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Admissible prediction in superpopulation models with random regression coefficients under matrix loss function ☆

Li-Wen Xu [a,*], Sheng-Hua Yu [b]

[a] College of Sciences, North China University of Technology, Beijing 100144, PR China

[b] College of Economics and Trade, Hunan University, Changsha 410079, PR China

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ABSTRACT

Admissible prediction problems in finite populations with arbitrary rank under matrix loss function are investigated. For the general random effects linear model, we obtained the necessary and sufficient conditions for a linear predictor of the linearly predictable variable to be admissible in the two classes of homogeneous linear predictors and all linear predictors and the class that contains all predictors, respectively. Moreover, we prove that the best linear unbiased predictors (**BLUP**s) of the population total and the finite population regression coefficient are admissible under different assumptions of superpopulation models respectively.

1. Introduction

For convenience, the following notations will be used in this paper. For a matrix $\boldsymbol{A}$, $\mathscr{M}(\boldsymbol{A}), \boldsymbol{A}', \boldsymbol{A}^-$ and $\boldsymbol{A}^+$ denote the column space, the transpose, any generalized inverse and the Moore–Penrose inverse, respectively, of $\boldsymbol{A}$, $\boldsymbol{A} > 0$ ($\boldsymbol{A} \geqslant 0$) means that $\boldsymbol{A}$ is a symmetric positive definite matrix (nonnegative definite matrix), $\boldsymbol{A} \geqslant \boldsymbol{B}$ ($\boldsymbol{A} \leqslant \boldsymbol{B}$) means that $\boldsymbol{A} - \boldsymbol{B} \geqslant 0$ ($\boldsymbol{B} - \boldsymbol{A} \geqslant 0$), the symbol, $\triangleq$, is used for 'defined as', $\boldsymbol{I}$ is an identity matrix with an appropriate order.

Let $\mathscr{P} = \{1, \ldots, N\}$ be the set of labels of the units of a finite population of size N, where N is known. Associated with the ith unit of $\mathscr{P}$, there are $p+1$ quantities: $y_i, x_{i1}, \ldots, x_{ip}$, where all but y_i are known, $i = 1, \ldots, N$. Denote $\boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N)'$, and $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N)'$, where $\boldsymbol{X}_i = (x_{i1}, \ldots, x_{ip})'$, $i = 1, \ldots, N$. We express the relationships among the variables by the linear model with stochastic regression coefficients

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}, \tag{1.1}$$

where $\boldsymbol{\beta}$ and $\boldsymbol{e}$ are $p \times 1$ and $N \times 1$ unobservable random vectors, respectively, with $E(\boldsymbol{\beta}) = \boldsymbol{A\alpha}$, $\mathrm{Cov}(\boldsymbol{\beta}) = \boldsymbol{U}$, $E(\boldsymbol{e}) = 0$, $\mathrm{Cov}(\boldsymbol{e}) = \boldsymbol{V}$, $E(\boldsymbol{\beta}\boldsymbol{e}') = \boldsymbol{W}$ and $E(\boldsymbol{e}\boldsymbol{\beta}') = \boldsymbol{W}'$, $\boldsymbol{X}$ and $\boldsymbol{A}$ are known $N \times p$ and $p \times k$ matrices respectively,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{U} & \boldsymbol{W} \\ \boldsymbol{W}' & \boldsymbol{V} \end{pmatrix} \geqslant \boldsymbol{0}$$

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* Corresponding author.
E-mail addresses: xulw2000@yahoo.com.cn (L.-W. Xu), yshh1966@163.com (S.-H. Yu).

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is a known $(N+p) \times (N+p)$ matrix, and α is a $k \times 1$ unknown superparameter vector. This model is usually called *superpopulation model* (cf. [6]). If $(\boldsymbol{\beta}', \boldsymbol{e}')'$ is a random vector of multivariate normal distribution, the model (1.1) will be written as

$$\boldsymbol{y} = X\boldsymbol{\beta} + \boldsymbol{e}, \quad (\boldsymbol{\beta}', \boldsymbol{e}')' \sim \mathscr{N}((\boldsymbol{\alpha}'\boldsymbol{A}', \boldsymbol{0}')', \boldsymbol{\Sigma}). \tag{1.2}$$

Since $\mathrm{Cov}(\boldsymbol{\beta}) = \boldsymbol{U} > 0$ is not necessary for the model (1.1) (or (1.2)), the above formulation includes, but is not limited to, the fixed effects, mixed effects and random models.

Considerable attention has been given in the past to the problem of making inferences from a sample about certain population quantity using the superpopulation approach (e.g., cf. [2–6,27,30–35,37,38]) to survey sampling. Under this perspective, according to conditionality principle (cf. [1]), the sampling plan is not relevant to the inference. Let $\theta(\boldsymbol{y})$ be some population quantity of interest which we want to predict in practice. Examples of such quantities are the population linear function $\boldsymbol{Qy}$, where $\boldsymbol{Q}$ is an $h \times N$ known matrix, and the population quadratic quantities like $\boldsymbol{y}'\boldsymbol{Hy}$, where $\boldsymbol{H} \geqslant 0$ is an $N \times N$ known matrix satisfying $\boldsymbol{HX} = 0$. Denote *the population total* $\boldsymbol{T_N} = \boldsymbol{1}_N'\boldsymbol{y} = \sum_{i=1}^{N} \boldsymbol{y_i}$, where $\boldsymbol{1}_N = (1, \ldots, 1)'$, and *the finite population regression coefficient*, $\boldsymbol{\beta}_N = (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}$, where $\boldsymbol{X}$ and $\boldsymbol{V}$ are of full column rank, and $\boldsymbol{\beta}$ is a fixed effect vector. Optimal prediction of $\boldsymbol{T}_N$ has been considered by Royall [17], Royall and Herson [18], and Pereira and Rodrigues [15]. All of these papers studied the *best linear unbiased predictor* (**BLUP**) of $\boldsymbol{T}_N$. Wang [20] considered the adaptive ridge-type predictors of $\boldsymbol{l}'\boldsymbol{y}$ in the presence of multicollinearity among the columns of $\boldsymbol{X}$, where $\boldsymbol{l}$ is an $N \times 1$ known vector. Bolfarine et al. [5] investigated the prediction of $\boldsymbol{T}_N$ under regression superpopulation model when explanatory variable vector is measured with error. The Bayes, minimax and best unbiased prediction of $\boldsymbol{\beta}_N$ has also been studied (see, e.g., [2–4]). For finite populations with arbitrary rank, under the quadratic loss function, the unique linear minimax predictor of the linear predictable variable $\boldsymbol{Qy}$ is obtained in the class of homogeneous linear predictors by Yu [32]. Xu and Wang [28] proved that this linear minimax predictor is also the unique minimax predictor of the linear predictable variable $\boldsymbol{Qy}$ in the class of all predictors under normality. Liu and Rong [13] studied the problem of quadratic prediction for the quadratic quantity $\boldsymbol{y}'\boldsymbol{Hy}$ in a general linear model. Liu and Rong [12] extended the problem of quadratic prediction from a general linear model to a multivariate general linear model.

Our objective is to study admissibility of linear predictors of a linearly predictable variable $\boldsymbol{Qy}$ in models (1.1) and (1.2). Similar problem has been received much attention in the theory of admissible estimator. Olsen et al. [14] provide seminal results in the characterization of admissible linear estimators in the general linear model. They described necessary conditions for the admissibility of unbiased linear estimators and showed that the admissible unbiased linear estimators form a minimal complete class of unbiased linear estimators. Their necessary conditions are demonstrably not sufficient. LaMotte [10] noted an extension of their characterization. Without the restriction to unbiasedness, Cohen [9] characterized admissible linear estimators of the mean vector while assuming a covariance matrix of the form $\boldsymbol{\sigma}^2\boldsymbol{I}$. Rao [16] accomplished the same characterization for models with mean vectors varying through a linear subspace and covariance matrix of the form $\boldsymbol{\sigma}^2\boldsymbol{V}$ with $\boldsymbol{V}$ known. Neither of these efforts appears to generalize to models in which the covariance matrix varies over more than one dimension. LaMotte [11] characterized admissible linear estimators while allowing for relations between elements of the mean vector and covariance matrix, and allowing the covariance matrix to vary in an arbitrary subset of nonnegative definite symmetric matrices. All these efforts but Rao [16] appear to characterize admissible linear estimators under the quadratic loss function. Rao [16] observed that an admissible linear estimator under the quadratic loss function is also an admissible linear estimator under the matrix loss function. However, the problem of admissible linear estimators under the matrix loss function has not been solved completely. Wu [22] provided the characterization of admissible linear estimators under the matrix loss function. Some other important references on the subject are Zontek [36], Stepniak [19], Chen et al. [7], Wu [23–25], Wu and Chen [26], and Chen and Zhan [8].

In the present paper, we present an efficient way to study the admissibility of an linear predictor. Our idea consists in an appropriate representation of the risk matrix function. In this way we reduce the general problem of the admissibility of predictors to an admissible estimation problem which can be solved by some known results. In consequence admissibility of linear predictors in superpopulation models (1.1) and (1.2) and the different predictor classes under the matrix loss function is characterized.

