}.$$

Then we get the continuous order of $N_{i,D}$ at t_j as follows:

$$\overline{p}_j^i := \begin{cases} d_j - r_j^{i,i+k}, & t_i \le t_j < t_{i+k}, \\ \overline{c}, & t_j = t_{i+k}, \end{cases}$$

where $\overline{c} := \overline{c}_{i,D}$ is iteratively obtained by the following recursion for $n = 0,\ 1,\ 2, \dots, D$:

$$\overline{c}_{i,n} := \begin{cases} -2, & d_i < D - n, \\ \begin{cases} -2, & t_i = t_{i+1}, \\ -1, & t_i \ne t_{i+1}, \end{cases} & d_i = D - n, \\ \begin{cases} \overline{c}_{i,n-1}, & d_{i+1} < D - n + 1, \\ \overline{c}_{i+1,n-1} + 1, & d_{i+1} \ge D - n + 1, \end{cases} & d_i > D - n. \end{cases} \tag{15}$$

In this section, the CDS-space over $\boldsymbol{T}$ and $\boldsymbol{G}$ is still denoted as $\Omega_{\boldsymbol{G}}[\boldsymbol{T}]$. The LSCCDS-subspace over $[t_i, t_{i+k}]$ is

$$\overline{\Gamma}_{\boldsymbol{G}}^{\overline{p}}[t_i, t_{i+k}] := \{u(t) \in \Omega_{\boldsymbol{G}}[\boldsymbol{T}] | u(t) = 0 \text{ if } t \notin [t_i, t_{i+k}],\ \text{and for } j \in [i, i+k],\ \text{the continuous order of } u(t) \text{ at the knot } t_j \text{ is greater than or equal to } \overline{p}_j^i\}.$$

4.1. A class of truncated functions

Like in Section 3.1, we define $\overline{F}_i = \overline{F}_i(t),\ i \in \mathbb{Z}$, as follows:

$$\overline{F}_i(t) := \begin{cases} 0, & t < t_i, \\ \overline{f}_j^i(t), & t_j \le t < t_{j+1},\ j = i,\ i+1, \dots, \end{cases} \tag{16}$$

where $\overline{f}_j^i = \overline{f}_j^i(t)$ is a function on $[t_j, t_{j+1})$ defined by

$$\overline{f}_j^i(t) := \begin{cases} (t - t_i)^{d_i - r_i^{i,+\infty} + 1}, & & t_j = t_i, \\ \overline{f}_{j - r_j^{i,j}}^i, & M_j^i \le d_j, & \\ \displaystyle\sum_{w=0}^{d_j} \frac{1}{w!} \overline{f}_{j - r_j^{i,j}}^{i\ (w)}(t_j)(t - t_j)^w, & M_j^i > d_j, & t_j > t_i, \end{cases} \tag{17}$$

and

$$M_j^i := \min\{d_i - r_i^{i,+\infty} + 1,\ d_{i+1},\ d_{i+2}, \dots, d_{j - r_j^{i,j}}\}.$$

From Formula (17), we see that the degree of $\overline{f}_i^i$ is determined by not only the degree d_i but also the multiplicity of the knot t_i. Assume that the current function is $\overline{f}_j^i$ on $[t_j, t_{j+1})$. Then the last function is $\overline{f}_{j-r_j^{i,j}}^i$ since the last nonzero knot interval is $[t_{j-r_j^{i,j}}, t_j)$. If the degree of $\overline{f}_{j-r_j^{i,j}}^i$ is less than or equal to the current degree d_j, then the current function is equal to the last function; otherwise, the current function is equal to the Taylor polynomial [25] of order d_j of the last function at t_j.

We give an example of such functions $\overline{F}_i,\ i \in \mathbb{Z}$, in Fig. 2. Here, the knot sequence is

$$\{\cdots \le t_1 < t_2 = t_3 = t_4 < t_5 = t_6 < t_7 < t_8 \le \cdots\}$$

and its corresponding degree sequence is

$$\{\dots, 2, 4, 4, 4, 5, 5, 3, \dots\}.$$

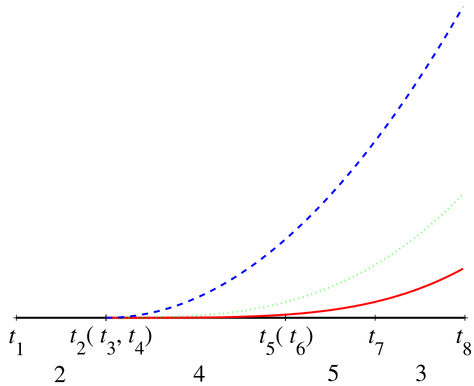

Fig. 2. Example of truncated functions for multiple knots.

