Lower Bounds for Merging Networks

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A lower bound theorem is established for the number of comparators in a merging network. Let $M(m, n)$ be the least number of comparators required in the $(m, n)$-merging networks, and let $C(m, n)$ be the number of comparators in Batcher’s $(m, n)$-merging network, respectively. We prove for $n \geq 1$ that $M(4, n) = C(4, n)$ for $n \equiv 0, 1, 3 \mod 4$, $M(4, n) \geq C(4, n) - 1$ for $n \equiv 2 \mod 4$, and $M(5, n) = C(5, n)$ for $n \equiv 0, 1, 5 \mod 8$. Furthermore Batcher’s $(6, 8k+6)$-, $(7, 8k+7)$-, and $(8, 8k+8)$-merging networks are optimal for $k \geq 0$. Our lower bound for $(m, n)$-merging networks, $m \leq n$, has the same terms as $C(m, n)$ has as far as $n$ is concerned. Thus Batcher’s $(m, n)$-merging network is optimal up to a constant number of comparators, where the constant depends only on $m$. An open problem posed by Yao and Yao (Lower bounds on merging networks, J. Assoc. Comput. Mach. 23, 566–571) is solved: $\lim_{n \to \infty} C(m, n)/n = [\log m/2 + m/2^{\log m}]$. © 2001 Academic Press

Key Words: merging network; odd–even merge; comparator; lower bound; open problem by Yao and Yao.

1. INTRODUCTION

An $(m, n)$-merging network (Knuth, 1998, p. 230) is that which merges $m$ elements $x_1 \leq x_2 \leq \cdots \leq x_m$ with $n$ elements $y_1 \leq y_2 \leq \cdots \leq y_n$ to form the sorted sequence $z_1 \leq z_2 \leq \cdots \leq z_{m+n}$. Batcher (1968) proposed odd–even merge, which gives an $(m, n)$-merging network with the least number of comparators known up to the present. (See Knuth (1998), pp. 223–224.) Let $C(m, n)$ be the number of comparators used in the odd–even merge for merging $m$ and $n$ elements. $C(m, n)$ is given (Knuth, 1998, p. 224) by

$$C(m, n) = \begin{cases} mn & (mn \leq 1) \\ C([m/2], [n/2]) + C([m/2], [n/2]) + [(m + n - 1)/2] & (mn > 1). \end{cases} \tag{1}$$

Let $M(m, n)$ be the minimum number of comparators among $(m, n)$-merging networks; we call such a minimum size network optimal. It is clear by definition that $M(m, n) = M(n, m)$, that $M(1, n) = C(1, n) = n$, and that $M(m, n) \leq C(m, n)$ for $m, n \geq 1$.

Yao and Yao (1976) proved that $M(2, n) = C(2, n) = [3n/2]$. Aigner and Schwarzkopf (1995) have shown that $M(3, n) = C(3, n) = \lfloor (7n+3)/4 \rfloor$, $M(4, n) \geq (11/6)n + 1$, and $M(5, n) \geq 2n + 1$. Miltersen et al. (1996) have analyzed the asymptotic behavior of $M(m, n)$, and have shown that $M(m, n) \geq ((m + n) \log_2(m + 1) - m \log_2 e)/2$ for $1 \leq m \leq n$. Leighton et al. (1995) have provided a different proof for the same lower bound up to a lower order term. Masuda and Iwata (1997) used an exhaustive search by computer to show that $M(4, 6) = 14$, and proved $M(4, 5) = 12$, $M(4, 8) = 17$. Tanno and Iwata (1997) also used a computer to show $M(4, 7) = 16$ and $M(5, 6) = 16$. Yamazaki et al. (2000) proved $M(6, 6) = 17$ by deriving a contradiction under an assumption that $M(6, 6) < 17$.

We will prove in this paper a lower bound theorem for merging networks in Section 2. The proof technique is an extension of that in Yao and Yao (1976). Their result that Batcher’s $(2, n)$-merging networks are optimal can be derived from our theorem. We give some lower bounds that can be obtained directly from the theorem for $n \geq 1$, $M(4, n) = C(4, n)$ for $n \equiv 0, 1, 3, \mod 4$, $M(4, n) \geq C(4, n) - 1$ for $n \equiv 2 \mod 4$; and Batcher’s $(5, 4k+1)$-, $(6, 8k+6)$-, $(7, 8k+7)$-merging networks are optimal for $k \geq 0$.

