Inverting linear combinations of identity and generalized Catalan matrices

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ABSTRACT

We introduce the notion of the generalized Catalan matrix as a kind of lower triangular Toeplitz matrix whose nonzero elements involve the generalized Catalan numbers. Inverse of the linear combination of the Pascal matrix with the identity matrix is computed in Aggarwala and Lamoureux (2002) [1]. In this paper, continuing this idea, we invert various linear combinations of the generalized Catalan matrix with the identity matrix. A simple and efficient approach to invert the Pascal matrix plus one in terms of the Hadamard product of the Pascal matrix and appropriate lower triangular Toeplitz matrices is considered in Yang and Liu (2006) [14]. We derive representations for inverses of linear combinations of the generalized Catalan matrix and the identity matrix, in terms of the Hadamard product which includes the Generalized Catalan matrix and appropriate lower triangular Toeplitz matrix.

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1. Introduction

The Catalan numbers are the terms of the sequence 1, 1, 2, 5, 14, 42, 132, ... where the initial term of is \( C_0 = 1 \), and the \( n \)th number in the sequence is given in terms of binomial coefficients by

\[
C_n = \frac{1}{n + 1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}, \quad n \geq 1.
\]  

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The Catalan numbers satisfy the recurrence relation
\begin{equation}
C_{n+1} = \sum_{i=0}^{n} C_i \cdot C_{n-i}, \quad n \geq 0.
\end{equation}

In [11] the authors introduced the notion of the Catalan matrix by inserting the Catalan numbers into a lower triangular Toeplitz matrix. We recall that a Toeplitz matrix is a matrix having constant entries along the diagonals.

The Catalan matrix of order \(n\), denoted by \(C_n[x] = [c_{ij}[x]]\), \(i, j = 1, \ldots, n\), is defined as
\begin{equation}
c_{ij}[x] = \begin{cases} x^{i-j}C_{i-j}, & i - j \geq 0, \\ 0, & i - j < 0, \end{cases}
\end{equation}
where \(C_n\) denotes the \(n\)th Catalan number.

In the literature many generalizations of Catalan numbers have been introduced. We use the following generalization of Catalan numbers (see, for example [7,9]), depending on the integer parameters \(a\) and \(b\):
\begin{equation}
C_n(a, b) = \frac{a}{a+bn} \binom{a+bn}{n}, \quad n \geq 0.
\end{equation}

The identity \(C_0(a, b) = 1\) evidently holds in the case \(a \neq 0\), and also we assume \(C_0(0, b) = 1\) throughout the paper. In the case \((a, b) = (1, 2)\) the generalized Catalan numbers reduce to Catalan numbers.

The convolution formula for generalized Catalan numbers [9]
\begin{equation}
\sum_{i=0}^{n} C_{i}(a, b)C_{n-i}(c, b) = C_n(a + c, b)
\end{equation}
represents a very useful recurrence relation. This formula is also known as the Hagen–Rothe convolution (see [7]).

In the following definition we introduce the concept of generalized Catalan matrix, which is obtained by arranging the generalized Catalan numbers (1.4) along the diagonals of a lower triangular Toeplitz matrix. In order to ensure the existence of the generalized Catalan matrix and of its powers with nonnegative exponents, throughout the paper we assume the default conditions \(a > 0, b \geq 0\) for the parameters \(a\) and \(b\).

**Definition 1.1.** Given the integers \(a > 0\) and \(b \geq 0\), the generalized Catalan matrix of order \(n\), denoted by \(c_{n}[a, b; x] = [c_{ij}[a, b; x]]\), \(i, j = 1, \ldots, n\), is the matrix having elements
\begin{equation}
c_{ij}[a, b; x] = \begin{cases} x^{i-j}C_{i-j}(a, b), & i - j \geq 0, \\ 0, & i - j < 0. \end{cases}
\end{equation}

The Catalan matrix [11] is the particular generalized Catalan matrix defined in Definition 1.1, in the case where \(a = 1, b = 2\).