The left of the paper is organized as follows. In the following parts of Section 1, some notions and lemma are provided. In Section 2, under the model (1.1) admissibility of a linear predictor in two classes of homogeneous linear predictors and all linear predictors are discussed and the corresponding necessary and sufficient conditions for a linear predictor to be admissible in the two classes of linear predictors are derived. In Section 3, under the model (1.2) a problem of admissibility of a linear predictor in the class of all predictors is mentioned and the corresponding necessary and sufficient conditions for a linear predictor to be admissible in the class of all predictors are obtained. Then, in Section 4, we apply our main results to several superpopulation models with fixed effects vector. Moreover, we prove that the **BLUP**s of $\boldsymbol{Qy}$, $\boldsymbol{T}_N$ and $\boldsymbol{\beta}_N$ are admissible in the above predictor classes under different assumptions of superpopulation models respectively.

In order to predict $\boldsymbol{Qy}$, we select a sample $\boldsymbol{s}$ of size $\boldsymbol{s}$ from $\mathscr{P}$ according to some specified sampling plan. Let $\boldsymbol{r} = \mathscr{P} - \boldsymbol{s}$ be the unobserved part of $\mathscr{P}$ of size $\boldsymbol{r}$. After the sample $\boldsymbol{s}$ has been selected, we may reorder the elements of $\boldsymbol{y}$ such that we have the corresponding partitions of $\boldsymbol{y}$, $\boldsymbol{X}$, $\boldsymbol{V}$ and $\boldsymbol{W}$, that is:

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y_s} \\ \boldsymbol{y_r} \end{pmatrix}, \quad \boldsymbol{X} = \begin{pmatrix} \boldsymbol{X_s} \\ \boldsymbol{X_r} \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} \boldsymbol{V_s} & \boldsymbol{V_{sr}} \\ \boldsymbol{V_{rs}} & \boldsymbol{V_r} \end{pmatrix}, \quad \text{and} \quad \boldsymbol{W} = \left(\boldsymbol{W_{ps}} \vdots \boldsymbol{W_{pr}} \right).$$

Partitioning $\boldsymbol{Q}$ into $\boldsymbol{Q} = (\boldsymbol{Q_s} \vdots \boldsymbol{Q_r})$, we may write $\boldsymbol{Qy} = \boldsymbol{Q_s y_s} + \boldsymbol{Q_r y_r}$. For the models (1.1) and (1.2), we have

$$\begin{aligned}\mathrm{Cov}(\boldsymbol{y}) &= (\boldsymbol{X} \vdots \boldsymbol{I})\boldsymbol{\Sigma}\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{I}\end{pmatrix} = \boldsymbol{XUX}' + \boldsymbol{W}'\boldsymbol{X}' + \boldsymbol{XW} + \boldsymbol{V}\\ &= \begin{pmatrix}\boldsymbol{X_s U X_s'} + \boldsymbol{W_{sp} X_s'} + \boldsymbol{X_s W_{ps}} + \boldsymbol{V_s} & \boldsymbol{X_s U X_r'} + \boldsymbol{W_{sp} X_r'} + \boldsymbol{X_r W_{ps}} + \boldsymbol{V_{sr}}\\ \boldsymbol{X_r U X_s'} + \boldsymbol{W_{rp} X_s'} + \boldsymbol{X_s W_{pr}} + \boldsymbol{V_{rs}} & \boldsymbol{X_r U X_r'} + \boldsymbol{W_{rp} X_r'} + \boldsymbol{X_r W_{pr}} + \boldsymbol{V_r}\end{pmatrix}\\ &\triangleq \begin{pmatrix}\boldsymbol{\Lambda_s} & \boldsymbol{\Lambda_{sr}}\\ \boldsymbol{\Lambda_{rs}} & \boldsymbol{\Lambda_r}\end{pmatrix} \triangleq \boldsymbol{\Lambda}.\end{aligned} \tag{1.3}$$

We may consider the following specific structures for $\boldsymbol{\Lambda}$:

Case 1. The matrix $\boldsymbol{\Lambda} = \boldsymbol{V}$ is a known arbitrary symmetric nonnegative definite matrix. Such a situation arises when $\boldsymbol{\beta}$ is a fixed effects vector.

Case 2. The matrix $\boldsymbol{\Lambda} = \boldsymbol{XUX}' + \boldsymbol{V} = \boldsymbol{X}_2\boldsymbol{U}_{22}\boldsymbol{X}_2' + \boldsymbol{V}$, where $\boldsymbol{U}_{22}$ is a known matrix and $\boldsymbol{X} = (\boldsymbol{X}_1 \vdots \boldsymbol{X}_2)$. Such a situation arises when we consider the mixed effects model

$$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{X}_2\boldsymbol{\xi} + \boldsymbol{\varepsilon}$$

where $\boldsymbol{\tau}$ is a fixed unknown parameter vector, and $\boldsymbol{\xi}$ and $\boldsymbol{\varepsilon}$ are all uncorrelated random vectors such that $E(\boldsymbol{\xi}) = 0,\ \mathrm{Cov}(\boldsymbol{\xi}) = \boldsymbol{U}_{22}$, and $\boldsymbol{\beta} = (\boldsymbol{\tau}', \boldsymbol{\xi}')'$.

Consider the following classes of linear predictors of $\boldsymbol{Qy}$:

$\mathscr{LH} = \{\boldsymbol{Ly_s} :\ \boldsymbol{L}$ is an $h \times s$ matrix$\}$, the class of all homogeneous linear predictors;

$\mathscr{LI} = \{\boldsymbol{Ly_s} + \boldsymbol{a} : \boldsymbol{L}$ is an $h \times s$ matrix, $\boldsymbol{a}$ is an $h \times 1$ vector$\}$, the class of all linear predictors.

Let $\boldsymbol{\delta}(\boldsymbol{y_s})$ be a predictor of $\boldsymbol{Qy}$. In this article, we will use the matrix loss function

$$\mathcal{L}(\boldsymbol{\delta}(\boldsymbol{y_s}), \boldsymbol{Qy}) = (\boldsymbol{\delta}(\boldsymbol{y_s}) - \boldsymbol{Qy})(\boldsymbol{\delta}(\boldsymbol{y_s}) - \boldsymbol{Qy})', \tag{1.4}$$

and corresponding risk function defined as $\mathscr{R}(\boldsymbol{\delta}(\boldsymbol{y_s}), \boldsymbol{Qy}) = E[\mathcal{L}(\boldsymbol{\delta}(\boldsymbol{y_s}), \boldsymbol{Qy})]$, where $E(\cdot)$ denotes expectation under the model (1.1) (or (1.2)).

Definition 1.1. The predictor $\boldsymbol{\delta}_1(\boldsymbol{y_s})$ is called as good as $\boldsymbol{\delta}_2(\boldsymbol{y_s})$ iff $\mathscr{R}(\boldsymbol{\delta}_1(\boldsymbol{y_s}), \boldsymbol{Qy}) \leqslant \mathscr{R}(\boldsymbol{\delta}_2(\boldsymbol{y_s}), \boldsymbol{Qy})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^k$, and $\boldsymbol{\delta}_1(\boldsymbol{y_s})$ is called better than $\boldsymbol{\delta}_2(\boldsymbol{y_s})$ iff $\boldsymbol{\delta}_1(\boldsymbol{y_s})$ is as good as $\boldsymbol{\delta}_2(\boldsymbol{y_s})$ and $\mathscr{R}(\boldsymbol{\delta}_1(\boldsymbol{y_s}), \boldsymbol{Qy}) \neq \mathscr{R}(\boldsymbol{\delta}_2(\boldsymbol{y_s}), \boldsymbol{Qy})$ at some $\boldsymbol{\alpha}_0$ in $\mathbb{R}^k$. Let $\mathscr{L}$ be a class of predictors. Then a predictor $\boldsymbol{\delta}(\boldsymbol{y_s})$ is said to be admissible for $\boldsymbol{Qy}$ in $\mathscr{L}$ iff $\boldsymbol{\delta}(\boldsymbol{y_s}) \in \mathscr{L}$ and there exists no predictor in $\mathscr{L}$ which is better than $\boldsymbol{\delta}(\boldsymbol{y_s})$.

Definition 1.2. $\boldsymbol{Qy}$ is called a linearly predictable variable, if there exists a linear predictor $\boldsymbol{Ly_s} + \boldsymbol{a}$ in $\mathscr{LI}$ such that $E(\boldsymbol{Ly_s} + \boldsymbol{a} - \boldsymbol{Qy}) = 0$ for all $\boldsymbol{\alpha} \in \mathbb{R}^k$.

Lemma 1.1. *$\boldsymbol{Qy}$ is a linearly predictable variable iff $\mathscr{M}(\boldsymbol{A}'\boldsymbol{X_r}'\boldsymbol{Q_r}') \subset \mathscr{M}(\boldsymbol{A}'\boldsymbol{X_s}')$.*

Proof. Its proof is obvious and therefore omitted here. □

2. Admissibility of a linear predictor in the class of linear predictors

In this section, we investigate the conditions of for a linear predictor to be admissible in the two linear predictor classes $\mathscr{LH}$ and $\mathscr{LI}$, respectively.

The following Lemma 2.1 is a direct consequence of Theorem 3.1 in [23].