In Section 3, we focus our attention on some $(m, n)$-merging networks, where $M(m, n) \geq C(m, n) - 1$ can be shown. For some pairs of $(m, n)$, we can prove that $M(m, n) = C(m, n)$. We will show that Batcher’s $(5, 8k)$- and $(8, 8k)$-merging networks are optimal for $k \geq 1$. In Section 4, we will evaluate...
our lower bound on $M(m, n)$ in general. For fixed $m$ and for any $n(m \leq n)$, the lower bound is the same as Batchter’s upper bound as far as $n$ is concerned. Thus Batchter’s $(m, n)$-merging network is optimal up to a constant number of comparators, where the constant depends only on $m$. We also give the exact value of $r_m = \lim_{n \to \infty} M(m, n)/n = \left\lceil \log m \right\rceil / 2 + m/2^{\left\lceil \log m \right\rceil}$, which solves the open problem posed by Yao and Yao (1976).

We call a horizontal line of input element $x_i$ of an $(m, n)$-merging network line $x_i$ (line $y_j$) for $1 \leq i < m$ (for $1 \leq j < n$, respectively). Assume that a comparator $\alpha$ connects line $a$ to line $b$ in the network, where line $a$ is placed upward compared with line $b$. We say that $\alpha$ is an upper endline and $b$ a lower endline of $\alpha$. The symbol $\uparrow a$ denotes one of the lines placed above line $a$ in the network, and $\downarrow a$ denotes one of the lines placed lower than line $a$. We denote a comparator by $[a : b]$, which connects lines $a$ and $b$ in the network, where line $a$ is placed above line $b$. See Fig. 1. If $x$ and $y$ are inputs to a comparator of the figure, then $\min\{x, y\}$ ($\max\{x, y\}$) is the output of the upper (lower, respectively) endline of the comparator. A subnetwork of a given merging network $N$ consists of a set $K$ of adjacent lines in $N$ plus the sets of all comparators that connect two lines in $K$.

The line $x_i$ (y_j) is always placed above the line $x_{i+1}$ (y_{j+1}) for $1 \leq i < m$ (for $1 \leq j < n$, respectively) in an $(m, n)$-merging network. We can place lines $x_1, x_2, \ldots, x_m$ of an $(m, n)$-merging network interspersed within lines $y_1, y_2, \ldots, y_n$; for any two differently interspersed input lines, there is a transformation from an $(m, n)$-merging network with interspersed input lines into a network with another interspersed input lines that preserves the number of comparators used, by an application of Exercise 16 of Knuth (1998, p. 238). For a line $z$ and a comparator $\alpha$ that is connected to line $z$, if there is not any comparator positioned to the left of $\alpha$ as far as line $z$ is concerned, i.e., $\alpha$ is the first comparator for input element $z$ to be compared, then we say that $\alpha$ is the leftmost comparator with respect to line $z$; furthermore if line $z$ is an upper endline (a lower endline) of $\alpha$ then we call $\alpha$ downward leftmost (upward leftmost, respectively) with respect to line $z$. A comparator $[a : b]$ that is downward leftmost with respect to line $a$ may at the same time also be upward leftmost with respect to line $b$. In Fig. 2, which shows a $(4, 8)$-merging network, $[y_2 : y_6]$ is the leftmost comparator with respect to line $y_6$, and it is upward leftmost with respect to the line; $[y_4 : y_8]$ is the downward leftmost comparator with respect to line $y_4$.

We sometimes identify a merging network with the set of comparators contained in the network. In a given merging network, if no exchange occurs by a comparator for every input, then we call

![Fig. 1](image1.png) A comparator.

![Fig. 2](image2.png) A $(4, 8)$-merging network.
the comparator redundant. Obviously, every optimal merging network does not contain any redundant comparator. Suppose that there is a line in a merging network with which two comparators \( \alpha \) and \( \beta \) are connected. If \( \alpha \) is positioned nearer than \( \beta \) to the input side then we write \( \alpha < \beta \). For sets \( A \) and \( B \), we denote the cardinality of \( A \) by \( |A| \), and the set \((A - B) \cup (B - A)\) by \( A \oplus B \). Logarithms are assumed to the base two throughout the paper.

