Homological Characteristics of Nakayama Algebras

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Communicated by Walter Feit

Received January 20, 1982

In a previous paper [3], we described a homological characterization of pro-uniserial rings. The purpose of this paper is to give a similar characterization of injec-uniserial rings and of Nakayama algebras. Throughout this paper, all rings are assumed to be finite-dimensional algebras over some fixed algebraically closed field $F$, and all modules are left modules.

If $R$ is a ring, an $R$-module is called uniserial iff it has a unique composition series. $R$ is called

1. pro-uniserial iff all of its indecomposable projective modules are uniserial,
2. injec-uniserial iff all of its indecomposable injective modules are uniserial,
3. uniserial, or Nakayama, iff it is both pro-uniserial and injec-uniserial.

If $M$ is an $R$-module, let $l(M)$ denote the composition length of $M$. If $M$ is uniserial, then the following are well known:

1. All submodules and quotient modules of $M$ are also uniserial.
2. If $0 \to M_1 \to M \to M_2 \to 0$ is any short exact sequence, then $l(M) = l(M_1) + l(M_2)$.
3. The only submodules of $M$ are those in its composition series, and hence its only quotients are those by submodules in its composition series. In particular, $M$ has a unique simple quotient, a unique simple submodule, and a unique maximal submodule. In fact, if $0 \leq k \leq l(M)$, then $M$ has a unique submodule and a unique quotient module having composition length $k$.

Let $J(R)$ denote the Jacobson radical of $R$, and if $M$ is an $R$-module, let $J(M) = J_1(M) = J(R) \cdot M$, $J_2(M) = J(R)^2 \cdot M$, etc. Since $R$ is an Artin
algebra, it has only a finite number, say \( r \), of isomorphism classes of simple modules. It is well known that \( R \) also has exactly \( r \) isomorphism classes of indecomposable projective modules and exactly \( r \) isomorphism classes of indecomposable injective modules. Moreover, each indecomposable projective module \( P \) has a unique maximal submodule which is equal to \( J(P) \), and, for each simple \( R \)-module \( S \), there is a unique (up to isomorphism) indecomposable projective \( R \)-module \( P \) with \( P/J(P) \cong S \). Similarly, each indecomposable injective \( R \)-module has a unique simple submodule, and hence a simple socle, and for each simple \( R \)-module \( S \), there is a unique (up to isomorphism) indecomposable injective \( R \)-module \( I \) with the socle of \( I \) (denoted henceforth by \( \text{soc} I \)) isomorphic to \( S \). Henceforth, if \( S \) is a simple \( R \)-module, \( P_S \) will denote the indecomposable projective \( R \)-module with \( P_S/J(P_S) \cong S \), and \( I_S \) will denote the indecomposable injective \( R \)-module with \( \text{soc} I_S \cong S \).

If \( M \) is a uniserial \( R \)-module with composition series \( M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n = 0 \), then we will say that the simple factor \( M_0/M_1 \) is at the top of \( M \), that \( M_1/M_2 \) lies beneath \( M_0/M_1 \) in \( M \), and so forth, and that \( M_n \) lies at the bottom of \( M \).

Suppose \( R \) is an injec-uniserial ring; if \( S \) and \( T \) are simple \( R \)-modules, we shall say that \( T \) is above \( S \), or \( S \) is beneath \( T \), iff \( T \cong \text{soc}(I_S/S) \). By an argument dual to that used in \(|3|\), it is easy to show that, if \( M \) is any uniserial \( R \)-module having \( S \) as a composition factor, then either \( S \) lies at the top of \( M \), or \( T \) lies above \( S \) in \( M \). It is clear that each simple \( R \)-module has at most one simple above it, although it may have several beneath it.

If \( R \) is injec-uniserial, we can now define the graph of \( R \) as follows: there is one node in the graph for each isomorphism class of simple \( R \)-modules, and, if \( S \) and \( T \) are two simple \( R \)-modules, then there is an arrow from the node corresponding to \( T \) to the node corresponding to \( S \) iff \( T \) is above \( S \). The graph will thus have the following form:
By the remark above, each node in the graph can have at most one arrow terminating at it, though it may have several originating there. We shall speak of the components of a graph in the obvious sense, and shall say a graph is connected iff it has only one component. Also, we shall say that a component of a graph is cyclic if it has at least one node which can be reached from itself by following arrows; if a component is not cyclic, we shall call it linear.