We show the functions $\overline{F}_2, \overline{F}_3$ and $\overline{F}_4$ by solid, dotted and dashed lines, respectively. On the 4-interval $[t_2, t_5)$,

$$\overline{F}_2 = (t-t_2)^4, \qquad \overline{F}_3 = (t-t_3)^3, \qquad \overline{F}_4 = (t-t_4)^2.$$

On the 5-interval $[t_5, t_7)$, $\overline{F}_2$, $\overline{F}_3$ and $\overline{F}_4$ are unchanged. We have

$$\overline{F}_2 = (t-t_2)^4, \qquad \overline{F}_3 = (t-t_3)^3, \qquad \overline{F}_4 = (t-t_4)^2.$$

On the 3-interval $[t_7, t_8)$, we still have

$$\overline{F}_3 = (t-t_3)^3, \qquad \overline{F}_4 = (t-t_4)^2,$$

but $\overline{F}_2$ is equal to the Taylor polynomial of order 3 of $(t-t_2)^4$ at t_7.

We easily derive the following properties of $\overline{F}_i,\ i \in \mathbb{Z}$, from their definitions.

Proposition 4.1. (1) *Each $\overline{F}_i$ on the interval $[t_j, t_{j+1})$ is a polynomial of degree less than or equal to d_j if $j \geq i$.*
(2) *Each $\overline{F}_i \in \Omega_{\boldsymbol{G}}[\boldsymbol{T}]$.*
(3) *The functions $\overline{F}_i,\ i \in \mathbb{Z}$, are linearly independent.*
(4) *If $j \geq i$, then the order of continuity of $\overline{F}_i$ at t_j is equal to $d_i - r_i^{i,+\infty}$ if $t_j = t_i$, and greater than or equal to d_j otherwise.*

4.2. The dimension of an LSCCDS-subspace

We give, like Lemma 3.1, the following lemma.

Lemma 4.1. *Let $s := r_i^{i,i+k}$. For any function $h = h(t) \in \Omega_{\boldsymbol{G}}[\boldsymbol{T}]$, if the right derivative $h^{(j)}(t_i+) = 0$ for $j = 0, 1, \ldots, d_i - s$, then there exist s numbers $a_l,\ i \leq l \leq i+s-1$, such that*

$$h(t) = \sum_{l=i}^{i+s-1} a_l \overline{F}_l(t), \quad t \in [t_i, t_{i+s}).$$

Proof. Since h on $[t_i, t_{i+r})$ is a polynomial function whose degree is less than or equal to d_i, it can be presented as

$$h = \sum_{l=0}^{d_i} b_l (t-t_i)^l$$

for some real number b_l. Because $h^{(j)}(t_i+) = 0$ for $j = 0, 1, \ldots, d_i - s$, we have $b_l = 0$ for $l = 0, 1, \ldots, d_i - s$. Therefore,

$$h = \sum_{l=d_i-s+1}^{d_i} b_l (t-t_i)^l = \sum_{l=i}^{i+s-1} b_{d_i+l-i-s+1} (t-t_i)^{d_i+l-i-s+1}.$$

According to the definitions of $\overline{F}_i,\ i \in \mathbb{Z}$, we have for $l = i, i+1, \ldots, i+s-1$,

$$\overline{F}_l = (t-t_l)^{d_l - r_l^{l,+\infty}+1} = (t-t_i)^{d_i+l-i-s+1}, \quad t \in [t_i, t_{i+s}),$$

and the claim follows with $a_l = b_{d_i+l-i-s+1}$. $\square$

From Formula (4), we have

$$r_{i+k_{i,n}}^{i,i+k_{i,n}} = \begin{cases} 1, & d_i < D - n, \\ \begin{cases} 2, & t_i = t_{i+1}, \\ 1, & t_i \neq t_{i+1}, \end{cases} & d_i = D - n. \end{cases}$$

Let $e := r_{i+k}^{i,i+k}$. Thus, we easily get:

Lemma 4.2. $k = \overline{c} + e + 1$.