2. LOWER BOUND THEOREM

Let \( m_1, m_2 \geq 1 \), and \( n \geq 2 \). Consider an optimal \((m_1 + m_2, n)\)-merging network \( N \), which consists of input lines \( x_1, x_2, \ldots, x_{m_1}, y_1, y_2, \ldots, y_n, x_{m_1+1}, \ldots, x_{m_1+m_2} \), such that \( x_1 \) is the top line, \( x_2 \) is the next to the top, then \( x_3, \ldots, x_{m_1}, y_1, y_2, \ldots, y_n, x_{m_1+1}, \ldots, x_{m_1+m_2-1} \), and \( x_{m_1+m_2} \) is the bottom line. Consider a subnetwork \( N_A \) of \( N \) consisting of lines \( x_1, x_2, \ldots, x_{m_1}, y_1 \). We note that \( N_A \) is an \((m_1, 1)\)-merging network, and that \( |N_A| \geq m_1 \). Similarly, a subnetwork \( N_B \) consisting of lines \( y_n, x_{m_1+1}, \ldots, x_{m_1+m_2} \) forms an \((1, m_2)\)-merging network, and \( |N_B| \geq m_2 \). Let \( N_1 \) \((N_2)\) be the set of upward \((downward, respectively\) leftmost comparators with respect to line \( y_i \) for some \( i \) \((2 \leq i \leq n - 1)\). We can assume that \( N_1 \) and \( N_2 \) are disjoint, because otherwise \( N \) contains a redundant comparator. Thus \( |N_1| + |N_2| = n - 2 \).

**Lemma 2.1.** Assume that input elements are supplied to \( N \) such that \( y_n < x_{m_1+1} \). The value on line \( y_i \) will always be greater than or equal to the input element \( y_j \) for all \( i \leq j \leq n \).

**Proof.** We can show by induction on \( j \) \((0 < j < n)\) that on this input the values that come on the lines below line \( y_{n-j} \) are greater than or equal to the input element \( y_{n-j} \). The proof of the induction is straightforward and is omitted. Thus only values that are greater than the element \( y_j \) may come to line \( y_i \) by comparators connected to the line. Hence, the lemma is proved.

**Lemma 2.2.** Let \( A \) be a network after deletion of the comparators from \( N \) that belong to \( N_B \) and to \( N_D \). Then \( |A| \geq M(m_1, n) \). Similarly, if \( B \) is a network constructed from \( N \) after deletion of the comparators of \( N_A \) and of \( N_U \), then \( |B| \geq M(m_2, n) \).

**Proof.** Let us consider the behavior of \( N \) under an input such that \( y_n < x_{m_1+1} \). \( N \) behaves like an \((m_1, n)\)-merging network, which consists of lines \( x_1, x_2, \ldots, x_{m_1}, y_1, y_2, \ldots, y_n \). Since the lower \( m_2 \) input elements of \( N \) do not move from their lines, and since exchange of elements never occurs by a comparator of \( N_B \), if we delete from \( N \) the comparators of \( N_B \) then the resultant network still acts as an \((m_1, n)\)-merging network. Let \( \alpha \) be a downward leftmost comparator with respect to line \( y_i \) for any \( 2 \leq i \leq n - 1 \). If the lower endline of \( \alpha \) is line \( x_j, m_1 + 1 \leq j \leq m_1 + m_2 \), then exchange of elements never occurs by \( \alpha \). If the lower endline of \( \alpha \) is line \( y_j \) for some \( j \leq n \), then by Lemma 2.1, the value on line \( y_i \) immediately before \( \alpha \) is greater than or equal to the input element \( y_j \). Thus exchange of elements does not occur by \( \alpha \). So deletion of \( \alpha \) from \( N \) does not prevent \( N \) from acting as an \((m_1, n)\)-merging network. After deletion of the comparators of \( N_B \) and of \( N_D \) from \( N \), the resultant network \( A \) behaves like an \((m_1, n)\)-merging network. Therefore \(|A| \geq M(m_1, n)\).

A similar argument holds for the network \( B \), and we can obtain that \(|B| \geq M(m_2, n)\).

By Lemma 2.2, \(|A| = |N| - (|N_B| + |N_D|) \geq M(m_1, n)\), and \(|B| = |N| - (|N_A| + |N_U|) \geq M(m_2, n)\).

Recall that \(|A \oplus B| = |N_A| + |N_B| + |N_U| + |N_D| \geq m_1 + m_2 + n - 2\). Thus by these inequalities we obtain

\[
2|N| \geq 2|A \cup B| \\
= |A| + |B| + |A \oplus B| \\
\geq M(m_1, n) + M(m_2, n) + (m_1 + m_2 + n - 2).
\]

Since \(|N|\) is an integer, we obtain the following lemma.
Lemma 2.3. For $m_1, m_2 \geq 1$ and $n \geq 2$,

$$M(m_1 + m_2, n) \geq [(M(m_1, n) + M(m_2, n) + m_1 + m_2 + n - 2)/2].$$

In the case of $n = 1$, both sides of the inequality of the above lemma are equal to $m_1 + m_2$. Thus we have proved our main theorem.