Lemma 2.1. *Consider the following model*

$$\boldsymbol{y_s} = \boldsymbol{X_s A\alpha} + \boldsymbol{e_s} \tag{2.1}$$

where $\boldsymbol{e_s}$ is an $s \times 1$ unobservable random vector, with $E(\boldsymbol{e_s}) = 0$, Cov $(\boldsymbol{e_s}) = \boldsymbol{\Lambda_s}$, $\boldsymbol{X_s}$, $\boldsymbol{A}$ and $\boldsymbol{\Lambda_s}$ are known $s \times p$, $p \times k$ and $s \times s$ matrices respectively, and $\boldsymbol{\alpha}$ is a $k \times 1$ unknown parameter vector. If $\boldsymbol{S\alpha}$ is linearly estimable under the model (2.1), then, under the loss function $(\boldsymbol{d} - \boldsymbol{S\alpha})(\boldsymbol{d} - \boldsymbol{S\alpha})'$, $\boldsymbol{Ly_s}$ is an admissible estimator for $\boldsymbol{S\alpha}$ in $\mathscr{LH}$ if and only if

(i) $\boldsymbol{L\Lambda_s} = \boldsymbol{LX_s AB\Lambda_s}$*(equivalently* $\mathscr{M}(\boldsymbol{\Lambda_s L}') \subset \mathscr{M}(\boldsymbol{X_s}A))$,

(ii) $\boldsymbol{LX_s A} = \boldsymbol{S}$; *or* $\boldsymbol{b}(\boldsymbol{LX_s A} - \boldsymbol{S})C(\boldsymbol{LX_s A} - \boldsymbol{S})' + \boldsymbol{LX_s ACA}'\boldsymbol{X_s}'\boldsymbol{L}' - \boldsymbol{SCS}' \geqslant 0$ *does not hold for all* $\boldsymbol{b} \in (0, 1)$ *when* $\boldsymbol{LX_s A} \neq \boldsymbol{S}$,

hereafter $\boldsymbol{T} = \boldsymbol{\Lambda_s} + \boldsymbol{X_s AA}'\boldsymbol{X_s}'$, $\boldsymbol{B} = (\boldsymbol{A}'\boldsymbol{X_s}'\boldsymbol{T}^+\boldsymbol{X_s A})^-\boldsymbol{A}'\boldsymbol{X_s}'\boldsymbol{T}^+$, *and* $\boldsymbol{C} = \boldsymbol{B\Lambda_s B}'$.

Theorem 2.1. *Under the model (1.1), let $\boldsymbol{Qy}$ be a linearly predictable variable. Then the following statements* 1°, 2°, and 3° *are equivalent:*

1° *$\boldsymbol{Ly_s}$ is an admissible predictor of $\boldsymbol{Qy}$ in the class $\mathscr{LH}$ under loss function (1.4);*

2° *$\boldsymbol{L}$ satisfies*

(i) $L\Lambda_s = LX_sAB\Lambda_s + (Q_s + Q_r\Lambda_{rs}\Lambda_s^+)(I - X_sAB)\Lambda_s$ (equivalently $\mathscr{M}(\Lambda_sL' - (\Lambda_s\vdots\Lambda_{sr})Q') \subset \mathscr{M}(X_sA)$),

(ii) $LX_sA = QXA$; or $b(LX_sA - QXA)C(LX_sA - QXA)' + (L - Q_s)X_sA \times CA'X_s'(L - Q_s)' - Q_rX_rACA'X_r'Q_r' - Q_r\Lambda_{rs}\Lambda_s^+X_sAC(LX_sA - QXA)' - (LX_sA - QXA)CA'X_s'\Lambda_s^+\Lambda_{sr}Q_r' \geqslant 0$ does not hold for all $b \in (0, 1)$ when $LX_sA \neq QXA$;

3° L satisfies

(i) $L\Lambda_s = LX_sAB\Lambda_s + (Q_s + Q_r\Lambda_{rs}T^+)(I - X_sAB)\Lambda_s$ (equivalently $\mathscr{M}(\Lambda_sL' - (\Lambda_s\vdots\Lambda_{sr})Q') \subset \mathscr{M}(X_sA)$),

(ii) $LX_sA = QXA$; or $f(b, L) \triangleq b(LX_sA - QXA)C(LX_sA - QXA)' + (L - Q_s)X_sACA'X_s'(L - Q_s)' - Q_rX_rACA'X_r'Q_r' - Q_r\Lambda_{rs}T^+X_sA(A'X_s'T^+X_sA)^-(LX_sA - QXA)' - (LX_sA - QXA)(A'X_s'T^+X_sA)^- \times A'X_s'T^+\Lambda_{sr}Q_r' \geqslant 0$ does not hold for all $b \in (0, 1)$ when $LX_sA \neq QXA$.

Proof. Note that $\mathscr{M}(\Lambda_{sr}) \subset \mathscr{M}(\Lambda_s)$, we have

$$\Lambda_{rs}\Lambda_s^+\Lambda_s = \Lambda_{rs}. \tag{2.2}$$

Therefore, we deduce that

$$\begin{aligned} E(Ly_s - Qy)(Ly_s - Qy)' &= (L - Q_s)\Lambda_s(L - Q_s)' - (L - Q_s)\Lambda_{sr}Q_r' - Q_r\Lambda_{rs}(L - Q_s)' \\ &\quad + (LX_sA - QXA)\alpha\alpha'(LX_sA - QXA)' \\ &= (L - Q_s - Q_r\Lambda_{rs}\Lambda_s^+)\Lambda_s(L - Q_s - Q_r\Lambda_{rs}\Lambda_s^+)' + Q_r\Lambda_rQ_r' - Q_r\Lambda_{rs}\Lambda_s^+\Lambda_{sr}Q_r' \\ &= E\{[(L - Q_s - Q_r\Lambda_{rs}\Lambda_s^+)y_s - (Q_rX_r - Q_r\Lambda_{rs}\Lambda_s^+X_s)A\alpha][(L - Q_s - Q_r\Lambda_{rs}\Lambda_s^+)y_s \\ &\quad - (Q_rX_r - Q_r\Lambda_{rs}\Lambda_s^+X_s)A\alpha]'\} + Q_r(\Lambda_r - \Lambda_{rs}\Lambda_s^+\Lambda_{sr})Q_r'. \end{aligned}$$

Noting that $Q_r(\Lambda_r - \Lambda_{rs}\Lambda_s^+\Lambda_{sr})Q_r'$ has no effect on the admissibility of a predictor, therefore, to prove that Ly_s is admissible for Qy in the class $\mathscr{LH}$ we need only to show that $(L - Q_s - Q_r\Lambda_{rs}\Lambda_s^+)y_s$ is an admissible estimator for $(Q_rX_r - Q_r\Lambda_{rs}\Lambda_s^+X_s)A\alpha$ under the model (2.1) and loss function $[d - (Q_rX_r - Q_r\Lambda_{rs}\Lambda_s^+ \times X_s)A\alpha][d - (Q_rX_r - Q_r\Lambda_{rs}\Lambda_s^+X_s)A\alpha]'$ in the class $\mathscr{LH}$. On the other hand, Qy is a linearly predictable variable, it follows from Lemma 1.1 that $\mathscr{M}(A'X_r'Q_r') \subset \mathscr{M}(A'X_s')$, and hence $\mathscr{M}(A'X_r'Q_r' - A'X_s'\Lambda_s^+\Lambda_{sr}Q_r') \subset \mathscr{M}(A'X_s')$. Therefore $(Q_rX_r - Q_r\Lambda_{rs}\Lambda_s^+X_s)A\alpha$ is linearly estimable under the model (2.1). These together with Lemma 2.1 imply that 1° is equivalent to 2°.

We now prove that 2° is equivalent to 3°. Noting $T = \Lambda_s + X_sAA'X_s'$ and $\mathscr{M}(\Lambda_s^+) = \mathscr{M}(\Lambda_s) \subset \mathscr{M}(T)$, it follows that

$$\begin{aligned} Q_r\Lambda_{rs}\Lambda_s^+(I - X_sAB)\Lambda_s &= Q_r\Lambda_{rs}\Lambda_s^+TT^+(I - X_sA(A'X_s'T^+X_sA)^-A'X_s'T^+)\Lambda_s \\ &= Q_r\Lambda_{rs}\Lambda_s^+\Lambda_sT^+(I - X_sA(A'X_s'T^+X_sA)^-A'X_s'T^+)\Lambda_s \\ &= Q_r\Lambda_{rs}T^+(I - X_sA(A'X_s'T^+X_sA)^-A'X_s'T^+)\Lambda_s \\ &= Q_r\Lambda_{rs}T^+(I - X_sAB)\Lambda_s. \end{aligned}$$

Hence the condition (i) in 2° is equivalent to the condition (i) in 3°.