If \( R \) is a ring, let \( R^{\text{op}} \) be its opposite ring; i.e., \( R^{\text{op}} \) has the same underlying set and addition as \( R \), but the multiplication is reversed. There is a well-known duality between the categories of \( R \)-modules and \( R^{\text{op}} \)-modules given by the contravariant functor \( D(M) = \text{Hom}_F(M, F) \), where \( F \) is the underlying field (see [2, Section I.9]). This functor preserves composition lengths, takes injectives to projectives and vice versa, and interchanges quotients and submodules. Thus, in particular, if \( R \) is inject-uniserial, then \( R^{\text{op}} \) is pro-uniserial, \( D(I_S) = P_{\text{proj} S} \) for each simple \( R \)-module \( S \), and the graph of \( R^{\text{op}} \) (as defined in [3]) is just the graph of \( R \) with the arrows reversed.

Recall that two rings are Morita equivalent iff their module categories are naturally isomorphic. By Theorem 22.1 of [1], it is clear that, if \( R \) and \( R' \) are two \( F \)-algebras, then \( R \) and \( R' \) are Morita equivalent iff \( R^{\text{op}} \) and \( R'^{\text{op}} \) are Morita equivalent.

With these observations, the following results follow immediately from the corresponding theorems about pro-uniserial rings in [3]:

I. Suppose we are given any finite directed graph having the property that no node has more than one arrow terminating there, and a length \( l(v) \) for each node \( v \), where each length is a positive integer and the list of lengths has the property that, if there is an arrow from \( v_1 \) to \( v_2 \) in the graph, then \( l(v_1) \geq l(v_2) - 1 \). Then there is an inject-uniserial ring having the given graph as its graph, and whose indecomposable injectives have the given lengths.

II. Two inject-uniserial rings are Morita equivalent iff they have the same graph, and corresponding indecomposable injectives have the same composition lengths.

Recall that, if \( R \) and \( R' \) are any two \( F \)-algebras, then they are said to be Poincaré equivalent iff there is a one-to-one correspondence between their simple modules which induces an \( F \)-algebra isomorphism between \( \text{Ext}(R) \) and \( \text{Ext}(R') \), where \( \text{Ext}(R) = \bigoplus S \sum T \text{Ext}^p_S(T, S) \) with \( S \) and \( T \) ranging over one representative from each isomorphism class of simple \( R \)-modules, and with a product equal to the Yoneda product, where it is defined, and zero elsewhere. With this in mind, we prove the following:

**Lemma 1.** If \( M \) and \( N \) are \( R \)-modules, where \( R \) is any \( F \)-algebra, then, for all \( n \geq 0 \), \( \text{Ext}^n_R(M, N) \cong \text{Ext}^n_{R^{\text{op}}}(D(N), D(M)) \).
Proof: Suppose
\[ \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0 \]
is a projective resolution of \( M \). Then
\[ 0 \longrightarrow D(M) \xrightarrow{D(d_0)} D(P_0) \xrightarrow{D(d_1)} D(P_1) \xrightarrow{D(d_2)} D(P_2) \longrightarrow \cdots \]
is an injective coresolution of \( D(M) \). Since \( D \) is a duality, \( \text{Hom}_R(P_i, N) \cong \text{Hom}_{R^{op}}(D(N), D(P_i)) \) for \( i \geq 0 \), and, if \( d_i^*: \text{Hom}_R(P_{i-1}, N) \to \text{Hom}_R(P_i, N) \) is the induced map, then the induced map \( D(d_i)^*: \text{Hom}_{R^{op}}(D(N), D(P_{i-1})) \to \text{Hom}_{R^{op}}(D(N), D(P_i)) \) is simply \( D(d_i^*) \). Further, since \( D \) is left and right exact, for \( i \geq 0 \), \( \ker D(d_i)^* = \ker D(d_i^*) \cong \ker d_i^* \), and \( \text{Im} D(d_i)^* = \text{Im} D(d_i^*) \cong \text{Im} d_i^* \), and so \( \text{Ext}_R^n(M, N) = (\ker d_{i+1}^*)/(\text{Im} d_i^*) \cong (\ker D(d_{i+1}^*)/(\text{Im} D(d_i^*)) = \text{Ext}_R^n(D(N), D(M)) \).