Theorem 2.1. For $m_1, m_2 \geq 1$ and $n \geq 1$,

$$M(m_1 + m_2, n) \geq [(M(m_1, n) + M(m_2, n) + m_1 + m_2 + n - 2)/2].$$

By setting $m_1 = \lfloor m/2 \rfloor$ and $m_2 = \lceil m/2 \rceil$, we obtain a simpler form of our main theorem.

Corollary 2.1. For $m \geq 2$ and $n \geq 1$,

$$M(m, n) \geq [(M(\lfloor m/2 \rfloor, n) + M(\lceil m/2 \rceil, n) + m + n - 2)/2].$$

In what follows, we apply Corollary 2.1 to show that some Batcher’s merging networks are optimal.

Corollary 2.2 (Yao and Yao, 1976). For $n \geq 1$, $M(2, n) = \lceil 3n/2 \rceil$.

Proof. Since $M(1, n) = n$, we obtain by Corollary 2.1 that

$$M(2, n) \geq [(M(1, n) + M(1, n) + n)/2] = \lceil 3n/2 \rceil = C(2, n).$$

Aigner and Schwarzkopf (1995) showed that Batcher’s $(3, n)$-merging networks are optimal. We can prove a part of their result.

Corollary 2.3 (Aigner and Schwarzkopf, 1995). For $n \geq 1$,

$$M(3, n) = C(3, n) \quad (n \equiv 0, 1, 3 \mod 4), \text{ and } C(3, n) - 1 \leq M(3, n) \leq C(3, n) \quad (n \equiv 2 \mod 4).$$

Proof. Since $M(1, n) = n$, and $M(2, n) = \lceil 3n/2 \rceil$ by Corollary 2.2,

$$M(3, n) \geq [(M(1, n) + M(2, n) + n + 1)/2] = \lceil (7n + 2)/4 \rceil.$$ By Eq. (1), $C(3, n) = \lceil (7n + 3)/4 \rceil$. Thus $M(3, n) = C(3, n)$ for $n \equiv 0, 1, 3 \mod 4$, and $C(3, n) - 1 \leq M(3, n) \leq C(3, n)$ for $n \equiv 2 \mod 4$. ■

The next theorem shows that Batcher’s $(4, n)$-merging networks are “almost” optimal.

Theorem 2.2. For $n \geq 1$,

$$M(4, n) = C(4, n) \quad (n \equiv 0, 1, 3 \mod 4), \text{ and } C(4, n) - 1 \leq M(4, n) \leq C(4, n) \quad (n \equiv 2 \mod 4).$$

Proof. Since $M(2, n) = \lceil 3n/2 \rceil$,

$$M(4, n) \geq [(M(2, n) + M(2, n) + n + 2)/2] = \lceil 3n/2 \rceil + \lceil n/2 \rceil + 1 = \begin{cases} 2n + 1 & \text{(for even } n) \\ 2n + 2 & \text{(for odd } n) \end{cases}.$$
By Eq. (1), we have \( C(4, n) = 2n + 1 \) for \( n \equiv 0 \) mod 4, and \( C(4, n) = 2n + 2 \) for \( n \equiv 1, 2, 3 \) mod 4. Hence the theorem is proved. 

**Theorem 2.3.**  For any \( k \geq 0 \), \( M(5, 4k + 1) = C(5, 4k + 1) \).

**Proof.** We obtain that \( M(5, 4k + 1) = \lceil (17k + 9)/2 \rceil \) by Corollary 2.1, and \( M(5, 4k + 1) = \lceil (17k + 9)/2 \rceil \) by Eq. (1). 

By similar calculations, we obtain the following theorem.

**Theorem 2.4.**  For any \( k \geq 0 \),

\[
M(6, 8k + 6) = C(6, 8k + 6), \quad \text{and} \\
M(7, 8k + 7) = C(7, 8k + 7).
\]

3. APPLICATION

**Lemma 3.1.** Any optimal \((2, 2k)\)-merging network consisting of lines \( x_1, x_2, y_1, y_2, \ldots, y_{2k} \) does not contain a comparator of the form \([y_{2k-1} : y_{2k}]\) for \( k \geq 1 \).