**Example 1.1.** The generalized Catalan matrix \(c_5[a, b; x]\), \(a > 0, b \geq 0\), of order 5 is equal to
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
ax & 1 & 0 & 0 & 0 \\
\frac{1}{2}a(a+2b-1)x^2 & ax & 1 & 0 & 0 \\
\frac{1}{8}a(a+3b-2)(a+3b-1)x^3 & \frac{1}{2}a(a+2b-1)x^2 & ax & 1 & 0 \\
\frac{1}{24}a(a+4b-3)(a+4b-2)(a+4b-1)x^4 & \frac{1}{8}a(a+3b-2)(a+3b-1)x^3 & \frac{1}{2}a(a+2b-1)x^2 & ax & 1
\end{bmatrix}.
\]

The generalized Pascal matrix, denoted by \(P_n[x] = [p_{ij}[x]]\), \(i, j = 1, \ldots, n\), is defined by
\begin{equation}
p_{ij}[x] = \begin{cases} x^{i-j}(i-1)(j-1), & i - j \geq 0, \\ 0, & i - j < 0. \end{cases}
\end{equation}
In the case $x = 1$ we use the simpler notation $\mathcal{P}_n = \mathcal{P}_n[1]$ for the usual Pascal matrix.
Various types of Pascal matrices have been studied in [1–3,15].

The problem of representing the inverse of the Pascal matrix and the inverse of linear combinations of the identity and the Pascal matrix is well studied in the literature. In particular, motivated by a problem from statistics, in [1] it is shown how to invert $I - \lambda \mathcal{P}_n[a]$. The goal of this paper is to extend the above results on Pascal matrices to the case of generalized Catalan matrices. We first provide an explicit representation of the $k$th power of the matrix $c_n[a, b; x]$, $k \geq 0$, which shows that the $k$th power of the generalized Catalan matrix is still a generalized Catalan matrix. Then we obtain an explicit representation of $(I - \lambda c_n[a, b; x])^{-1}$, as well as the expansion of the resolvent $(\lambda I - c_n[a, b; x])^{-1}$ in terms of the Gamma function and the Fox–Wright hypergeometric function. Finally, we construct the matrix $\Delta_n$ satisfying $(I - \lambda c_n[a, b; x])^{-1} = c_n[a, b; x] \circ \Delta_n$, where $\Delta_n$ is a lower triangular Toeplitz matrix with nonzero elements defined in terms of the Gamma function, the Pochhammer function and the Fox–Wright hypergeometric function.

The analogous results valid for Catalan matrices are particular cases of our more general results.

The paper is organized as follows. In Section 2 we recall some definitions and some results on Pascal matrices. Section 3 contains the main results on generalized Catalan matrices. In Section 4 we draw the conclusions.

2. Definitions and motivation

For the sake of completeness we recall the definitions of the generalized hypergeometric functions, the Euler polynomials and the Bernoulli numbers as well as the notion of the Hadamard product of matrices.

The generalized hypergeometric function is given by

\[
pFq(a_1, \ldots, a_p; b_1, \ldots, b_q; \lambda) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{\lambda^k}{k!},
\]

where

\[
(a)_n = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}
\]

is the well–known Pochhammer function (also known as rising factorial notation) and $\Gamma(n)$ is the Euler gamma function (see, for example [10,13]).

The next definition of the Fox–Wright generalization $p \Psi_q$ of the hypergeometric function $pFq$ is frequently used in the literature (see, for example [6,12]):

\[
p \Psi_q \left[ \begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ (\beta_1, B_1), \ldots, (\beta_q, B_q); \\ \lambda \end{array} \right] = p \Psi_q \left[ \begin{array}{c} (\alpha_j, A_j); 1_p; \\ (\beta_j, B_j); 1_q; \\ \lambda \end{array} \right] := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1+A_1k) \cdots \Gamma(\alpha_p+A_pk)}{\Gamma(\beta_1+B_1k) \cdots \Gamma(\beta_q+B_qk)} \frac{\lambda^k}{k!},
\]

where $A_j > 0$, $j = 1, \ldots, p$, $B_j > 0$, $j = 1, \ldots, q$ for suitably bounded values of $|\lambda|$. In particular, when $A_j = 1$, $j = 1, \ldots, p$ and $B_j = 1$, $j = 1, \ldots, q$, we have the following obvious relationship

\[
pFq(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; \lambda) = \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_q)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_p)} \cdot p \Psi_q \left[ \begin{array}{c} (\alpha_j, 1); 1_p; \\ (\beta_j, 1); 1_q; \\ \lambda \end{array} \right].
\]

The Euler polynomials $E_n(x)$ are defined by the generating function (see [4,5])

\[
\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2e^{tx}}{e^t + 1}.
\]

The Bernoulli numbers $B_n$ are defined by (see [4,5])

\[
\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}.
\]
The Hadamard product $A \circ B$ of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the matrix obtained by entry-wise multiplication of matrices $A$ and $B$: $(A \circ B)_{ij} = a_{ij}b_{ij}$.