Observing $T = \Lambda_s + X_sAA'X_s'$, $\mathscr{M}(\Lambda_s^+) = \mathscr{M}(\Lambda_s) \subset \mathscr{M}(T) = \mathscr{M}(T^+)$, $T^+T\Lambda_s^+ = \Lambda_s^+$, $\mathscr{M}(A'X_s'L' - A'X'Q') \subset \mathscr{M}(A'X_s') = \mathscr{M}(A'X_s'T^+X_sA)$ and (2.2), we deduce by substituting Λ_s, Λ_s^+ and $C = B\Lambda_sB'$ into the follows that

$$\begin{aligned} (LX_sA - QXA)CA'X_s'\Lambda_s^+\Lambda_{sr} &= (LX_sA - QXA)(A'X_s'T^+X_sA)^-A'X_s'T^+\Lambda_sT^+X_sA(A'X_s'T^+X_sA)^-A'X_s'\Lambda_s^+\Lambda_{sr} \\ &= (LX_sA - QXA)(A'X_s'T^+X_sA)^-A'X_s'T^+(T - X_sAA'X_s')T^+X_sA(A'X_s'T^+X_sA)^-A'X_s'T^+T\Lambda_s^+\Lambda_{sr} \\ &= (LX_sA - QXA)(A'X_s'T^+X_sA)^-A'X_s'T^+TT^+X_sA(A'X_s'T^+X_sA)^-A'X_s'T^+T\Lambda_s^+\Lambda_{sr} \\ &\quad - (LX_sA - QXA)(A'X_s'T^+X_sA)^-A'X_s'T^+X_sAA'X_s'T^+X_sA(A'X_s'T^+X_sA)^-A'X_s'T^+T\Lambda_s^+\Lambda_{sr} \\ &= (LX_sA - QXA)(A'X_s'T^+X_sA)^-(A'X_s'T^+X_sA)(A'X_s'T^+X_sA)^-A'X_s'T^+T\Lambda_s^+\Lambda_{sr} \\ &\quad - (LX_sA - QXA)(A'X_s'T^+X_sA)^-(A'X_s'T^+X_sA)(A'X_s'T^+X_sA)(A'X_s'T^+X_sA)^-A'X_s'T^+T\Lambda_s^+\Lambda_{sr} \\ &= (LX_sA - QXA)(A'X_s'T^+X_sA)^-A'X_s'T^+T\Lambda_s^+\Lambda_{sr} - (LX_sA - QXA)A'X_s'T^+T\Lambda_s^+\Lambda_{sr} \\ &= (LX_sA - QXA)(A'X_s'T^+X_sA)^-A'X_s'T^+(\Lambda_s + X_sAA'X_s')\Lambda_s^+\Lambda_{sr} - (LX_sA - QXA)A'X_s'T^+T\Lambda_s^+\Lambda_{sr} \\ &= (LX_sA - QXA)(A'X_s'T^+X_sA)^-A'X_s'T^+\Lambda_s\Lambda_s^+\Lambda_{sr} + (LX_sA - QXA)(A'X_s'T^+X_sA)^-(A'X_s'T^+X_sA)A'X_s'\Lambda_s^+\Lambda_{sr} \\ &\quad - (LX_sA - QXA)A'X_s'T^+T\Lambda_s^+\Lambda_{sr} \\ &= (LX_sA - QXA)(A'X_s'T^+X_sA)^-A'X_s'T^+\Lambda_{sr} + (LX_sA - QXA)A'X_s'\Lambda_s^+\Lambda_{sr} - (LX_sA - QXA)A'X_s'\Lambda_s^+\Lambda_{sr} \\ &= (LX_sA - QXA)(A'X_s'T^+X_sA)^-A'X_s'T^+\Lambda_{sr}. \end{aligned}$$

Thus the condition (ii) in 2° is equivalent to the condition (ii) in 3°. The proof is completed.

For the model (1.1), in the case of $W = 0$, i.e., β and e are uncorrelated, from (1.3) we have

$$\mathrm{Cov}(y) = \begin{pmatrix} X_sUX_s' + V_s & X_sUX_r' + V_{sr} \\ X_rUX_s' + V_{rs} & X_rUX_r' + V_r \end{pmatrix} \triangleq \begin{pmatrix} \Lambda_s & \Lambda_{sr} \\ \Lambda_{rs} & \Lambda_r \end{pmatrix}.$$

Therefore we have the following corollary. □

Corollary 2.1. *Under the model (1.1), let $\boldsymbol{\beta}$ and $\boldsymbol{e}$ be uncorrelated, and $\boldsymbol{Qy}$ be a linearly predictable variable. Then the following statements 1°, 2°, and 3° are equivalent:*

1° $\boldsymbol{Ly_s}$ *is an admissible predictor of* $\boldsymbol{Qy}$ *in the class* $\mathscr{LH}$ *under loss function (1.4);*
2° $\boldsymbol{L}$ *satisfies*

(i) $\boldsymbol{L\Delta_s} = \boldsymbol{LX_sAB_1\Delta_s} + (\boldsymbol{Q_s} + \boldsymbol{Q_r\Delta_{rs}\Delta_s^+})(\boldsymbol{I} - \boldsymbol{X_sAB_1})\boldsymbol{\Delta_s}$ (equivalently $\mathscr{M}(\boldsymbol{\Delta_sL'} - (\boldsymbol{\Delta_s}\vdots\boldsymbol{\Delta_{sr}})\boldsymbol{Q'}) \subset \mathscr{M}(\boldsymbol{X_sA})$),
(ii) $\boldsymbol{LX_sA} = \boldsymbol{QXA}$; or $\boldsymbol{b}(\boldsymbol{LX_sA} - \boldsymbol{QXA})\boldsymbol{C_1}(\boldsymbol{LX_sA} - \boldsymbol{QXA})' + (\boldsymbol{L} - \boldsymbol{Q_s})\boldsymbol{X_sAC_1} \times \boldsymbol{A'X_s'}(\boldsymbol{L} - \boldsymbol{Q_s})' - \boldsymbol{Q_rX_rAC_1A'X_r'Q_r'} - \boldsymbol{Q_r\Delta_{rs}\Delta_s^+X_sAC_1}(\boldsymbol{LX_sA} - \boldsymbol{QXA})' - (\boldsymbol{LX_sA} - \boldsymbol{QXA})\boldsymbol{C_1A'X_s'\Delta_s^+\Delta_{sr}Q_r'} \geqslant 0$ *does not hold for all* $\boldsymbol{b} \in (0, 1)$ *when* $\boldsymbol{LX_sA} \neq \boldsymbol{QXA}$;

3° $\boldsymbol{L}$ *satisfies*

(i) $\boldsymbol{L\Delta_s} = \boldsymbol{LX_sAB_1\Delta_s} + (\boldsymbol{Q_s} + \boldsymbol{Q_r\Delta_{rs}T_1^+})(\boldsymbol{I} - \boldsymbol{X_sAB_1})\boldsymbol{\Delta_s}$(equivalently $\mathscr{M}(\boldsymbol{\Delta_sL'} - (\boldsymbol{\Delta_s}\vdots\boldsymbol{\Delta_{sr}})\boldsymbol{Q'}) \subset \mathscr{M}(\boldsymbol{X_sA})$),
(ii) $\boldsymbol{LX_sA} = \boldsymbol{QXA}$; or $f_1(\boldsymbol{b}, \boldsymbol{L}) \triangleq \boldsymbol{b}(\boldsymbol{LX_sA} - \boldsymbol{QXA})\boldsymbol{C_1}(\boldsymbol{LX_sA} - \boldsymbol{QXA})' + (\boldsymbol{L} - \boldsymbol{Q_s})\boldsymbol{X_sAC_1A'X_s'}(\boldsymbol{L} - \boldsymbol{Q_s})' - \boldsymbol{Q_rX_rAC_1A'X_r'Q_r'} - \boldsymbol{Q_r\Delta_{rs}T_1^+X_sA}(\boldsymbol{A'X_s'T_1^+X_sA})^-(\boldsymbol{LX_sA} - \boldsymbol{QXA})' - (\boldsymbol{LX_sA} - \boldsymbol{QXA})(\boldsymbol{A'X_s'T_1^+X_sA})^-\boldsymbol{A'X_s'T_1^+\Delta_{sr}Q_r'} \geqslant \boldsymbol{0}$ *does not hold for all* $\boldsymbol{b} \in (0, 1)$ *when* $\boldsymbol{LX_sA} \neq \boldsymbol{QXA}$,

hereafter $\boldsymbol{T_1} = \boldsymbol{\Delta_s} + \boldsymbol{X_sAA'X_s'}$, $\boldsymbol{B_1} = (\boldsymbol{A'X_s'T_1^+X_sA})^-\boldsymbol{A'X_s'T_1^+}$ *and* $\boldsymbol{C_1} = \boldsymbol{B_1\Delta_sB_1'}$.

For the model (1.1), when $\boldsymbol{\beta}$ is a fixed effects vector, and hence $\boldsymbol{U} = \boldsymbol{0}$, from (1.3) we have

$$\mathrm{Cov}(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{V_s} & \boldsymbol{V_{sr}} \\ \boldsymbol{V_{rs}} & \boldsymbol{V_r} \end{pmatrix}.$$

Therefore we have the following corollary.