**Proof.** Assume that \( N \) is an optimal \((2, 2k)\)-merging network consisting of lines \( x_1, x_2, y_1, y_2, \ldots, y_{2k} \), and that \( N \) contains \( \alpha = [y_{2k-1} : y_{2k}] \). Since \( M(2, 2k) = 3k \) and \( M(2, 2k - 1) = 3k - 1 \) by Corollary 2.2, \( \alpha \) is the only one comparator that connects to line \( y_{2k} \). Suppose an input that satisfies \( y_{2k} < x_1 \) is given to \( N \). Since the element \( x_2 \) comes down to line \( y_{2k} \), there is a comparator \( \beta = [\uparrow y_{2k-1} : y_{2k-1}] < \alpha \). There is also a comparator \( \gamma = [\uparrow y_{2k-1} : y_{2k-1}] \neq \beta, \alpha < \gamma \), since the element \( x_1 \) comes down to line \( y_{2k-1} \). See Fig. 3.

Now consider a subnetwork \( \hat{N} \) of \( N \) consisting of lines \( x_1, x_2, y_1, y_2, \ldots, y_{2k-1} \). We note that \( \hat{N} \) is an optimal \((2, 2k-1)\)-merging network, since \( |\hat{N}| = 3k - 1 \). On input satisfying \( y_{2k-1} < x_2 \), the element \( x_2 \) moves down to line \( y_{2k-1} \) by the comparator \( \beta \). Thus in this case no element is exchanged by \( \gamma \). On input satisfying \( x_2 \leq y_{2k-1} \), no element is exchanged by \( \gamma \) either. Thus the optimal merging network \( \hat{N} \) contains a redundant comparator \( \gamma \)—a contradiction. 

**Lemma 3.2.**  Every optimal \((4, 8k)\)-merging network consisting of lines \( x_1, x_2, x_3, x_4, y_1, y_2, \ldots, y_{8k} \) does not contain a comparator of the form \([y_{8k-1} : y_{8k}]\) for \( k \geq 1 \).

**Proof.** Since \( M(4, 8k) = 16k + 1 \) and \( M(4, 8k - 1) = 16k \) by Theorem 2.2, the lemma can be proved similarly as the previous lemma. 

**Lemma 3.3.**  For \( k \geq 1 \), \( M(5, 8k) = C(5, 8k) \).

**Proof.** By Eq. (1), \( C(5, 8k) = 17k + 3 \), while Corollary 2.1 implies that \( M(5, 8k) \geq 17k + 2 \). For a contradictory discussion, we assume that \( M(5, 8k) = C(5, 8k) - 1 = 17k + 2 \). Let \( N \) be an optimal \((5, 8k)\)-merging network consisting of lines \( x_1, x_2, y_1, y_2, \ldots, y_{8k}, x_3, x_4, x_5 \), the line \( x_1 \) be the top, and the line \( y_5 \) be the bottom. See Fig. 4.

Assume that \( N_A \) (or \( N_B \)) is a subnetwork of \( N \) consisting of the top three lines \( x_1, x_2, y_1 \) (the bottom four lines \( y_{8k}, x_3, x_4, x_5 \), respectively). Suppose that \( N_U \) (or \( N_D \)) is the set of upward (downward, respectively) leftmost comparators with respect to line \( y_i \) for \( i \geq 2 \leq 8 \). If \( A(B) \) is a network after deletion

![FIG. 3. An optimal (2, 2k)-merging network N.](image-url)
of the comparators both in $N_B$ and in $N_D$ (both in $N_A$ and in $N_U$) from $N$, then $A$ ($B$) behaves like a $(2, 8k)$-merging network ($3, 8k$)-merging network, and $|A| \geq M(2, 8k)$ ($|B| \geq M(3, 8k)$, respectively). The set $A \oplus B$ contains at least $N_A \cup N_B \cup N_U \cup N_D$. Then by a similar discussion made in the proof of Lemma 2.3,

$$2|N| \geq 2|A \cup B| = |A| + |B| + |A \oplus B| \\
\geq M(2, 8k) + M(3, 8k) + (2 + 3 + 8k - 2) \\
= 34k + 4 \\
= 2M(5, 8k).$$

Since $N$ is an optimal $(5, 8k)$-merging network, $|N| = M(5, 8k)$. Thus $|A| = M(2, 8k)$, $|B| = M(3, 8k)$, and $|A \oplus B| = 8k + 3$, that is, $|N_A| = 2$, $|N_B| = 3$. It implies that every comparator of $N$ belongs to exactly one of $N_A$, $N_B$, $N_U$, $N_D$, or of $A \cap B$.