In [1,14], the inverse of the Pascal matrix is represented as the Hadamard product $P_n^{-1} = P_n \circ I_n$, where $I_n$ is the $n \times n$ lower triangular matrix defined by

$$(I_n)_{ij} = \begin{cases} (-1)^{i-j}, & i \leq j, \\ 0, & i < j. \end{cases}$$

Moreover, the matrix $(P_n + I_n)^{-1}$ is the Hadamard product $P_n \circ \Delta_n$, where $\Delta_n$ is the $n \times n$ lower triangular matrix defined by [14]

$$(\Delta_n)_{ij} = \begin{cases} \frac{i}{2}E_{i-j}(0), & i \leq j, \\ 0, & i < j. \end{cases}$$

or

$$(\Delta_n)_{ij} = \begin{cases} \frac{1}{2}(1-2^{i-j+1})E_{i-j+1}, & i \leq j, \\ 0, & i < j. \end{cases}$$

Aggarwala and Lamoureux [1] have proved that elements $(\Delta_n)_{ij}$ are values of the Dirichlet eta function evaluated at negative integers, or more generally, certain polylogarithm functions evaluated at the number $-1$.

The goal of this paper is to extend the above results on Pascal matrices to the case of generalized Catalan matrices.

These representations might be useful in applications in control engineering, where it is needed to calculate the determinant and adjoint polynomials of the matrix $(\lambda I - A)^{-1}$.

3. Inverting linear combinations of generalized Catalan matrices with identity matrix

In order to accomplish the above described goals of the present article, a formula for the $k$th power of the generalized Catalan matrix is derived.

In the paper [1] authors have shown that the $k$th power of the Pascal matrix is again the Pascal matrix, proving the fact

$$P[s]^k = P[ks],$$

for any integer $k$. In the case $k \geq 0$, we prove that $k$th power of the generalized Catalan matrix $C_n[a, b; x]$, $a > 0, b \geq 0$ satisfies the same property with respect to the first parameter.

**Lemma 3.1.** For arbitrary integer $k \geq 0$, the $k$th power of the generalized Catalan matrix $C_n[a, b; x]$, $a > 0, b \geq 0$ is equal to

$$C_n[a, b; x]^k = C_n[ak, b; x].$$

**Proof.** In the case $k = 0$ it suffices to verify $C_n[a, b; x]^0 = I_n = C_n[0, b; x]$. The statement in the case $k > 0$ can be verified by mathematical induction, where the inductive step follows from the Hagen–Rotte convolution (1.5). \(\square\)

**Example 3.1.** The $k$th power of the generalized Catalan matrix $C_5[a, b; x]$, $a > 0, b \geq 0$, in the case $k \geq 0$, is equal to

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
akx & 1 & 0 & 0 & 0 \\
\frac{1}{2}ak(2b + ak - 1)x^2 & akx & 1 & 0 & 0 \\
\frac{1}{2}ak(3b + ak - 2)(3b + ak - 1)x^3 & \frac{1}{2}ak(2b + ak - 1)x^2 & akx & 1 & 0 \\
\frac{1}{2}ak(4b + ak - 3)(4b + ak - 2)(4b + ak - 1)x^4 & \frac{1}{2}ak(3b + ak - 2)(3b + ak - 1)x^3 & \frac{1}{2}ak(2b + ak - 1)x^2 & akx & 1
\end{bmatrix}$$
The following theorem contains our main result: explicit inversion of the linear combination $I - \lambda C_n[a, b; x]$.

**Theorem 3.1.** For an arbitrary small value of the parameter $\lambda$ satisfying $|\lambda| < 1/\|C_n[a, b; x]\|$ and selected integers $i, j \leq n$, an arbitrary $(i, j)$th element of the inverse $(I - \lambda C_n[a, b; x])^{-1}$, $a > 0$, $b \geq 0$, can be expressed as in the following

$$(I - \lambda C_n[a, b; x])^{-1}_{ij} = \begin{cases} \delta_{ij} + x^{i-j} \cdot \frac{\lambda}{(i-j)!} \cdot 2\Psi_1 \left[ (2, 1), (a + b(i-j), a); (a + (b-1)(i-j) + 1, a); \lambda \right], & i \geq j, \\ 0, & i < j, \\ \sum_{k=0}^{\infty} \lambda^k \cdot x^{i-j} \cdot \frac{a}{(i-j)!} \sum_{k=0}^{\infty} (a + (i-j)b + ak - (i-j-1))_{i-j-1} (k+1)\lambda^k, \quad & i = j, \\ 0, & i \geq j + 1, \\ \sum_{k=0}^{\infty} \lambda^k, & i < j. \end{cases} \quad (3.2)$$

where $\delta_{ij}$ is the Kronecker delta symbol.