Lemma 4.3. *Given integers i, q and l such that $l \geq r_i^{i,+\infty}$ and $i \leq q \leq i + l$, the determinant*

$$\begin{vmatrix} \overline{F}_i(t_q) & \overline{F}_{i+1}(t_q) & \cdots & \overline{F}_{i+l}(t_q) \\ \overline{F}_i'(t_q) & \overline{F}_{i+1}'(t_q) & \cdots & \overline{F}_{i+l}'(t_q) \\ \vdots & \vdots & \vdots & \vdots \\ \overline{F}_i^{(l)}(t_q) & \overline{F}_{i+1}^{(l)}(t_q) & \cdots & \overline{F}_{i+l}^{(l)}(t_q) \end{vmatrix} \neq 0.$$

The lemma is similar to Lemma 3.3, so we do not prove it here. From Lemmas 4.2 and 4.3, we get the dimension of $\overline{\Gamma}_{\boldsymbol{G}}^{\overline{p}}[t_i, t_{i+k}]$.

Theorem 4.1. *The dimension of the linear space $\overline{\Gamma}_{\boldsymbol{G}}^{\overline{p}}[t_i, t_{i+k}]$ is 1.*

Proof. Assume that $u = u(t)$ is an arbitrary function in $\overline{\Gamma}_{\boldsymbol{G}}^{\overline{p}}[t_i, t_{i+k}]$. Thus, for $j = i,\ i + 1, \ldots, i + k - e$, we have

$$u^{(l)}(t_j-) = u^{(l)}(t_j) = u^{(l)}(t_j+), \quad l = 0, 1, \ldots, d_j - r_j^{i,i+k}.$$

From Lemma 4.1, the function u on $[t_i, t_{i+s})$ is represented as the linear combination of $\{\overline{F}_l\}_{l=i}^{i+s-1}$. That is, there are real numbers $\{a_l\}_{l=i}^{i+s-1}$ such that

$$u = \sum_{l=i}^{i+s-1} a_l \overline{F}_l.$$

Let $v = r_{i+s}^{i,i+k}$. Consider the function $u - \sum_{l=i}^{i+s-1} a_l \overline{F}_l$ on $[t_i, t_{i+s+v})$. According to Lemma 4.1, we see that there are real numbers $\{a_l\}_{l=i+s}^{i+s+v-1}$ such that on the interval $[t_i, t_{i+s+v})$,

$$u - \sum_{l=i}^{i+s-1} a_l \overline{F}_l = \sum_{l=i+s}^{i+s+v-1} a_l \overline{F}_l.$$

That is,

$$u = \sum_{l=i}^{i+s+v-1} a_l \overline{F}_l, \quad t \in [t_i, t_{i+s+v-1}).$$

Recursively, we deduce that there exist $k - e + 1$ real numbers $a_l,\ i \leq l \leq i + k - e$, such that

$$u = \sum_{l=i}^{i+k-e} a_l \overline{F}_l, \quad t \in [t_i, t_{i+k}).$$

Then we focus on the continuity of u at the knot t_{i+k}. From Lemma 4.2, we have the following system of linear equations:

$$\begin{pmatrix} \overline{F}_i(t_{i+k}) & \overline{F}_{i+1}(t_{i+k}) & \cdots & \overline{F}_{i+\overline{c}+1}(t_{i+k}) \\ \overline{F}_i'(t_{i+k}) & \overline{F}_{i+1}'(t_{i+k}) & \cdots & \overline{F}_{i+\overline{c}+1}'(t_{i+k}) \\ \vdots & \vdots & \vdots & \vdots \\ \overline{F}_i^{(\overline{c})}(t_{i+k}) & \overline{F}_{i+1}^{(\overline{c})}(t_{i+k}) & \cdots & \overline{F}_{i+\overline{c}+1}^{(\overline{c})}(t_{i+k}) \end{pmatrix} \begin{pmatrix} a_i \\ a_{i+1} \\ \vdots \\ a_{i+\overline{c}+1} \end{pmatrix} = \mathbf{0}. \tag{18}$$

By a proof like that of Theorem 3.1, the dimension of the solution space of (18) is equal to 1. This means that the dimension of $\overline{\Gamma}_{\boldsymbol{G}}^{\overline{p}}[t_i, t_{i+k}]$ is 1. □

4.3. The determinantal representation

The theorem for determinantal representations, like Theorem 3.2, is given as follows.