Now consider a subnetwork $N_B'$ of $N$ consisting of the bottom five lines $y_{8k-1}, y_{8k}, x_3, x_4, x_5$. $N_B'$ forms $(2, 3)$-merging network, and $|N_B'| \geq M(2, 3) = 5$. $|N_B| = 3$ and $|N_B'| \geq 5$ imply that there are at least two comparators in $N_B'$, whose upper endlines are $y_{8k-1}$. By Lemma 3.1, $A$ does not contain a comparator of the form $[y_{8k-1} : y_{8k}]$. One possible comparator in $N_B'$ of the form $[y_{8k-1} \downarrow y_{8k-1}]$ may be a downward leftmost one with respect to line $y_{8k-1}$. Since every comparator of $N$ is in $N_A$, $N_B$, $A \cap B$, or in one of the leftmost ones with respect to $y_i$, $2 \leq i \leq 8k - 1$, there are at most four comparators in $N_B'$—a contradiction. ■

Combining the above lemma with Theorem 2.3, we derive the following theorem.

**Theorem 3.1.** For $n \geq 1$, $n \equiv 0, 1, 5 \mod 8$, $M(5, n) = C(5, n)$.

Tarui (1999) gave the author a suggestion to prove the next result.

**Theorem 3.2.** For $k \geq 1$, $M(8, 8k) = C(8, 8k)$.

**Proof.** We can obtain $20k + 4 \leq M(8, 8k) \leq 20k + 5 = C(8, 8k)$ by Corollary 2.1 and by Eq. (1). For a contradictory discussion, we assume that $M(8, 8k) = C(8, 8k) - 1 = 20k + 4$.

Let $N$ be an optimal $(8, 8k)$-merging network consisting of lines $x_1, x_2, x_3, x_4, y_1, y_2, \ldots, y_{8k}, x_5, x_6, x_7, x_8$. Now let us delete from $N$ the downward leftmost comparators with respect to line $y_i$ for $i$ ($2 \leq i \leq 8k - 1$), and the comparators contained in the bottom five lines. We call the resultant network $A$. We can show that $A$ is an optimal $(4, 8k)$-merging network, and that there are exactly four comparators in the bottom five lines $y_{8k}, x_5, x_6, x_7, x_8$ of $N$ by the same argument described in the proof of Lemma 3.3. By Lemma 3.2.2, $A$ does not contain a comparator of the form $[y_{8k-1} : y_{8k}]$. A similar discussion made in the proof of Lemma 3.3 will lead us to a contradiction that a subnetwork consisting of the bottom six lines of $N$ contains at most five comparators, where the subnetwork should contain at least $M(2, 4) = 6$ comparators. ■
Lemma.

Definition of the function $L$

Let us define for $m$ general.

Case 1. $m = 2k$ (so $k \geq 1$, $\lceil \log m \rceil = \lceil \log k \rceil + 1$). By the inductive hypothesis and by definition,

$$L(m, n + 2^\lceil \log m \rceil) = L(k, n + 2^\lceil \log k \rceil + 1) + k + \lceil n/2 \rceil + 2^\lceil \log k \rceil - 1$$

$$= L(k, n) + 2(k + \lceil \log k \rceil 2^\lceil \log k \rceil - 1) + k + \lceil n/2 \rceil - 1 + 2^\lceil \log k \rceil$$

$$= L(k, n) + k + \lceil n/2 \rceil - 1 + 2k + (1 + \lceil \log k \rceil)2^\lceil \log k \rceil$$

$$= (2L(k, n) + 2k + n - 2)/2 + 2k + \lceil \log(2k) \rceil 2^\lceil \log(2k) \rceil - 1$$

$$= L(m, n) + m + \lceil \log m \rceil 2^\lceil \log m \rceil - 1.$$  

Case 2. $m = 2k + 1$ (so $k \geq 1$, $\lceil \log(2k + 1) \rceil = \lceil \log(k + 1) \rceil + 1$).

$$L(m, n + 2^\lceil \log m \rceil) = \left\lceil (L(k, n + 2^\lceil \log(k + 1) \rceil + 1) + L(k + 1, n + 2^\lceil \log(k + 1) \rceil + 1) + (2k + 1) + n + 2^\lceil \log(k + 1) \rceil + 1 - 2)/2 \right\rceil.$$  


Table 1 gives the best lower bounds and upper bounds on $M(m, n)$ with $m, n \leq 10$.

### 4. Behavior of $M(m, n)$

We discuss in this section a lower bound of the function $M(m, n)$. To analyze the behavior of $M(m, n)$, let us define for $n \geq 1$

$$L(m, n) = \begin{cases} n & (m = 1) \\ \left\lceil (L\lfloor m/2 \rfloor, n) + L\lfloor m/2 \rfloor, n + m + n - 2)/2 \right\rceil & (m \geq 2). \end{cases}$$

We note that $M(m, n) \geq L(m, n)$ for $m, n \geq 1$ by Corollary 2.1. Also note that $L(m, n) \neq L(n, m)$ in general.