Corollary 2.2. *Under the model (1.1), let $\boldsymbol{\beta}$ be a fixed effects vector, and $\boldsymbol{Qy}$ be a linearly predictable variable. Then the following statements 1°, 2°, and 3° are equivalent:*

1° $\boldsymbol{Ly_s}$ *is an admissible predictor of* $\boldsymbol{Qy}$ *in the class* $\mathscr{LH}$ *under loss function (1.4);*
2° $\boldsymbol{L}$ *satisfies*

(i) $\boldsymbol{LV_s} = \boldsymbol{LX_sB_2V_s} + (\boldsymbol{Q_s} + \boldsymbol{Q_rV_{rs}V_s^+})(\boldsymbol{I} - \boldsymbol{X_sB_2})\boldsymbol{V_s}$ (equivalently $\mathscr{M}(\boldsymbol{V_sL'} - (\boldsymbol{V_s}\vdots\boldsymbol{V_{sr}})\boldsymbol{Q'}) \subset \mathscr{M}(\boldsymbol{X_s})$),
(ii) $\boldsymbol{LX_s} = \boldsymbol{QX}$; or $\boldsymbol{b}(\boldsymbol{LX_s} - \boldsymbol{QX})\boldsymbol{C_2}(\boldsymbol{LX_s} - \boldsymbol{QX})' + (\boldsymbol{L} - \boldsymbol{Q_s})\boldsymbol{X_sC_2X_s'}(\boldsymbol{L} - \boldsymbol{Q_s})' - \boldsymbol{Q_rX_rC_2X_r'Q_r'} - \boldsymbol{Q_rV_{rs}V_s^+X_sC_2}(\boldsymbol{LX_s} - \boldsymbol{QX})' - (\boldsymbol{LX_s} - \boldsymbol{QX})\boldsymbol{C_2X_s'V_s^+V_{sr}Q_r'} \geqslant \boldsymbol{0}$ *does not hold for all* $\boldsymbol{b} \in (0, 1)$ *when* $\boldsymbol{LX_s} \neq \boldsymbol{QX}$;

3° $\boldsymbol{L}$ *satisfies*

(i) $\boldsymbol{LV_s} = \boldsymbol{LX_sB_2V_s} + (\boldsymbol{Q_s} + \boldsymbol{Q_rV_{rs}T_2^+})(\boldsymbol{I} - \boldsymbol{X_sB_2})\boldsymbol{V_s}$ (equivalently $\mathscr{M}(\boldsymbol{V_sL'} - (\boldsymbol{V_s}\vdots\boldsymbol{V_{sr}})\boldsymbol{Q'}) \subset \mathscr{M}(\boldsymbol{X_s})$),
(ii) $\boldsymbol{LX_s} = \boldsymbol{QX}$; or $f_2(\boldsymbol{b}, \boldsymbol{L}) \triangleq \boldsymbol{b}(\boldsymbol{LX_s} - \boldsymbol{QX})\boldsymbol{C_2}(\boldsymbol{LX_s} - \boldsymbol{QX})' + (\boldsymbol{L} - \boldsymbol{Q_s})\boldsymbol{X_sC_2X_s'} \times (\boldsymbol{L} - \boldsymbol{Q_s})' - \boldsymbol{Q_rX_rC_2X_r'Q_r'} - \boldsymbol{Q_rV_{rs}T_2^+X_s}(\boldsymbol{X_s'T_2^+X_s})^-(\boldsymbol{LX_s} - \boldsymbol{QX})' - (\boldsymbol{LX_s} - \boldsymbol{QX})(\boldsymbol{X_s'T_2^+X_s})^-\boldsymbol{X_s'T_2^+V_{sr}Q_r'} \geqslant \boldsymbol{0}$ *does not hold for all* $\boldsymbol{b} \in (0, 1)$ *when* $\boldsymbol{LX_s} \neq \boldsymbol{QX}$,

hereafter $\boldsymbol{T_2} = \boldsymbol{V_s} + \boldsymbol{X_sX_s'}$, $\boldsymbol{B_2} = (\boldsymbol{X_s'T_2^+X_s})^-\boldsymbol{X_s'T_2^+}$, *and* $\boldsymbol{C_2} = \boldsymbol{B_2V_sB_2'}$.

Remark 2.1. Theorem 1 in [31] is a special case of the above Corollary 2.2.

Remark 2.2. Each condition in Theorem 2.1, Corollaries 2.1 and 2.2 is invariant with respect to the choice of the involved generalized inverse.

We now discuss admissibility of a linear predictor $\boldsymbol{Ly_s} + \boldsymbol{a}$ to be admissible in the class $\mathscr{LI}$.

Lemma 2.2. *Under the assumptions of the model (2.1), if $\boldsymbol{S\alpha}$ is linearly estimable under the model (2.1), then, under the loss function $(\boldsymbol{d} - \boldsymbol{S\alpha})(\boldsymbol{d} - \boldsymbol{S\alpha})'$, $\boldsymbol{Ly_s} + \boldsymbol{a}$ is admissible estimator for $\boldsymbol{S\alpha}$ in $\mathscr{LH}$ if and only if*

1° *the condition* (i) *in Lemma 2.1 holds,*
2° $\boldsymbol{LX_sA} = \boldsymbol{S}$, $\boldsymbol{a} = \boldsymbol{0}$; *or* $\boldsymbol{b}(\boldsymbol{LX_sA} - \boldsymbol{S})\boldsymbol{C}(\boldsymbol{LX_sA} - \boldsymbol{S})' + \boldsymbol{LX_sACA'X_s'L'} - \boldsymbol{SCS'} \geqslant \boldsymbol{0}$ *does not hold for all* $\boldsymbol{b} \in (0, 1)$ *when* $\boldsymbol{LX_sA} \neq \boldsymbol{S}$,

where $\boldsymbol{C}$ is defined as in Lemma 2.1.

Similar to the method of proving the Theorem 2.1, we get

Theorem 2.2. *Under the model (1.1), let $\boldsymbol{Qy}$ be a linearly predictable variable. Then $\boldsymbol{Ly_s} + \boldsymbol{a}$ is an admissible predictor of $\boldsymbol{Qy}$ in the class $\mathscr{LI}$ under loss function (1.4) if and only if $\boldsymbol{L}$ and $\boldsymbol{a}$ satisfy*

1° *the condition* (i) *of the statement* 3° *in Theorem 2.1 holds,*
2° $\boldsymbol{LX_sA} = \boldsymbol{QXA}$, $\boldsymbol{a} = 0$; *or* $f(\boldsymbol{b}, \boldsymbol{L}) \geqslant \boldsymbol{0}$ *does not hold for all* $\boldsymbol{b} \in (0, 1)$ *when* $\boldsymbol{LX_sA} \neq \boldsymbol{QXA}$,

where $f(\boldsymbol{b}, \boldsymbol{L})$ is defined as in Theorem 2.1.

Corollary 2.3. *Under the model (1.1), let $\boldsymbol{\beta}$ and $\boldsymbol{e}$ be uncorrelated, and $\boldsymbol{Qy}$ be a linearly predictable variable. Then $\boldsymbol{Ly_s} + \boldsymbol{a}$ is an admissible predictor of $\boldsymbol{Qy}$ in the class $\mathscr{LI}$ under loss function (1.4) if and only if $\boldsymbol{L}$ and $\boldsymbol{a}$ satisfy*

1° *the condition* (i) *of the statement* 2° *or* 3° *in* Corollary 2.1 *holds,*
2° $\boldsymbol{LX_sA} = \boldsymbol{QXA}$, $\boldsymbol{a} = \boldsymbol{0}$; *or* $f_1(\boldsymbol{b}, \boldsymbol{L}) \geqslant \boldsymbol{0}$ *does not hold for all* $\boldsymbol{b} \in (0, 1)$ *when* $\boldsymbol{LX_sA} \neq \boldsymbol{QXA}$,

where $f_1(\boldsymbol{b}, \boldsymbol{L})$ *is defined as in* Corollary 2.1.

Corollary 2.4. *Under the model* (1.1)*, let* $\boldsymbol{\beta}$ *be a fixed effects vector, and* $\boldsymbol{Qy}$ *be a linearly predictable variable. Then* $\boldsymbol{Ly_s} + \boldsymbol{a}$ *is an admissible predictor of* $\boldsymbol{Qy}$ *in the class* $\mathscr{L}\mathscr{I}$ *under loss function* (1.4) *if and only if* $\boldsymbol{L}$ *and* $\boldsymbol{a}$ *satisfy*

1° *the condition* (i) *of the statement* 2° *or* 3° *in* Corollary 2.2 *holds,*
2° $\boldsymbol{LX_s} = \boldsymbol{QX}$, $\boldsymbol{a} = \boldsymbol{0}$; *or* $f_2(\boldsymbol{b}, \boldsymbol{L}) \geqslant \boldsymbol{0}$ *does not hold for all* $\boldsymbol{b} \in (0, 1)$ *when* $\boldsymbol{LX_s} \neq \boldsymbol{QX}$,

where $f_2(\boldsymbol{b}, \boldsymbol{L})$ *is defined as in* Corollary 2.2.

Remark 2.3. The above Corollary 2.4 is Theorem 3 in [31].

3. Admissibility of a linear predictor in the class of all predictors

In this section, we shall investigate the conditions of for a linear predictor to be admissible in the class of all predictors. Unless otherwise stated, to say a predictor to be admissible means that it is admissible in the class of all predictors in the following.

Lemma 3.1 below is quoted from Theorem 2.3 in [8].