In the above proof, we used the well-known fact that \( \text{Ext}_R^n(S, T) \) may be computed using an injective coresolution of the second variable; i.e., if
\[ \cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} S \longrightarrow 0 \]
is a projective resolution of \( S \) and
\[ 0 \longrightarrow T \xrightarrow{d_n} J_0 \xrightarrow{d_1} J_1 \xrightarrow{d_2} J_2 \xrightarrow{d_3} \cdots \]
is an injective coresolution of \( T \), then for each \( n \geq 0 \), \( (\ker \partial_{n+1}^*)/(\text{Im} \partial_n^*) \) is naturally isomorphic to \( (\ker d_{n+1}^*)/(\text{Im} d_n^*) \), where \( \partial_n^*: \text{Hom}_R(P_{n-1}, T) \to \text{Hom}_R(P_n, T) \) and \( d_n^*: \text{Hom}_R(S, J_{n-1}) \to \text{Hom}_R(S, J_n) \) are the induced maps. In fact, if \( f \in \ker \partial_{n+1}^* \), then we can find maps \( \gamma_i \), \( 0 \leq i \leq n \), so that the following commutes:
\[ \cdots \xrightarrow{f} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} P_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} S \xrightarrow{0} 0 \]
\[ 0 \longrightarrow T \xrightarrow{d_n} J_0 \xrightarrow{d_1} J_1 \xrightarrow{d_2} J_2 \xrightarrow{d_3} \cdots \]
Moreover, \( \gamma_n \in \ker d_{n+1}^* \), and its class in \( (\ker d_{n+1}^*)/(\text{Im} d_n^*) \) depends only on the class of \( f \) in \( (\ker \partial_{n+1}^*)/(\text{Im} \partial_n^*) \). Thus \( \Phi: (\ker \partial_{n+1}^*)/(\text{Im} \partial_n^*) \to (\ker d_{n+1}^*)/(\text{Im} d_n^*) \) defined by \( \Phi(f) = \gamma_n \) is a well-defined function; and indeed, \( \Phi \) is a natural isomorphism between these \( F \)-modules. For proofs, see [5, Theorems III.6.4 and III.8.2].
Now suppose $S$, $T$, and $U$ are $R$-modules, and $0 \rightarrow T \rightarrow d_0 J_0 \rightarrow d_1 J_1 \rightarrow d_2 \cdots$ and $0 \rightarrow U \rightarrow d_0 I_0 \rightarrow d_1 I_1 \rightarrow d_2 \cdots$ are injective coresolutions of $T$ and $U$, respectively. If $[f'] \in \text{Ext}^p_R(S, T)$ and $[g'] \in \text{Ext}^q_R(T, U)$, where the Ext's are taken to be defined via the injective coresolutions, so that $f' \in \ker d_{n+1}$ and $g' \in \ker d_{n+1}$, then we can define $[f'] \cdot [g']$ by noting that there exist maps $\beta_i, 0 \leq i \leq n$, so that the following commutes:

\[
\begin{array}{ccccccc}
S & \rightarrow & T & \rightarrow & J_0 & \rightarrow & J_1 & \rightarrow & \cdots & \rightarrow & J_{n-2} & \rightarrow & J_{n-1} & \rightarrow & J_n \\
\downarrow & & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & d_0 J_0 & \rightarrow & d_1 J_1 & \rightarrow & d_2 J_2 & \rightarrow & \cdots & \rightarrow & d_{n-2} J_{n-2} & \rightarrow & d_{n-1} J_{n-1} & \rightarrow & d_n J_n \\
\end{array}
\]

Then $[f'] \cdot [g']$ can be defined to be the class of $\beta_n \circ f'$ in $\text{Ext}^{n+m}_R(S, U)$; by arguments exactly dual to those used to establish the Yoneda product it can be shown that $[\beta_n \circ f']$ is well defined and depends only on the classes of $f'$ and $g'$. Our next lemma shows that, in fact, this operation is just another realization of the Yoneda product.

**Lemma 2.** If $[f] \in \text{Ext}^p_R(S, T)$ and $[g] \in \text{Ext}^q_R(T, U)$, defined projectively, then $\Phi([f]) \cdot \Phi([g]) = \Phi([f] \otimes [g])$, where $\otimes$ is the Yoneda product and $\Phi$ and $\cdot$ are as defined above.