**Proof.** It is known that if $\| \cdot \|$ is a matrix norm and if $\|A\| < 1, A \in \mathbb{R}^{n \times n}$, then $I - A$ is invertible and $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$ (see, for example [8]). The generalized Catalan matrix $C_n[a, b; x]$ has the finite matrix norm, so for any value of the parameter $\lambda$ satisfying $|\lambda| < 1/\|C_n[a, b; x]\|$ we have $\|\lambda C_n[a, b; x]\| < 1$, and it is possible to express the inverse of the matrix $I - \lambda C_n[a, b; x]$ as the following infinite matrix sum

$$(I - \lambda C_n[a, b; x])^{-1} = \sum_{k=0}^{\infty} \lambda^k C_n[a, b; x]^k.$$  

In view of (3.1) we have

$$(I - \lambda C_n[a, b; x])^{-1}_{ij} = \begin{cases} \delta_{ij} + x^{i-j} \sum_{k=1}^{\infty} \lambda^k C_{i-j}(ka, b), & i \geq j, \\ 0, & i < j. \end{cases} \quad (3.4)$$

Using the following relation in the case $i \geq j, k \geq 1$

$$\frac{x^{i-j} C_{i-j}(ka, b)}{C_{i-j}(a, b)} = \frac{x^{i-j} \cdot \frac{ka}{ka+b(i-j)}}{C_{i-j}(a, b)} \cdot \frac{\lambda}{(i-j)!} \cdot \frac{(ka+b(i-j))!(ka+b(i-j)-1)!}{(ka+b(i-j)+1)!(ka+b(i-j)-1)!},$$

and later combining (3.4) and (3.5) we immediately obtain

$$(I - \lambda C_n[a, b; x])^{-1}_{ij} = \delta_{ij} + C_{i-j}(a, b) x^{i-j} \sum_{k=1}^{\infty} \lambda^k \frac{(a + b(i-j))_{(k-1)a}}{(a + (b-1)(i-j) + 1)_{(k-1)a}},$$

$$= \delta_{ij} + C_{i-j}(a, b) x^{i-j} \sum_{k=1}^{\infty} \frac{k!}{(a + (b-1)(i-j) + 1)_{(k-1)a}} \frac{(a + b(i-j))_{(k-1)a}}{(k-1)!} \lambda^{k-1},$$

$$= \delta_{ij} + C_{i-j}(a, b) x^{i-j} \sum_{k=1}^{\infty} \frac{(2)_{k-1} (a + b(i-j))_{(k-1)a}}{(a + (b-1)(i-j) + 1)_{(k-1)a}} \lambda^{k-1}. \quad (3.5)$$

The last equation can be later written as
We now continue the transformations using

\[
\sum_{k=0}^{\infty} \frac{(2)_k}{(a + (b - 1)(i - j) + 1)_k} \frac{\lambda^k}{k!} = \frac{\Gamma(a + (b - 1)(i - j) + 1)}{\Gamma(2) \Gamma(a + b(i - j))} \sum_{k=0}^{\infty} \frac{\Gamma(2 + k) \Gamma(a + b(i - j) + ak)}{\Gamma(a + (b - 1)(i - j) + 1 + ak)} \frac{\lambda^k}{k!}
\]

which leads to

\[
(I - \lambda c_n[a, b; x])^{-1}_{ij} = \delta_{ij} + x^{i-j} \cdot C_{j-i}(a, b) \cdot \lambda \cdot \frac{\Gamma(a + (b - 1)(i - j) + 1)}{\Gamma(a + b(i - j))} \cdot 2 \Psi_1 \left[ \begin{array}{c} (2, 1), (a + b(i - j), a); \\ (a + (b - 1)(i - j) + 1, a); \\ \lambda \end{array} \right], \quad (3.6)
\]

in the case \( i \geq j \). Now, the confirmation of the first case in (3.2) follows from the following transformations