Lemma 3.1. *Consider the following model*

$$\boldsymbol{y_s} = \boldsymbol{X_sA\alpha} + e_s, \quad \boldsymbol{e_s} \sim \mathscr{N}(0, \boldsymbol{\Lambda_s}), \tag{3.1}$$

where $\boldsymbol{e_s}$ *is an* $s \times 1$ *unobservable random vector,* $\boldsymbol{X_s}$, $\boldsymbol{A}$ *and* $\boldsymbol{\Lambda_s}$ *are known* $s \times p$, $p \times k$ *and* $s \times s$ *matrices, respectively, and* $\boldsymbol{\alpha}$ *is a* $k \times 1$ *unknown parameter vector. If* $\boldsymbol{S\alpha}$ *is linearly estimable under the model* (3.1)*, then, under the loss function* $(\boldsymbol{d} - \boldsymbol{S\alpha})(\boldsymbol{d} - \boldsymbol{S\alpha})'$, $\boldsymbol{Ly_s} + \boldsymbol{a}$ *is an admissible estimator for* $\boldsymbol{S\alpha}$ *in the class of all estimators if and only if*

(i) $\boldsymbol{L\Lambda_s} = \boldsymbol{LX_sAB\Lambda_s}$ *(equivalently* $\mathscr{M}(\boldsymbol{\Lambda_s}L') \subset \mathscr{M}(\boldsymbol{X_s}A)$*),*
(ii) $\boldsymbol{LX_sA} = \boldsymbol{S}$, $\boldsymbol{a} = \boldsymbol{0}$; *or* $\boldsymbol{b}(\boldsymbol{LX_sA} - \boldsymbol{S})\boldsymbol{C}(\boldsymbol{LX_sA} - \boldsymbol{S})' + \boldsymbol{LX_sACA'X_s'L'} - \boldsymbol{SCS'} \geqslant \boldsymbol{0}$ *does not hold for all* $\boldsymbol{b} \in (0, 1)$ *when* $\boldsymbol{LX_sA} \neq \boldsymbol{S}$,

where $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *are defined as in* Lemma 2.1.

Lemma 3.2. *Under the model* (1.2) *and the loss function* (1.4)*, if each element of the risk function of predictor* $\delta(\boldsymbol{y_s})$ *of* $\boldsymbol{Qy}$ *is finite everywhere, then* $\delta(\boldsymbol{y_s})$ *is an admissible predictor of* $\boldsymbol{Qy}$ *if and only if* $\delta(\boldsymbol{y_s}) - (\boldsymbol{Q_s} + \boldsymbol{Q_r\Lambda_{rs}\Lambda_s^+})\boldsymbol{y_s}$ *is an admissible estimator of* $(\boldsymbol{Q_rX_r} - \boldsymbol{Q_r\Lambda_{rs}\Lambda_s^+X_s})\boldsymbol{A\alpha}$ *under the model* (3.1) *and loss function* $[\boldsymbol{d} - (\boldsymbol{Q_rX_r} - \boldsymbol{Q_r\Lambda_{rs}\Lambda_s^+X_s})\boldsymbol{A\alpha}][\boldsymbol{d} - (\boldsymbol{Q_rX_r} - \boldsymbol{Q_r\Lambda_{rs}\Lambda_s^+X_s})\boldsymbol{A\alpha}]'$ *in the class of all estimators.*

Proof. From (1.3), we have

$$\mathrm{Cov}\begin{pmatrix}\boldsymbol{y_r}\\ \boldsymbol{y_s}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\Lambda_r} & \boldsymbol{\Lambda_{rs}}\\ \boldsymbol{\Lambda_{sr}} & \boldsymbol{\Lambda_s}\end{pmatrix}.$$

Thus it follows from Theorems 1.8 and 2.1 of Chapter 2 in Wang [21] that

$$E(\boldsymbol{y_r}|\boldsymbol{y_s}) = \boldsymbol{X_rA\alpha} + \boldsymbol{\Lambda_{rs}\Lambda_s^+}(\boldsymbol{y_s} - \boldsymbol{X_sA\alpha}),\ a.e. \tag{3.2}$$

$$\mathrm{Cov}(\boldsymbol{y_r}|\boldsymbol{y_s}) = \boldsymbol{\Lambda_r} - \boldsymbol{\Lambda_{rs}\Lambda_s^+\Lambda_{sr}}. \tag{3.3}$$

Since each element of the risk function of predictor $\delta(\boldsymbol{y_s})$ of $\boldsymbol{Qy}$ is finite everywhere, we obtain

$$\begin{aligned} E[\delta(\boldsymbol{y_s}) - \boldsymbol{Q_sy_s} - \boldsymbol{Q_r}\boldsymbol{E}(\boldsymbol{y_r}|\boldsymbol{y_s})][\boldsymbol{Q_ry_r} - \boldsymbol{Q_r}\boldsymbol{E}(\boldsymbol{y_r}|\boldsymbol{y_s})]' &= \boldsymbol{E}\{\boldsymbol{E}[\delta(\boldsymbol{y_s}) - \boldsymbol{Q_sy_s} - \boldsymbol{Q_r}\boldsymbol{E}(\boldsymbol{y_r}|\boldsymbol{y_s})][\boldsymbol{Q_ry_r} - \boldsymbol{Q_r}\boldsymbol{E}(\boldsymbol{y_r}|\boldsymbol{y_s})]'|\boldsymbol{y_s}\} \\ &= \boldsymbol{E}[\delta(\boldsymbol{y_s}) - \boldsymbol{Q_sy_s} - \boldsymbol{Q_r}\boldsymbol{E}(\boldsymbol{y_r}|\boldsymbol{y_s})][\boldsymbol{Q_r}\boldsymbol{E}(\boldsymbol{y_r}|\boldsymbol{y_s}) - \boldsymbol{Q_r}\boldsymbol{E}(\boldsymbol{y_r}|\boldsymbol{y_s})]' \\ &= \boldsymbol{0}. \end{aligned}$$

Combining it with (3.2) and (3.3), we derive

$$\begin{aligned} E[\delta(\boldsymbol{y_s}) - \boldsymbol{Qy}][\delta(\boldsymbol{y_s}) - \boldsymbol{Qy}]' &= \boldsymbol{E}[\delta(\boldsymbol{y_s}) - \boldsymbol{Q_sy_s} - \boldsymbol{Q_r}\boldsymbol{E}(\boldsymbol{y_r}|\boldsymbol{y_s})][\delta(\boldsymbol{y_s}) - \boldsymbol{Q_sy_s} - \boldsymbol{Q_r}\boldsymbol{E}(\boldsymbol{y_r}|\boldsymbol{y_s})]' \\ &\quad + \boldsymbol{E}\,[\boldsymbol{Q_ry_r} - \boldsymbol{Q_r}\boldsymbol{E}(\boldsymbol{y_r}|\boldsymbol{y_s})][\boldsymbol{Q_ry_r} - \boldsymbol{Q_r}\boldsymbol{E}(\boldsymbol{y_r}|\boldsymbol{y_s})]' \\ &= \boldsymbol{E}\{[\delta(\boldsymbol{y_s}) - (\boldsymbol{Q_s} + \boldsymbol{Q_r\Lambda_{rs}\Lambda_s^+})\boldsymbol{y_s} - (\boldsymbol{Q_rX_r} - \boldsymbol{Q_r\Lambda_{rs}\Lambda_s^+X_s})\boldsymbol{A\alpha}] \\ &\quad [\delta(\boldsymbol{y_s}) - (\boldsymbol{Q_s} + \boldsymbol{Q_r\Lambda_{rs}\Lambda_s^+})\boldsymbol{y_s} - (\boldsymbol{Q_rX_r} - \boldsymbol{Q_r\Lambda_{rs}\Lambda_s^+X_s})\boldsymbol{A\alpha}]'\} \\ &\quad + \boldsymbol{Q_r}(\boldsymbol{\Lambda_r} - \boldsymbol{\Lambda_{rs}\Lambda_s^+\Lambda_{sr}})\boldsymbol{Q_r'}. \end{aligned} \tag{3.4}$$

Noting that the second term on the right-hand side of the equality (3.4) does not affect our discussion on the admissibility of the predictor, and hence the results in Lemma 3.2 follow.

From Lemmas 3.1 and 3.2, we immediately obtain the following. □

Theorem 3.1. *Under the model (1.2), let $\boldsymbol{Qy}$ be a linearly predictable variable. Then $\boldsymbol{Ly_s}+\boldsymbol{a}$ is an admissible predictor of $\boldsymbol{Qy}$ under loss function (1.4) if and only if $\boldsymbol{L}$ and $\boldsymbol{a}$ satisfy the conditions* 1° *and* 2° *in Theorem 2.2.*

Corollary 3.1. *Under the model (1.2), let $\boldsymbol{\beta}$ and $\boldsymbol{e}$ be uncorrelated, $\boldsymbol{Qy}$ be a linearly predictable variable. Then $\boldsymbol{Ly_s}+\boldsymbol{a}$ is an admissible predictor of $\boldsymbol{Qy}$ under loss function (1.4) if and only if $\boldsymbol{L}$ and $\boldsymbol{a}$ satisfy the conditions* 1° *and* 2° *in Corollary 2.3.*

Corollary 3.2. *Under the model (1.2), let $\boldsymbol{\beta}$ be a fixed effects vector, $\boldsymbol{Qy}$ be a linearly predictable variable. Then $\boldsymbol{Ly_s}+\boldsymbol{a}$ is an admissible predictor of $\boldsymbol{Qy}$ under loss function (1.4) if and only if $\boldsymbol{L}$ and $\boldsymbol{a}$ satisfy the conditions* 1° *and* 2° *in Corollary 2.4.*

4. Admissibility of the best linear unbiased predictors

In Sections 2 and 3, we have considered admissibility of linear predictors in the superpopulation models (1.1) and (1.2) under the matrix loss function. The necessary and sufficient conditions for a linear predictor to be admissible in the classes of homogeneous and all linear predictors and the class of all predictors are obtained, respectively. The class of admissible predictors is generally big. Therefore, what is of much practical interest is: are commonly used decision rules such as the best linear unbiased predictor (**BLUP**) (cf. [3]) admissible? In the following, we will answer this question with a theorem and two examples.