**Proof.** Consider diagram 1. (Note that $\sigma_m = \Phi(g).$)

Here, $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow S \rightarrow 0$ and $\cdots \rightarrow Q_2 \rightarrow Q_1 \rightarrow Q_0 \rightarrow T \rightarrow 0$ are projective resolutions of $S$ and $T$, respectively, and $0 \rightarrow T \rightarrow J_0 \rightarrow J_1 \rightarrow J_2 \rightarrow \cdots \rightarrow 0 \rightarrow U \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots$ are injective coresolutions of $T$ and $U$, respectively. Also, for $0 \leq i \leq m$, $\alpha_i : P_{n+i} \rightarrow Q_i$ and these define $[f] \otimes [g]$; for $0 < i < n$, $\gamma_i : P_{n+i} \rightarrow J_i$ and these define $\Phi([f])$; for $0 \leq i \leq m$, $\sigma_i : Q_{m+i} \rightarrow I_i$ and these define $\Phi([g])$; and for $0 \leq i \leq n$, $\beta_i : J_i \rightarrow I_{n+i}$ and these define $\Phi([f]) \cdot \Phi([g])$. Since each square and triangle commutes, so does the whole diagram. To find $\Phi([f] \otimes [g])$, we need to find maps $\tau_i : P_{n+m+i} \rightarrow I_i$, $0 \leq i \leq m + n$, so that the following commutes:

\[
\begin{array}{ccccccc}
P_{n+m} & \rightarrow & P_{n+m-1} & \rightarrow & \cdots & \rightarrow & P_1 & \rightarrow & P_0 & \rightarrow & S & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & U & \rightarrow & I_0 & \rightarrow & \cdots & \rightarrow & I_{n+m-2} & \rightarrow & I_{n+m-1} & \rightarrow & I_{n+m}. \\
\end{array}
\]
Diagram 1
\[ \Phi(\lfloor f \rfloor \otimes \lfloor g \rfloor) \] will then be the class of \( \tau_{n+m} \). However, from diagram I, it is clear that, for \( 0 \leq i \leq m-1 \), we can choose \( \tau_i = \sigma_i \circ \alpha_{m-i-1} \), and, for \( m \leq i \leq m+n \), we can choose \( \tau_i = \beta_{i-m} \circ \gamma_{i-m} \). Then the required diagram does commute, and \( \tau_{n+m} = \beta_{m} \circ \gamma_{m} \), which is a representative of \( \Phi(\lfloor f \rfloor) \cdot \Phi(\lfloor g \rfloor) \). The lemma follows.

The following is now an immediate consequence of Lemmas 1 and 2:

**Lemma 3.** Suppose \( R \) and \( R' \) are two \( F \)-algebras. Then \( R \) and \( R' \) are Poincaré equivalent iff \( R^{op} \) and \( R'^{op} \) are Poincaré equivalent.

Now suppose \( R \) is an injec-uniserial ring. If \( S \) is a simple \( R \)-module, let \( \mathcal{J}(S) = \text{soc}(I_x(J(I_x))/I_x(J(I_x))) \); in other words, \( \mathcal{J}(S) \) is the simple module above the top of \( I_x \). We divide the simple \( R \)-modules into three groups as follows:

1. \( S \) is of Type I (or \( S \in \mathcal{J}_1 \)) if it is possible to start at \( S \) in the graph of \( R \) and, by following arrows, reach \( S \) again.
2. \( S \) is of Type II (or \( S \in \mathcal{J}_2 \)) if \( S \in \mathcal{J}_1 \) but \( \mathcal{J}(S) \in \mathcal{J}_1 \).
3. \( S \) is of Type III (or \( S \in \mathcal{J}_3 \)) if \( S \notin \mathcal{J}_1 \) and \( \mathcal{J}(S) \notin \mathcal{J}_1 \).

Also, we define the cycle length of \( R \) to be the number of isomorphism classes of simple \( R \)-modules of Type I.

Using the above lemmas and definitions, the following results follow immediately from the corresponding theorems about pro-uniserial rings in [3]:

III. Suppose \( R \) is an injec-uniserial ring.

(A) (i) If \( S \) and \( T \) are simple \( R \)-modules, and \( n \geq 0 \), then either \( \text{Ext}_R^n(S, T) = 0 \) or \( \text{Ext}_R^n(S, T) \cong F \).