**Lemma 4.1.** For $m, n \geq 1$, $L(m, n + 2^\lceil \log m \rceil) = L(m, n) + m + \lceil \log m \rceil 2^\lceil \log m \rceil - 1$.

**Proof.** We prove the lemma by induction on $m \geq 1$. Let $P(m)$ denote the statement of the lemma.

**Basis.** If $m = 1$ the lemma holds, since both sides of the equation are equal to $n + 1$. Thus $P(1)$.

**Induction step.** For $m \geq 2$, we assume $P(1)$ through $P(m - 1)$, and we will prove $P(m)$. By definition of the function $L$,

$$L(m, n + 2^\lceil \log m \rceil) = \left\lceil (L\lfloor m/2 \rfloor, n + 2^\lceil \log m \rceil) + L\lfloor m/2 \rfloor, n + 2^\lceil \log m \rceil) + m + n + 2^\lceil \log m \rceil - 2)/2 \right\rceil.$$  

**Case 1.** $m = 2k$ (so $k \geq 1$, $\lceil \log m \rceil = \lceil \log k \rceil + 1$). By the inductive hypothesis and by definition,

$$L(m, n + 2^\lceil \log m \rceil) = L(k, n + 2^\lceil \log k \rceil + 1) + k + \lceil n/2 \rceil + 2^\lceil \log k \rceil - 1$$

$$= L(k, n) + 2(k + \lceil \log k \rceil 2^\lceil \log k \rceil - 1) + k + \lceil n/2 \rceil - 1 + 2^\lceil \log k \rceil$$

$$= L(k, n) + k + \lceil n/2 \rceil - 1 + 2k + (1 + \lceil \log k \rceil)2^\lceil \log k \rceil$$

$$= (2L(k, n) + 2k + n - 2)/2 + 2k + \lceil \log(2k) \rceil 2^\lceil \log(2k) \rceil - 1$$

$$= L(m, n) + m + \lceil \log m \rceil 2^\lceil \log m \rceil - 1.$$  

**Case 2.** $m = 2k + 1$ (so $k \geq 1$, $\lceil \log(2k + 1) \rceil = \lceil \log(k + 1) \rceil + 1$).

$$L(m, n + 2^\lceil \log m \rceil) = \left\lceil (L(k, n + 2^\lceil \log(k + 1) \rceil + 1) + L(k + 1, n + 2^\lceil \log(k + 1) \rceil + 1) + (2k + 1) + n + 2^\lceil \log(k + 1) \rceil + 1 - 2)/2 \right\rceil.$$
Subcase 2-1. \( k \) is an integral power of 2, i.e., \( k = 1, 2, 4, 8, \ldots \) (so \( \lfloor \log(k+1) \rfloor = \lfloor \log k \rfloor + 1, k = 2^{\lfloor \log k \rfloor} \)):

\[
L(m, n + 2^{\lfloor \log m \rfloor}) = \left[ (L(k, n) + 4(k + \lfloor \log k \rfloor)2^{\lfloor \log k \rfloor-1}) + L(k + 1, n) + 2(k + 1) + \lfloor \log(k+1) \rfloor 2^{\lfloor \log(k+1) \rfloor-1}) + (2k + 1) + n - 2 + 2^{\lfloor \log k \rfloor+1})/2 \right] \\
= \left[ ((L(k, n) + L(k + 1, n) + (2k + 1) + n)/2) + (2k + 1) + (2\lfloor \log k \rfloor + 3)2^{\lfloor \log k \rfloor} + k \right] \\
= L(2k + 1, n) + (2k + 1) + (\lfloor \log k \rfloor + 2)2^{\lfloor \log k \rfloor+1} \\\n= L(m, n) + m + \lfloor \log m \rfloor 2^{\lfloor \log m \rfloor-1}.
\]