Theorem 4.1. *Consider the superpopulation models (1.1) and (1.2), where $\boldsymbol{\beta}$ is a fixed effects vector. Let $\boldsymbol{Qy}$ be a linearly predictable variable. Denote*

$$\boldsymbol{L}_0\boldsymbol{y_s}=\boldsymbol{Q_s y_s}+\boldsymbol{Q_r X_r}\widetilde{\boldsymbol{\beta}}_{\boldsymbol{s}}+\boldsymbol{Q_r V_{rs}}\boldsymbol{T}_2^{+}(\boldsymbol{y_s}-\boldsymbol{X_s}\widetilde{\boldsymbol{\beta}}_{\boldsymbol{s}}),$$

*which is the best linear unbiased predictor (**BLUP**) of $\boldsymbol{Qy}$ (see Theorem 2.2 in [34]), where*

$$\widetilde{\boldsymbol{\beta}}_{\boldsymbol{s}}=(\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{X}_s)^{-}\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{y}_s,$$
$$\boldsymbol{L}_0=\boldsymbol{Q_s}+\boldsymbol{Q_r X_r}(\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{X}_s)^{-}\boldsymbol{X}_s'\boldsymbol{T}_2^{+}+\boldsymbol{Q_r V_{rs}}\boldsymbol{T}_2^{+}(\boldsymbol{I}-\boldsymbol{X_s}\mathbf{B}_2),$$

and $\boldsymbol{T}_2=\boldsymbol{V_s}+\boldsymbol{X_s X_s'}$ and $\mathbf{B}_2=(\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{X}_s)^{-}\boldsymbol{X}_s'\boldsymbol{T}_2^{+}$ are defined as in Corollary 2.2. Then the following statements 1° *and* 2° *hold:*

1° *For the model (1.1) with a fixed effects vector $\boldsymbol{\beta}$, $\boldsymbol{L}_0\boldsymbol{y_s}$ is an admissible predictor of $\boldsymbol{Qy_s}$ in $\mathscr{LI}$ under the loss function (1.4);*
2° *For the model (1.2) with a fixed effects vector $\boldsymbol{\beta}$, $\boldsymbol{L}_0\boldsymbol{y_s}$ is an admissible predictor of $\boldsymbol{Qy_s}$ in the class of all predictors under the loss function (1.4).*

Proof. First, since $\boldsymbol{L}_0\boldsymbol{y_s}$ is the best linear unbiased predictor of $\boldsymbol{Qy}$, we obtain $\boldsymbol{L}_0\boldsymbol{X_s}=\boldsymbol{QX}$. Let $\boldsymbol{L}_0\boldsymbol{y_s}=\boldsymbol{L}_0\boldsymbol{y_s}+\boldsymbol{0}$. Then it is easy to verify that $\boldsymbol{L}_0\boldsymbol{y_s}$ satisfies condition 2° in Corollary 2.4.

In the following, we shall verify that $\boldsymbol{L}_0$ also satisfies the condition 1° in Corollary 2.4, i.e., the condition (i) of the statement 2° or 3° in Corollary 2.2.

Applying the method of proving Theorem 2.1 and using the definitions of $\boldsymbol{L}_0$, and $\boldsymbol{T}_2$ and $\boldsymbol{B}_2$, we deduce that

$$\begin{aligned}\boldsymbol{L}_0\boldsymbol{V_s}&=[\boldsymbol{Q_s}+\boldsymbol{Q_r X_r}(\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{X}_s)^{-}\boldsymbol{X}_s'\boldsymbol{T}_2^{+}+\boldsymbol{Q_r V_{rs}}\boldsymbol{T}_2^{+}(\boldsymbol{I}-\boldsymbol{X_s}\mathbf{B}_2)]\boldsymbol{V_s}\\&=[\boldsymbol{Q_s}+\boldsymbol{Q_r X_r}(\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{X}_s)^{-}\boldsymbol{X}_s'\boldsymbol{T}_2^{+}]\boldsymbol{V_s}+\boldsymbol{Q_r V_{rs}}\boldsymbol{T}_2^{+}(\boldsymbol{I}-\boldsymbol{X_s}\mathbf{B}_2)\boldsymbol{V_s}\\&=\boldsymbol{Q_s V_s}+\boldsymbol{Q_r X_r}(\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{X}_s)^{-}\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{V_s}+\boldsymbol{Q_r V_{rs}}\boldsymbol{V}_s^{+}(\boldsymbol{I}-\boldsymbol{X_s}\mathbf{B}_2)\boldsymbol{V_s},\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{L}_0\boldsymbol{X_s}\mathbf{B}_2\boldsymbol{V_s}+(\boldsymbol{Q_s}+\boldsymbol{Q_r V_{rs}}\boldsymbol{V}_s^{+})(\boldsymbol{I}-\boldsymbol{X_s}\mathbf{B}_2)\boldsymbol{V_s}&=[\boldsymbol{Q_s}+\boldsymbol{Q_r X_r}(\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{X}_s)^{-}\boldsymbol{X}_s'\boldsymbol{T}_2^{+}+\boldsymbol{Q_r V_{rs}}\boldsymbol{T}_2^{+}(\boldsymbol{I}-\boldsymbol{X_s}\mathbf{B}_2)]\boldsymbol{X_s}\mathbf{B}_2\boldsymbol{V_s}\\&\quad+\boldsymbol{Q_s}(\boldsymbol{I}-\boldsymbol{X_s}\mathbf{B}_2)\boldsymbol{V_s}+\boldsymbol{Q_r V_{rs}}\boldsymbol{V}_s^{+}(\boldsymbol{I}-\boldsymbol{X_s}\mathbf{B}_2)\boldsymbol{V_s}\\&=\boldsymbol{Q_s X_s}\mathbf{B}_2\boldsymbol{V_s}+\boldsymbol{Q_r X_r}(\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{X}_s)^{-}\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{X_s}\mathbf{B}_2\boldsymbol{V_s}\\&\quad+\boldsymbol{Q_s}(\boldsymbol{I}-\boldsymbol{X_s}\mathbf{B}_2)\boldsymbol{V_s}+\boldsymbol{Q_r V_{rs}}\boldsymbol{V}_s^{+}(\boldsymbol{I}-\boldsymbol{X_s}\mathbf{B}_2)\boldsymbol{V_s}\\&=\boldsymbol{Q_s V_s}+\boldsymbol{Q_r X_r}(\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{X}_s)^{-}\boldsymbol{X}_s'\boldsymbol{T}_2^{+}\boldsymbol{V_s}+\boldsymbol{Q_r V_{rs}}\boldsymbol{V}_s^{+}(\boldsymbol{I}-\boldsymbol{X_s}\mathbf{B}_2)\boldsymbol{V_s}\\&=\boldsymbol{L}_0\boldsymbol{V_s}.\end{aligned}$$

Thus the condition (i) of the statement 2° (equivalently 3°) in Corollary 2.2 holds, and hence $\boldsymbol{L}_0$ satisfies the condition 1° in Corollary 2.4. According to Corollaries 2.4 and 3.2, Theorem 4.1 holds. □

Example 4.1. Consider the superpopulation models

$$\boldsymbol{y}=\mathbf{1}_N\boldsymbol{\beta}+\boldsymbol{e},\qquad E(\boldsymbol{e})=0,\qquad \mathrm{Cov}(\boldsymbol{e})=\boldsymbol{I}\tag{4.1}$$

and

$$\boldsymbol{y}=\mathbf{1}_N\boldsymbol{\beta}+\boldsymbol{e},\qquad \boldsymbol{e}\sim\mathscr{N}(\boldsymbol{0},\boldsymbol{I}),\tag{4.2}$$

where $\boldsymbol{\beta}$ is a fixed effects scalar. It is obvious that the population total $\boldsymbol{T}_N = \boldsymbol{1}'_N\boldsymbol{y}$ is linearly predictable. Denote $\hat{\boldsymbol{T}}_N = \frac{N}{s}\boldsymbol{1}'_s\boldsymbol{y}_s$. Then $\hat{\boldsymbol{T}}_N$ is the best linear unbiased predictor (**BLUP**) of $\boldsymbol{T}_N$ (cf. Example 2.2 in [29]). $\hat{\boldsymbol{T}}_N$ is an admissible predictor of $\boldsymbol{T}_N$ under loss function (1.4). Then, from Theorem 4.1, the following statements 1° and 2° hold:

1° For the model (4.1), $\hat{\boldsymbol{T}}_N$ is an admissible predictor of $\boldsymbol{T}_N$ in the class of all linear predictors under loss function (1.4);

2° For the model (4.2), $\hat{\boldsymbol{T}}_N$ is an admissible predictor of $\boldsymbol{T}_N$ in the class of all predictors under loss function (1.4).