(ii) If \( S \) and \( T \) lie in different components of the graph of \( R \), then \( \text{Ext}_R^n(S, T) = 0 \) for all \( n \geq 0 \).

(B) The following are equivalent:

(i) \( R \) has a connected linear graph.

(ii) There is a unique indecomposable injective \( R \)-module \( I_x \) with \( l(I_x) = 1 \).

(iii) There is a unique simple \( R \)-module \( S \) with \( \text{Ext}_R^n(T, S) = 0 \) for all simple \( R \)-modules \( T \) and all \( n > 0 \).

IV. If \( R \) and \( R' \) are two injec-uniserial rings with connected, linear graphs, then \( R \) and \( R' \) are Poincaré equivalent iff they have the same graph and corresponding indecomposable injectives have the same composition lengths.

V. Any injec-uniserial ring is Morita equivalent to a direct sum of injec-uniserial rings with connected graphs.
VI. Suppose $R$ and $R'$ are two inject-uniserial rings having connected, cyclic graphs.

(A) If $R$ and $R'$ are Poincaré equivalent, then they have the same graph, and corresponding simples have the same types. Let $S \in_R \text{Mod}$ denote the simple corresponding to the simple $S \in_R \text{Mod}$. If $S \in_R \mathcal{G}$, then $l(I_S) = l(I_S')$, and if $k$ is the cycle length and $S \in_R \mathcal{G}'$, then $l(I_S) \equiv l(I_S') \pmod{k}$. Also, there is an integer $m$ such that if $S \in_R \mathcal{G}$ then $l(I_S) = l(I_S') + mk$; $m$ is independent of which $S \in_R \mathcal{G}$ is examined.

(B) Suppose $R$ and $R'$ have the same graph; let $S \in_R \text{Mod}$ and $S' \in_R \text{Mod}$ denote simple modules corresponding to the same node. Let $k$ be the cycle length, and suppose further that there is an integer $m$ such that

(a) if $S \in_R \mathcal{G} \cup \mathcal{G}'$, then $l(I_S) = l(I_S') + mk$.
(b) if $S \in_R \mathcal{G}$, then $l(I_S) = l(I_S')$.
(c) if $S \in_R \mathcal{G}$, then $l(I_S) \geq 2k + 1$ and $l(I_S') \geq 2k + 1$.
(d) if $S \in_R \mathcal{G}$, then each simple in $\mathcal{G}$ appears as a composition factor in $I_S$ and $I_S'$ at least twice.

Then $R$ and $R'$ are Poincaré equivalent.

VII. An $F$-algebra $R$ is inject-uniserial iff for each simple $R$-module $T$, \[ \sum_{S \in_R \text{Mod}} \dim_F \text{Ext}^1_R(S, T) \leq 1 \], where $\mathcal{G} \subset_R \text{Mod}$ contains exactly one representative from each isomorphism class of simple $R$-modules.

Now suppose that $R$ is a Nakayama algebra. Since $R$ is certainly inject-uniserial, it has a graph associated with it as described above, in which each node corresponds to an isomorphism class of simple $R$-modules, and there is an arrow from $S$ to $T$, where $S$ and $T$ are simple $R$-modules, iff $S \cong \text{soc}(I_t/\text{soc} I_t)$. However, since $R$ is also pro-uniserial, there is another graph associated with $R$ (see [3]) in which each node again corresponds to an isomorphism class of simple $R$-modules, but in which there is an arrow from $S$ to $T$ iff $T \cong \text{soc}(I_t/\text{soc} I_t)$. We want to show that these two graphs are the same; since they certainly have the same nodes, the following will suffice:

**Lemma 4.** If $R$ is a Nakayama algebra and $S$ and $T$ are simple $R$-modules, then $S \cong \text{soc}(I_t/\text{soc} I_t)$ iff $T \cong \text{soc}(I_t/\text{soc} I_t)$.