Theorem 4.2. *Let $w := r_i^{i,+\infty}$ and*

$$\overline{\varphi}_i = \overline{\varphi}_i(t) := \begin{vmatrix} \overline{F}_i(t) & \overline{F}_{i+1}(t) & \cdots & \overline{F}_{i+\overline{c}+1}(t) \\ \overline{F}_i(t_{i+k}) & \overline{F}_{i+1}(t_{i+k}) & \cdots & \overline{F}_{i+\overline{c}+1}(t_{i+k}) \\ \overline{F}_i'(t_{i+k}) & \overline{F}_{i+1}'(t_{i+k}) & \cdots & \overline{F}_{i+\overline{c}+1}'(t_{i+k}) \\ \vdots & \vdots & \vdots & \vdots \\ \overline{F}_i^{(\overline{c})}(t_{i+k}) & \overline{F}_{i+1}^{(\overline{c})}(t_{i+k}) & \cdots & \overline{F}_{i+\overline{c}+1}^{(\overline{c})}(t_{i+k}) \end{vmatrix}. \tag{19}$$

Then we have

$$N_{i,D} = \overline{\alpha}_i \overline{\varphi}_i, \tag{20}$$

where

$$\overline{\alpha}_i := \frac{(-1)^{d_i-w+1} \begin{vmatrix} \overline{\boldsymbol{A}}_{i+w-d_i-1}^i & \cdots & \overline{\boldsymbol{A}}_{i-1}^i & \overline{\boldsymbol{A}}_{i+1}^i & \cdots & \overline{\boldsymbol{A}}_{i+w-1}^i \end{vmatrix}}{\begin{vmatrix} \overline{\varphi}_{i+w-d_i-1}(t_i) & \overline{\varphi}_{i+w-d_i}(t_i) & \cdots & \overline{\varphi}_{i+w-1}(t_i) \\ \overline{\boldsymbol{A}}_{i+w-d_i-1}^i & \overline{\boldsymbol{A}}_{i+w-d_i}^i & \cdots & \overline{\boldsymbol{A}}_{i+w-1}^i \end{vmatrix}} \tag{21}$$

and

$$\overline{\boldsymbol{A}}_j^i := \begin{pmatrix} \overline{\varphi}_j'(t_i+) \\ \overline{\varphi}_j''(t_i+) \\ \vdots \\ \overline{\varphi}_j^{(d_i)}(t_i+) \end{pmatrix}. \tag{22}$$

Proof. A simple proof is given here since it is similar to the proof of Theorem 3.2. Firstly, each *CD*-spline basis function is represented as

$$N_{i,D} = \overline{\alpha}_i \overline{\varphi}_i$$

because the function $\overline{\varphi}_i \in \overline{\Gamma}_{\boldsymbol{G}}^{\overline{p}}[t_i, t_{i+k}]$.

Then we use the normalized property of *CD*-spline basis functions. On the nonzero interval $[t_i, t_{i+w})$, we have $\sum_{j=i+w-d_i-1}^{i+w-1} N_{j,D}(t) \equiv 1$. Hence, we have the following system of linear equations:

$$\begin{cases} \displaystyle\sum_{j=i+w-d_i-1}^{i+w-1} \overline{\alpha}_j \overline{\varphi}_j(t_i) = 1, \\ \displaystyle\sum_{j=i+w-d_i-1}^{i+w-1} \overline{\alpha}_j \overline{\varphi}_j'(t_i+) = 0, \\ \vdots \\ \displaystyle\sum_{j=i+w-d_i-1}^{i+w-1} \overline{\alpha}_j \overline{\varphi}_j^{(d_i)}(t_i+) = 0. \end{cases} \tag{23}$$

The coefficient matrix of (23) is full rank. So we use Cramer's rule [26] to get $\overline{\alpha}_i$ as follows:

$$\begin{aligned} \overline{\alpha}_i &= \frac{\begin{vmatrix} \overline{\varphi}_{i+w-d_i-1}(t_i) & \cdots & \overline{\varphi}_{i-1}(t_i) & 1 & \overline{\varphi}_{i+1}(t_i) & \cdots & \overline{\varphi}_{i+w-1}(t_i) \\ \overline{\boldsymbol{A}}_{i+w-d_i-1}^i & \cdots & \overline{\boldsymbol{A}}_{i-1}^i & \mathbf{0} & \overline{\boldsymbol{A}}_{i+1}^i & \cdots & \overline{\boldsymbol{A}}_{i+w-1}^i \end{vmatrix}}{\begin{vmatrix} \overline{\varphi}_{i+w-d_i-1}(t_i) & \overline{\varphi}_{i+w-d_i}(t_i) & \cdots & \overline{\varphi}_{i+w-1}(t_i) \\ \overline{\boldsymbol{A}}_{i+w-d_i-1}^i & \overline{\boldsymbol{A}}_{i+w-d_i}^i & \cdots & \overline{\boldsymbol{A}}_{i+w-1}^i \end{vmatrix}} \\ &= \frac{(-1)^{d_i-w+1} \begin{vmatrix} \overline{\boldsymbol{A}}_{i+w-d_i-1}^i & \cdots & \overline{\boldsymbol{A}}_{i-1}^i & \overline{\boldsymbol{A}}_{i+1}^i & \cdots & \overline{\boldsymbol{A}}_{i+w-1}^i \end{vmatrix}}{\begin{vmatrix} \overline{\varphi}_{i+w-d_i-1}(t_i) & \overline{\varphi}_{i+w-d_i}(t_i) & \cdots & \overline{\varphi}_{i+w-1}(t_i) \\ \overline{\boldsymbol{A}}_{i+w-d_i-1}^i & \overline{\boldsymbol{A}}_{i+w-d_i}^i & \cdots & \overline{\boldsymbol{A}}_{i+w-1}^i \end{vmatrix}}. \end{aligned}$$

The theorem is proved. □

5. Some results

In this section, we give some results deduced from the explicit representations of *CD*-spline basis functions.

Firstly, we present $N_{i,D}$ as a linear combination of some functions $\overline{F}_j$. On the basis of (22) and the properties of determinants [26], we give the following theorem as a corollary of Theorem 4.2.

Theorem 5.1. *Let*

$$\overline{\lambda}_j := \begin{vmatrix} \overline{F}_i(t_{i+k}) & \cdots & \overline{F}_{j-1}(t_{i+k}) & \overline{F}_{j+1}(t_{i+k}) & \cdots & \overline{F}_{i+\overline{c}+1}(t_{i+k}) \\ \overline{F}_i'(t_{i+k}) & \cdots & \overline{F}_{j-1}'(t_{i+k}) & \overline{F}_{j+1}'(t_{i+k}) & \cdots & \overline{F}_{i+\overline{c}+1}'(t_{i+k}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \overline{F}_i^{(\overline{c})}(t_{i+k}) & \cdots & \overline{F}_{j-1}^{(\overline{c})}(t_{i+k}) & \overline{F}_{j+1}^{(\overline{c})}(t_{i+k}) & \cdots & \overline{F}_{i+\overline{c}+1}^{(\overline{c})}(t_{i+k}) \end{vmatrix} \tag{24}$$

and

$$\overline{\beta}_j := (-1)^{j-i}\overline{\lambda}_j\overline{\alpha}_i \tag{25}$$

for $j = i, i+1, \ldots, i+\overline{c}+1$. *Then we have*

$$N_{i,D}(t) = \sum_{j=i}^{i+\overline{c}+1} \overline{\beta}_j\overline{F}_j(t). \tag{26}$$

Secondly, we answer the question posed in Section 1. In this paper, we have found some changeable degree truncated power functions $\overline{F}_i$, $i \in \mathbb{Z}$. They are linearly independent and can represent every *CD*-spline basis function. Thus they form a basis for CDS-space. From the process of explicit representations, we have the following remark concerning the CDS-space.

Remark 5.1. The CDS-space over the knot sequence $\boldsymbol{T}$ and degree sequence $\boldsymbol{G}$

$$\begin{aligned}\Omega_{\boldsymbol{G}}[\boldsymbol{T}] = \{&\text{piecewise function } u(t) \text{ over } \boldsymbol{T} \mid u(t) \text{ limited to each knot interval } [t_i, t_{i+1}) \\ &\text{is a polynomial function of degree} \le d_i;\ \text{the continuous order at each knot } t_i \text{ of } u(t) \\ &\text{is greater than or equal to } d_i - m_i,\ \text{where } m_i \text{ is the multiplicity of } t_i\}.\end{aligned}$$

Thirdly, in this process of explicit representation, we only use a few properties of *CD*-spline basis functions. These properties are normalization, local support, continuous order and basis properties. All of them are easily obtained from the integral definitions. So our method for explicit representations is also feasible for all the integral defined basis functions mentioned in Section 1.

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