Example 4.2. Consider the superpopulation models

$$\boldsymbol{y} = X\boldsymbol{\beta} + e, \qquad E(\boldsymbol{e}) = 0, \qquad \mathrm{Cov}(\boldsymbol{e}) = \boldsymbol{V} \tag{4.3}$$

and

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + e, \quad \boldsymbol{e} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{V}), \tag{4.4}$$

where $\boldsymbol{\beta}$ is a fixed effects vector, $\boldsymbol{V}$ is a positive definite matrix, and $\boldsymbol{X_s}$, $\boldsymbol{X_r}$ and $\boldsymbol{X}$ are all of full column rank. Let

$$\hat{\boldsymbol{\beta}}_N = (\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{y}_s$$

be the weighted least squares estimator of $\boldsymbol{\beta}$. Then $\hat{\boldsymbol{\beta}}_{\boldsymbol{N}}$ is the best linear unbiased predictor (**BLUP**) of the finite population regression coefficient $\boldsymbol{\beta_N} = (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}$ (e.g., cf. [34,4]). We conclude that the following statements 1° and 2° hold:

1° For the model (4.3), $\hat{\boldsymbol{\beta}}_{\boldsymbol{N}}$ is an admissible predictor of $\boldsymbol{\beta}_N$ in the class of all linear predictors under loss function (1.4);

2° For the model (4.4), $\hat{\boldsymbol{\beta}}_{\boldsymbol{N}}$ is an admissible predictor of $\boldsymbol{\beta_N}$ in the class of all predictors under loss function (1.4).

Proof. To begin, recall that

$$(\boldsymbol{A} + \boldsymbol{CBD})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{C}(\boldsymbol{B}^{-1} + \boldsymbol{DA}^{-1}\boldsymbol{C})^{-1}\boldsymbol{DA}^{-1},$$

for matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ with appropriate orders. Thus we obtain that

$$\boldsymbol{T}_2^+ = \boldsymbol{T}_2^{-1} = \boldsymbol{V}_s^{-1} - \boldsymbol{V}_s^{-1}\boldsymbol{X}_s(\boldsymbol{I} + \boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1},$$
$$\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s(\boldsymbol{I} + \boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1} = \boldsymbol{I} - (\boldsymbol{I} + \boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}.$$

Thus

$$\begin{aligned}\boldsymbol{X}'_s\boldsymbol{T}_2^+ = \boldsymbol{X}'_s\boldsymbol{T}_2^{-1} &= \boldsymbol{X}'_s\boldsymbol{V}_s^{-1} - \boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s(\boldsymbol{I} + \boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\\ &= \boldsymbol{X}'_s\boldsymbol{V}_s^{-1} - [\boldsymbol{I} - (\boldsymbol{I} + \boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}]\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\\ &= (\boldsymbol{I} + \boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}.\end{aligned} \tag{4.5}$$

Since $\boldsymbol{X_s}$ is of full column rank, it follows from (4.5) that

$$\begin{aligned}\boldsymbol{X}_s(\boldsymbol{X}'_s\boldsymbol{T}_2^+\boldsymbol{X}_s)^-\boldsymbol{X}'_s\boldsymbol{T}_2^+ &= \boldsymbol{X}_s(\boldsymbol{X}'_s\boldsymbol{T}_2^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{T}_2^{-1}\\ &= \boldsymbol{X}_s[(\boldsymbol{I} + \boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s]^{-1}(\boldsymbol{I} + \boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\\ &= \boldsymbol{X}_s(\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}.\end{aligned} \tag{4.6}$$

Denote $\boldsymbol{L}_1 = (\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}$ and $\boldsymbol{Q}^* = (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{V}^{-1}$, then $\hat{\boldsymbol{\beta}}_{\boldsymbol{N}} = \boldsymbol{L}_1\boldsymbol{y}_s$ and $\boldsymbol{\beta_N} = \boldsymbol{Q}^*\boldsymbol{y}$. Hereafter partition $\boldsymbol{Q}^*$ into $\boldsymbol{Q}^* = (\boldsymbol{Q}_s^* \vdots \boldsymbol{Q}_r^*)$. Because $\boldsymbol{L}_1\boldsymbol{y}_s = \hat{\boldsymbol{\beta}}_{\boldsymbol{N}}$ is the best linear unbiased predictor, we easily obtain $\boldsymbol{L}_1\boldsymbol{X_s} = \boldsymbol{Q}^*\boldsymbol{X}$. Noting that $\boldsymbol{L}_1\boldsymbol{y}_s = \boldsymbol{L}_1\boldsymbol{y}_s + 0$, it follows that $\boldsymbol{L}_1$ and $\boldsymbol{a}_1 = 0$ satisfy condition 2° in Corollary 2.4. In the following, we shall verify that $\boldsymbol{L}_1$ also satisfies the condition 1° in Corollary 2.4, i.e., the condition (i) of the statement 2° or 3° in Corollary 2.2. By an approach similar to that of in Theorem 4.1, we deduce from (4.6) that

$$\begin{aligned}\boldsymbol{L}_1\boldsymbol{V}_s &= (\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{V}_s\\ &= (\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s,\end{aligned}$$

and

$$\begin{aligned}&\boldsymbol{L}_1\mathrm{B}_2\boldsymbol{V}_s + (\boldsymbol{Q}_s^* + \boldsymbol{Q}_r^*\boldsymbol{V}_{rs}\boldsymbol{V}_s^+)(\boldsymbol{I} - \mathrm{B}_2)\boldsymbol{V}_s = \boldsymbol{L}_1\boldsymbol{X}_s(\boldsymbol{X}'_s\boldsymbol{T}_2^+\boldsymbol{X}_s)^-\boldsymbol{X}'_s\boldsymbol{T}_2^+\boldsymbol{V}_s + (\boldsymbol{Q}_s^* + \boldsymbol{Q}_r^*\boldsymbol{V}_{rs}\boldsymbol{V}_s^+)(\boldsymbol{I} - \boldsymbol{X}_s(\boldsymbol{X}'_s\boldsymbol{T}_2^+\boldsymbol{X}_s)^-\boldsymbol{X}'_s\boldsymbol{T}_2^+)\boldsymbol{V}_s\\ &= \boldsymbol{L}_1\boldsymbol{X}_s(\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{V}_s + (\boldsymbol{Q}_s^* + \boldsymbol{Q}_r^*\boldsymbol{V}_{rs}\boldsymbol{V}_s^{-1})(\boldsymbol{I} - \boldsymbol{X}_s(\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1})\boldsymbol{V}_s\\ &= (\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s + (\boldsymbol{Q}_s^*\boldsymbol{V}_s + \boldsymbol{Q}_r^*\boldsymbol{V}_{rs})\boldsymbol{V}_s^{-1}(\boldsymbol{I} - \boldsymbol{X}_s(\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1})\boldsymbol{V}_s\\ &= (\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s + (\boldsymbol{Q}_s^* \vdots \boldsymbol{Q}_r^*)\begin{pmatrix}\boldsymbol{V}_s\\ \boldsymbol{V}_{rs}\end{pmatrix}\boldsymbol{V}_s^{-1}(\boldsymbol{I} - \boldsymbol{X}_s(\boldsymbol{X}'_s\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}'_s\boldsymbol{V}_s^{-1})\boldsymbol{V}_s\end{aligned}$$

$$
\begin{aligned}
&= (\boldsymbol{X}_s'\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}_s' + (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{V}^{-1}\begin{pmatrix}\boldsymbol{V}_s\\ \boldsymbol{V}_{rs}\end{pmatrix}\boldsymbol{V}_s^{-1}(\boldsymbol{I} - \boldsymbol{X}_s(\boldsymbol{X}_s'\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}_s'\boldsymbol{V}_s^{-1})\boldsymbol{V}_s \\
&= (\boldsymbol{X}_s'\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}_s' + (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\begin{pmatrix}\boldsymbol{I}_s\\ \boldsymbol{0}\end{pmatrix}\boldsymbol{V}_s^{-1}(\boldsymbol{I} - \boldsymbol{X}_s(\boldsymbol{X}_s'\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}_s'\boldsymbol{V}_s^{-1})\boldsymbol{V}_s \\
&= (\boldsymbol{X}_s'\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}_s' + (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}_s'\boldsymbol{V}_s^{-1}(\boldsymbol{I} - \boldsymbol{X}_s(\boldsymbol{X}_s'\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}_s'\boldsymbol{V}_s^{-1})\boldsymbol{V}_s \\
&= (\boldsymbol{X}_s'\boldsymbol{V}_s^{-1}\boldsymbol{X}_s)^{-1}\boldsymbol{X}_s' \\
&= \boldsymbol{L}_1\boldsymbol{V}_s.
\end{aligned}
$$

Thus the condition (i) of the statement 2° (equivalently 3°) in Corollary 2.2 holds, and hence $\boldsymbol{L}_1$ satisfies the condition 1° in Corollary 2.4. According to Corollaries 2.4 and 3.2, the proof is completed. □

Remark 4.1. For the model (1.1) (or (1.2)), in the case of $\boldsymbol{\Sigma} = \sigma^2\boldsymbol{\Sigma}_0$, we may obtain the same conclusions as before, where $\boldsymbol{\Sigma}_0$ is a known matrix, and $\sigma^2 > 0$ is also a superparameter.

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