**Proof:** Suppose $S \cong \text{soc}(I_t/\text{soc} I_t)$. Let $\pi_1: I_t \to I_t/\text{soc} I_t$ and $\pi_2: I_t/\text{soc} I_t \to (I_t/\text{soc} I_t)/\text{soc}(I_t/\text{soc} I_t)$ be the projections, and let $M = \ker \pi_2 \circ \pi_1$. Then $M \subset I_t$, so $M$ is uniserial, $l(M) = 2$, $\text{soc}(M) = J(M) \cong T$, and $M/\text{soc} M = M/J(M) \cong S$. If $l(P_S) = 1$, then $S$ is projective, and so, since $S$ is a quotient of $M$, it must be a direct summand. But since $M$ is uniserial, it cannot have any direct summands. Hence, $l(P_S) \geq 2$. Let $J(P_S)/J_2(P_S) = U \neq 0$, so that $U$ lies beneath $S$ in $P_S$. By
Theorem 1 of [3], since \( T \) lies beneath \( S \) in the uniserial module \( M \), we must have \( T \cong U \).

Now suppose \( T \cong J(P_3)/J_2(P_3) \). Let \( N = P_3/J_2(P_3) \); then \( l(N) = 2 \) and \( T \cong J(N) \). If \( l(T) = 1 \), then \( T \) is injective. Since \( T \) is a submodule of \( N \), it must be a direct summand. But since \( N \) is a quotient of a uniserial module, it must be uniserial, and hence can have no direct summands. Hence \( l(T) \geq 2 \).

Let \( \text{soc}(I_T/\text{soc} I_T) = V \neq 0 \). Since \( N \) is a uniserial module and \( S \) is above \( T \) in \( N \), \( S \cong V \), as observed earlier.

Thus, when \( R \) is Nakayama, we may speak of the graph of \( R \) without ambiguity. Moreover, since such an \( R \) is injec-uniserial, no node in the graph may have more than one arrow terminating there, and, since \( R \) is also pro-uniserial, no node may have more than one arrow originating there (see [3]). This motivates the following:

**Theorem 1.** Given any finite directed graph having the property that no node has more than one arrow terminating there or originating there, and any list of composition lengths for indecomposable injectives having the property that if \( v_1 \) and \( v_2 \) are nodes in the graph, and there is an arrow from \( v_1 \) to \( v_2 \), then \( l(v_2) \geq l(v_2) - 1 \), then there is a Nakayama algebra having the given graph whose indecomposable injectives have the given composition lengths.

**Proof.** As noted above, there is certainly an injec-uniserial ring, call it \( R \), with the required properties, so we need only show that this ring is also pro-uniserial. By Theorem 9 of [3], \( R \) is pro-uniserial iff for each simple \( R \)-module \( S \), \( \sum_{r \in r} \text{dim}_r \text{Ext}^k_k(S, T) \leq 1 \), where \( r \subseteq \text{Mod} \) contains exactly one representative from each isomorphism class of simple \( R \)-modules. As noted earlier, we know that, for any simples \( S \) and \( T \), either \( \text{Ext}^k_k(S, T) = 0 \) or \( \text{Ext}^k_k(S, T) \cong F \). So, to show that \( R \) is pro-uniserial, it is sufficient to show that, for each simple \( R \)-module \( S \), there is at most one (up to isomorphism) simple \( R \)-module \( T \) with \( \text{Ext}^k_k(S, T) \cong F \).

We begin by constructing the first part of an injective coresolution for the simple \( R \)-module \( T \). Clearly, we may take \( I_0 = I_T \) and \( d_0 : T \to I_0 \) to be the inclusion. If \( I = I_T \), then we may take \( I_n = 0 \) for \( n \geq 1 \). Otherwise, the cokernel of \( d_0 \) is \( I_T/T \), a uniserial \( R \)-module. Let \( U \) be the simple at the bottom of \( I_T/T \), so that \( U = \text{soc}(I_T/T) \). As observed earlier, \( I_T/T \) must be isomorphic to a submodule of \( I_T \). Choose \( I_1 = I_U \) and \( d_1 : I_0 \to I_1 \) to be the projection onto \( I_T/T \) followed by inclusion. If \( d_1 \) is onto, we may take \( I_n = 0 \) for \( n \geq 1 \); otherwise, coker \( d_1 \) is a uniserial \( R \)-module. Let \( V \) be the simple at the bottom of coker \( d_1 \). Then, as above, coker \( d_1 \) is a submodule of \( I_T \), and so we may take \( I_2 = I_V \) and \( d_2 : I_1 \to I_2 \) to be the projection \( I_1 \to \text{coker} d_1 \) followed by the inclusion. We can now compute \( \text{Ext}^k_k(S, T) \), where \( S \) is any simple \( R \)-module.
Consider the complex

\[ 0 \to \text{Hom}_R(S, I_0) \xrightarrow{d_{1*}} \text{Hom}_R(S, I_1) \xrightarrow{d_{2*}} \text{Hom}_R(S, I_2) \]

where \( d_{1*} \) and \( d_{2*} \) are the induced maps. By definition, \( \text{Ext}_R^1(S, T) = (\ker d_{2*})/(\text{Im} d_{1*}) \). Certainly, if \( I_1 = 0 \), \( \text{Hom}_R(S, I_1) = 0 \), so \( \text{Ext}_R^1(S, T) = 0 \).