\[
C_{j-i}(a, b) \cdot \frac{\Gamma(a + (b - 1)(i - j) + 1)}{\Gamma(a + b(i - j))} \cdot \frac{\Gamma(a + b(i - j) - (i - j - 1))}{\Gamma(a + b(i - j))} = \frac{\Gamma(a + b(i - j) - (i - j - 1))}{\Gamma(a + b(i - j))},
\]

which can be verified in both cases \( i = j \) and \( i \geq j + 1 \) using well-known relations \( \Gamma(z + 1) = z \Gamma(z) \) and \( \Gamma(n) = (n - 1)! \) for an arbitrary positive integer \( n \). The proof can be completed by verifying (3.3) by means of the identity

\[
2 \Psi_1 \left[ \begin{array}{c} (2, 1), (a + b(i - j), a); \\ (a + (b - 1)(i - j) + 1, a); \\ \lambda \end{array} \right] = \left\{ \begin{array}{ll} \frac{1}{a} \sum_{k=0}^{\infty} \lambda^k, & i = j, \\ \sum_{k=0}^{\infty} (a + (i - j)b + ak - (i - j - 1))_{i-j-1}(k + 1)\lambda^k, & i \geq j + 1. \end{array} \right. \quad (3.7)
\]

which is valid in the case \( i \geq j \). \( \square \)

**Example 3.2.** Using the results of Theorem 3.1 we compute the inverse of the matrix \( I - \lambda C_3[a, b; x] \) in the case \( |\lambda| < 1/\|C_3[a, b; x]\| \). Nonzero entries of the inverse matrix are as follows

\[
(I - \lambda C_3[a, b; x])^{-1}_{ij} = \sum_k \lambda^k,
\]

\[
(I - \lambda C_3[a, b; x])^{-1}_{j+1,i} = x\lambda a \sum_k (k + 1)\lambda^k,
\]

\[
(I - \lambda C_3[a, b; x])^{-1}_{j+2,i} = x^2 \lambda a \frac{1}{2} \left( (a + 2b - 1) \sum_k (k + 1)\lambda^k + a\lambda \sum_k (k + 1)(k + 2)\lambda^k \right).
\]

All three sums included in the right hand sides in (3.8) are convergent in the case \( |\lambda| < 1 \). Using

\[
\sum_k \lambda^k \to \frac{1}{1 - \lambda}, \quad \sum_k (k + 1)\lambda^k \to \frac{1}{(1 - \lambda)^2}, \quad \sum_k (k + 1)(k + 2)\lambda^k \to \frac{2}{(1 - \lambda)^3},
\]

We get
and applying necessary algebraic transformations, it is possible to verify that the inverse is equal to
\[
(I - \lambda C_n a, b; x)^{-1} = \begin{bmatrix}
\frac{1}{1 - \lambda} & 0 & 0 \\
\frac{\lambda \alpha}{x(1 - \lambda) \gamma} & 1 - \lambda & 0 \\
\frac{x^2 \lambda a(\alpha a + 2i - 2b + 2a + 1)}{2(1 - \lambda)^2} & \frac{\lambda \alpha}{x(1 - \lambda) \gamma} & 1 - \lambda
\end{bmatrix}
\]
for every parameter \( \lambda \) satisfying \(|\lambda| < \min\{1, 1/\|C_n a, b; x\|\} \).

In the particular case \( a = 1, b = 2 \) we obtain the following result for the Catalan matrix.

**Corollary 3.1.** An arbitrary \((i,j)\)th element of the inverse \((I - \lambda C_n a, b; x)^{-1}\) is given by
\[
(I - \lambda C_n a, b; x)^{-1}_{ij} = \begin{cases}
\delta_{ij} + x^{i-j} \cdot C_{i-j} \cdot \lambda \cdot 2F_1(2, 2(i-j) + 1; i - j + 2; \lambda), & i \geq j, \\
0, & i < j.
\end{cases}
\]

for every parameter \( \lambda \) satisfying \(|\lambda| < 1/\|C_n a, b; x\|\) and \(i,j \leq n\).

Applying results of Theorem 3.1 we obtain the next explicit representation for the resolvent of the generalized Catalan matrix.