Subcase 2-2. \( k \) is not an integral power of 2 (so \( \lfloor \log(k+1) \rfloor = \lfloor \log k \rfloor \)):

\[
L(m, n + 2^{\lfloor \log m \rfloor}) = \left[ (L(k, n) + 4(k + \lfloor \log k \rfloor)2^{\lfloor \log k \rfloor-1}) + L(k + 1, n) + 2(k + 1) + \lfloor \log(k+1) \rfloor 2^{\lfloor \log(k+1) \rfloor-1}) + (2k + 1) + n - 2 + 2^{\lfloor \log k \rfloor+1})/2 \right] \\
= \left[ ((L(k, n) + L(k + 1, n) + (2k + 1) + n)/2) + (2k + 1) + (\lfloor \log k \rfloor)2^{\lfloor \log k \rfloor-1} + \lfloor \log(k+1) \rfloor 2^{\lfloor \log k \rfloor-1} + 2^{\lfloor \log k \rfloor} \right] \\
= L(2k + 1, n) + (2k + 1) + (\lfloor \log k \rfloor + 1)2^{\lfloor \log k \rfloor} \\\n= L(m, n) + m + \lfloor \log m \rfloor 2^{\lfloor \log m \rfloor-1}.
\]

We have proved \( P(m) \). Therefore the lemma is proved.

By repeated applications of Lemma 4.1, we obtain the next corollary. The corollary gives our lower bound on \( M(m, n) \) for general \( m, n \).

**Corollary 4.1.** For \( m, n \geq 1 \),

\[
M(m, n) \geq L(m, n) = L(m, n - 2^{\lfloor \log m \rfloor} \lfloor n/2^{\lfloor \log m \rfloor} \rfloor - 1) \\
+ (m + \lfloor \log m \rfloor 2^{\lfloor \log m \rfloor-1}) \lfloor n/2^{\lfloor \log m \rfloor} \rfloor - 1).
\]

Assume that \( m \leq n \) hereafter in the paper. By Knuth (1998, p. 225), \( C(m, n + 2^{\lfloor \log m \rfloor}) = C(m, n) + m + \lfloor \log m \rfloor 2^{\lfloor \log m \rfloor-1} \). Now let \( m' = n - 2^{\lfloor \log m \rfloor} \lfloor (n - m)/2^{\lfloor \log m \rfloor} \rfloor \). Note that \( m \leq m' < m + 2^{\lfloor \log m \rfloor} \). Then we can derive that

\[
C(m, n) = C(m, m') + (m + \lfloor \log m \rfloor 2^{\lfloor \log m \rfloor-1}) \lfloor (n - m)/2^{\lfloor \log m \rfloor} \rfloor.\tag{2}
\]

Meanwhile by Lemma 4.1, we obtain that

\[
L(m, n) = L(m, m') + (m + \lfloor \log m \rfloor 2^{\lfloor \log m \rfloor-1}) \lfloor (n - m)/2^{\lfloor \log m \rfloor} \rfloor.\tag{3}
\]

**Theorem 4.1.** For \( m \leq n \),

\[
C(m, n) - M(m, n) \leq C(m, n) - L(m, n) = C(m, m') - L(m, m').
\]

Theorem 4.1 implies that Batchter’s \((m, n)\)-merging network is optimal excluding at most \( C(m, m') - L(m, m') \) comparators, where the number of comparators is \( O(1) \) with respect to \( n \).

Yao and Yao (1976) posed the problem of determining the value of \( \rho_m = \lim_{n \to \infty} M(m, n)/n \) and showed that for each \( m \), the limit exists and

\[
\frac{\log(m + 1)}{2} \leq \rho_m \leq \frac{\lfloor \log m \rfloor}{2} + \frac{m}{2^{\lfloor \log m \rfloor}} = \lim_{n \to \infty} \frac{C(m, n)}{n}.
\]
The last equality is by Eq. (2). Miltersen et al. (1996) improved the lower bound

\[ r_m \geq \frac{1}{2} \left( \log(m + 1) + (1 + \theta - 2^\theta) \right), \quad \theta = \lceil \log(m + 1) \rceil - \log(m + 1). \]

Now we can give the exact real number of \( r_m \) for each \( m \geq 1 \) to solve this open problem.

**Theorem 4.2.** For \( m \geq 1 \),

\[ r_m = \frac{\lceil \log m \rceil}{2} + \frac{m}{2^{\lceil \log m \rceil}}. \]

**Proof.** Since \( L(m, n) \leq M(m, n) \leq C(m, n) \),

\[ \lim_{n \to \infty} \frac{L(m, n)}{n} \leq r_m \leq \lim_{n \to \infty} \frac{C(m, n)}{n}. \]

Both \( \lim_{n \to \infty} C(m, n)/n \) and \( \lim_{n \to \infty} L(m, n)/n \) are equal to \( \lceil \log m \rceil / 2 + m / 2^{\lceil \log m \rceil} \) by Eqs. (2) and (3) respectively. Hence the theorem is proved. 

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**References**

Tarui, J. (1999), Personal communication.