Also, by Lemma 3, if \( U \neq S \), \( \text{Hom}_R(S, I_U) = 0 \), so again \( \text{Ext}_R^1(S, T) = 0 \). So suppose \( U = S \) and \( 0 \neq I_1 = I_U = I_S \). By Lemma 3, \( \text{Hom}_R(S, I_1) \cong F \). If \( I_2 = 0 \) or if \( I_2 = I_U \) and \( V \neq S \), then \( \text{Hom}_R(S, I_2) = 0 \), so \( d_{2*} \) is the zero map.

So suppose also \( V = S \) and \( 0 \neq I_2 = I_V = I_S \). Then \( d_2 : I_S \to I_S \). If \( \ker d_2 = 0 \), then, since \( \ker d_2 = \text{Im} d_1 \), \( \text{Im} d_1 = 0 \Rightarrow \ker d_1 = I_0 = \text{Im} d_0 = I_0 = T \Rightarrow I_n = 0 \) for \( n \geq 1 \), a contradiction. Hence we must have \( \ker d_2 \neq 0 \).

Suppose \( f \in \text{Hom}_R(S, I_2) \). Then \( \text{Im} f = \text{soc} I_3 = \text{the unique simple submodule of} I_3 \); since \( \ker d_2 \neq 0 \), it must contain a simple submodule. Hence \( \text{Im} f \leq \ker d_2 \).

But then \( d_{2*}(f) = d_2 \circ f = 0 \), so \( d_{2*} \) is the zero map. Thus, in all cases, \( d_{2*} \) is the zero map.

Similarly, if \( T \neq S \), \( \text{Hom}_R(S, I_0) = 0 \) and so \( d_{1*} \) is the zero map; if \( T \cong S \), \( \text{Hom}_R(S, I_0) \cong F \) but, by reasoning similar to the above, \( d_{1*} \) is still zero.

Thus, whenever \( \text{Hom}_R(S, I_1) \neq 0 \), we have \( \text{Hom}_R(S, I_1) \cong F \) and \( d_{1*} \) and \( d_{2*} \) are zero maps. Hence, in this case, we have \( \ker d_{2*} = \text{Hom}_R(S, I_1) \cong F \) and \( \text{Im} d_{1*} = 0 \), so \( \text{Ext}_R^1(S, T) \cong \text{Hom}_R(S, I_1) \cong F \).

To summarize, we see that

\[
\dim_R \text{Ext}_R^1(S, T) = \begin{cases} 0 & \text{if } T = I_T \text{ or } S \cong U \\ 1 & \text{if } T \neq I_T \text{ and } S \cong U. \end{cases}
\]

From the construction of the coresolution for \( T \), we see that \( U \) was chosen to be that simple isomorphic to \( \text{soc}(I_T/T) \); in other words, \( U \) is that simple having the property that there is an arrow in the graph of \( R \) originating at \( U \) and terminating at \( T \). Thus \( \dim_R \text{Ext}_R^1(S, T) = 1 \) iff there is an arrow from \( S \) to \( T \) in the graph of \( R \); the dimension is zero otherwise. But by the hypotheses of this theorem, there can be at most one arrow originating at \( S \), and so \( \dim_R \text{Ext}_R^1(S, T) = 1 \) for at most one (up to isomorphism) simple \( T \) with the dimension zero for all other simples. Thus certainly \( \sum_{T \in \mathcal{S}} \dim_R \text{Ext}_R^1(S, T) \leq 1 \) for all simple \( R \)-modules \( S \). As noted earlier, this suffices to prove the theorem.