**Corollary 3.2.** For any parameter \( \lambda \) satisfying \(|\lambda| > \|C_n a, b; x\|\), positive integers \( i,j \leq n \) and parameters \( a > 0, b \geq 0 \), an arbitrary \((i,j)\)th element of the resolvent \((\lambda I - C_n a, b; x)^{-1}\) can be expressed as in the following
\[
(\lambda I - C_n a, b; x)^{-1}_{ij} = \begin{cases}
\frac{1}{\lambda} \delta_{ij} + x^{i-j} \cdot C_{i-j} \cdot \frac{2F_1(2, 2(i-j) + 1; i - j + 2; \frac{1}{\lambda})}{\Gamma(\alpha a + 2i - 2b + 2a + 1)} \cdot 2\Psi_1((a + b - 1)(i-j) + 1, a), & i \geq j, \\
0, & i < j.
\end{cases}
\]

**Proof.** Since the condition \( \|\frac{1}{\lambda} C_n a, b; x\| < 1 \) is satisfied, the proof follows from
\[
(\lambda I - C_n a, b; x)^{-1} = \frac{1}{\lambda} \left( I - \frac{1}{\lambda} C_n a, b; x \right)^{-1}
\]
and Theorem 3.1. \( \Box \)

**Corollary 3.3.** An arbitrary \((i,j)\)th element of the inverse \((\lambda I - C_n a, b; x)^{-1}\) can be expressed explicitly by
\[
(\lambda I - C_n a, b; x)^{-1}_{ij} = \begin{cases}
\frac{1}{\lambda} \delta_{ij} + x^{i-j} \cdot C_{i-j} \cdot \frac{2F_1(2, 2(i-j) + 1; i - j + 2; \frac{1}{\lambda})}{\Gamma(\alpha a + 2i - 2b + 2a + 1)} \cdot 2\Psi_1((a + b - 1)(i-j) + 1, a), & i \geq j, \\
0, & i < j.
\end{cases}
\]

for any parameter \( \lambda \) satisfying \(|\lambda| > \|C_n a, b; x\|\) and for indices \( i,j \leq n \).

Now we find the matrix \( \Delta_n \) satisfying \((I - \lambda C_n a, b; x)^{-1} = C_n a, b; x \circ \Delta_n\).

**Theorem 3.2.** For the parameter \( \lambda \) satisfying \(|\lambda| < \|C_n a, b; x\|\) and the parameters \( a > 0, b \geq 0 \) the inverse \((I - \lambda C_n a, b; x)^{-1}\) can be expressed as
\[
(I - \lambda C_n a, b; x)^{-1} = C_n a, b; x \circ \Delta_n,
\]
where
\[
(\Delta_n)_{ij} = \begin{cases}
\lambda^{1-\delta_{ij}} \cdot \frac{\Gamma(a + b - 1)(i-j) + 1)}{\Gamma(a + b(i-j))} \cdot 2\Psi_1((a + b - 1)(i-j) + 1, a), & i \geq j, \\
0, & i < j.
\end{cases}
\]
Proof. In the case $i > j$ the proof immediately follows from identity (3.6) of Theorem 3.1. On the other hand, in the case $i = j$, formula (3.11) reduces to
\[
(\triangle_n)_{i,i} = \lambda^1 \delta_{i,i} \cdot \frac{\Gamma(a + 1)}{\Gamma(a)} \cdot 2\Psi_1 \left[\begin{array}{c}
(2,1), (a,a); \\
(a+1, a);
\end{array}\right].
\]

Now, the identities
\[
(\triangle_n)_{i,i} = \sum_{k=0}^{\infty} \lambda^k
\]
(3.12)
\[
(I - \lambda C_n[a, b; x])^{-1}_{i,i} = \sum_{k=0}^{\infty} \lambda^k
\]
(3.13)
follow from (3.7) and (3.3), respectively. Thus, the identity
\[
(I - \lambda C_n[a, b; x])^{-1}_{i,i} = c_{ij}[a, b; x] \cdot (\triangle_n)_{i,i}
\]
is true, since $c_{ij}[a, b; x] = 1$. □

4. Conclusion

We have introduced the definition of generalized Catalan matrix $C_n[a, b; x]$, as a lower triangular Toeplitz matrix whose nonzero entries involve the generalized Catalan numbers.

Motivated by the main result in [1], where the linear combination $I - \lambda P_n[x]$ of the Pascal matrix with the identity matrix is inverted, in the present paper we invert linear combinations $I - \lambda C_n[a, b; x]$ and $\lambda I - C_n[a, b; x]$ of the generalized Catalan matrix with the identity matrix. In particular, the inverse of various linear combinations of the identity and the generalized Catalan matrices are expressed in terms of the generalized Catalan numbers, the Euler Gamma function and the Fox–Wright hypergeometric function.

Analogous results for the Catalan matrix, shown in [11], are derived as corollaries.

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