Since a Nakayama algebra is certainly injec-uniserial, it follows from (I) and from the Appendix to [3] that every Nakayama algebra is Morita equivalent to a direct sum of Nakayama algebras having connected graphs. If a Nakayama algebra has a connected, linear graph, then (III)(B) and (IV) apply, and nothing stronger can be said. However, if a Nakayama algebra has a connected, cyclic graph, it is clear that each of its simple modules must...
be of Type I. The following is now a consequence of (VI) and of Theorems 7 and 8 of [3]:

**Theorem 2.** Suppose $R$ and $R'$ are two Nakayama algebras having connected cyclic graphs.

(A) If $R$ and $R'$ are Poincaré equivalent, then they have the same graph. Let $S' \in R_{\text{Mod}}$ denote the simple corresponding to the simple $S \in R_{\text{Mod}}$. Then there is an integer $m$ such that, for any simple $R$-module $S$, $l(I_S) = l(I_{S'}) + mk$ and $l(P_S) = l(P_{S'}) + mk$, where $k$ is the cycle length.

(B) Suppose $R$ and $R'$ have the same graph; let $S' \in R_{\text{Mod}}$ and $S' \in R_{\text{Mod}}$ denote simple modules corresponding to the same node. Let $k$ be the cycle length, and suppose either

(a) there is an integer $m$ such that, for any simple $R$-module $S$, $l(I_S) = l(I_{S'}) + mk$, and

(b) for any simple $R$-module $S$, $l(I_S) \geq 2k + 1$ and $l(I_{S'}) \geq 2k + 1$.

or

(a') there is an integer $m$ such that, for any simple $R$-module $S$, $l(P_S) = l(P_{S'}) + mk$, and

(b') for any simple $R$-module $S$, $l(P_S) \geq 2k + 1$ and $l(P_{S'}) \geq 2k + 1$.

Then $R$ and $R'$ are Poincaré equivalent.

**Proof.** (B) is an immediate consequence of (VI) and of Theorem 8 of [3]. We need a brief argument to establish (A): by (VI), there is an integer $m$ such that $l(I_S) = l(I_{S'}) + mk$ for all simples $S \in R_{\text{Mod}}$; by Theorem 7 of [3], there is an integer $n$ such that $l(P_S) = l(P_{S'}) + nk$. We must show that $n = m$.

Let $T$ be a simple $R$-module having the property that, for any simple $S \in R_{\text{Mod}}$, $l(T) \geq l(I_S)$. Since $R$ has only finitely many non-isomorphic simple modules, there certainly is such a $T$. Let $U = I_T/J(T)$ be the simple factor at the top of $I_T$. Since $R$ is pro-uniserial, it follows from Lemma 1 of [3] that $I_T$ must be a quotient of $P_T$, and so $l(P_T) \geq l(I_T)$. Let $V = \text{soc} P_T$ be the simple at the bottom of $P_T$. Since $R$ is injec-uniserial, $P_T$ must be a submodule of $I_T$, and so $l(I_T) \geq l(P_T)$, as noted in the proof of Theorem 1. Thus we have $l(I_T) \geq l(P_T) \geq l(I_T)$. But by the choice of $T$, $l(I_T) \geq l(I_T)$. Hence we must have $l(I_T) = l(P_T) = l(I_T)$. Since $P_T$ is a submodule of $I_T$ and $I_T$ is a quotient of $P_T$, this implies that $I_T = P_T = I_T$, and so $V = T$. By similar reasoning, since $T'$ must have the property that $l(I_{T'}) \geq l(I_{T'})$ for all simples $S' \in R_{\text{Mod}}$, we get $P_{T'} = I_{T'}$. Thus we have $l(I_{T'}) + mk = l(I_T) = l(P_T) = l(P_{T'}) + nk$, and so $m = n$.

Finally, we can show the following:

**Theorem 3.** An $F$-algebra $R$ is Nakayama iff for each simple $R$-module
\[ S, \sum_{T \in \mathcal{Y}} \dim_F \text{Ext}_R^1(S, T) \leq 1 \quad \text{and} \quad \sum_{T \in \mathcal{Y}} \dim_F \text{Ext}_R^1(T, S) \leq 1, \quad \text{where} \quad \mathcal{Y} \subseteq_R \text{Mod contains exactly one representative from each isomorphism class of simple } R\text{-modules.} \]

**Proof.** This is an immediate consequence of (VII) and of Theorem 9 of [3].

**ACKNOWLEDGMENT**

The author is grateful to Professor Jonathan Alperin for his suggestions.

**REFERENCES**