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The classical adjoint

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Abstract

This paper summarizes the historical background of the notion of the classical adjoint as outlined by Muir, and provides applications of the adjoint to various studies of generalized invertibility of matrices over commutative rings. Specifically, in this setting, the classical adjoint is used to provide a novel proof of von Neumann's 1936 observation that every matrix over a regular ring is regular, and to provide a necessary and sufficient condition for the existence of the Moore–Penrose inverse of a given matrix. In particular, a representation of the Moore–Penrose inverse is given that leads to an immediate proof of Moore's 1920 formula specifying the entries of his "reciprocal" in terms of determinants.
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Given a square matrix A, the transposed matrix of cofactors of A is called the *classical adjoint* A^{ad} of A. Specifically, the (i, j) entry of the adjoint of an $n \times n$ matrix A is

$$(A^{\mathrm{ad}})_{ij} = (-1)^{i+j}|A_{\hat{j}\hat{i}}|,$$

where $|A_{\hat{j}\hat{i}}|$ is the determinent of the $(n-1) \times (n-1)$ submatrix of A obtained by the deletion of row j and column i. (See, for example, [9, p. 20] or [10, p. 16].) If $n = 1$, then A^{ad} is the 1×1 identity matrix. The classical adjoint is sometimes called the adjugate of A and is often denoted by adj A.

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The most familiar property of the adjoint is that

$$A^{\mathrm{ad}}A = |A|I = AA^{\mathrm{ad}},$$

which is obtained by use of the Laplace expansion of the determinant $|A|$ of A. In particular, A is invertible iff $|A|$ is invertible, and in this case,

$$A^{-1} = |A|^{-1}A^{\mathrm{ad}}.$$

The purpose of this paper is to provide some insight into the historical background and extended role of the adjoint. In particular, we offer some applications of the adjoint that are not standard fare, yet many are accessible to students of linear algebra. Many of the results are not new, but the use of the classical adjoint in some of the proofs offers a different perspective.

1. Historical sketch

Information about the early history of the classical adjoint is provided by Muir. Specifically, Muir's first reference to the idea is given in pages 64–65 of [13], where he reviews some observations about quadratic forms from the fifth chapter of Gauss' 1801 *Disquisitiones Arithmeticae*. In particular, Gauss considered the ternary quadratic form

$$axx + a'x'x' + a''x''x'' + 2bx'x'' + 2b'xx'' + 2b''xx'$$

in the variables x, x', x''. He denoted this form by

$$f = \begin{pmatrix} a & a' & a'' \\ b & b' & b'' \end{pmatrix},$$

and then introduced another form

$$F = \begin{pmatrix} A & A' & A'' \\ B & B' & B'' \end{pmatrix},$$

which he called the *adjunctam* of f, by defining

$$\begin{array}{lll} A = bb - a'a'' & A' = b'b' - aa'' & A'' = b''b'' - aa' \\ B = ab - b'b'' & B' = a'b' - bb'' & B'' = a''b'' - bb'. \end{array}$$

Moreover, he defined what he termed the *determinantem* of the form f as

$$D = abb + a'b'b' + a''b''b'' - aa'a'' - 2bb'b''.$$

He also observed that the determinantem of F is the square of D, and that the adjunctam of F is D times the form f.

Today we would usually represent the quadratic form f by the symmetric matrix

$$A = \begin{pmatrix} a & b'' & b' \\ b'' & a' & b \\ b' & b & a'' \end{pmatrix},$$

and note that

$$A^{\mathrm{ad}} = \begin{pmatrix} \begin{vmatrix} a' & b \\ b & a'' \end{vmatrix} & -\begin{vmatrix} b'' & b' \\ b & a'' \end{vmatrix} & \begin{vmatrix} b'' & b' \\ a' & b \end{vmatrix} \\ -\begin{vmatrix} b'' & b \\ b' & a'' \end{vmatrix} & \begin{vmatrix} a & b' \\ b' & a'' \end{vmatrix} & -\begin{vmatrix} a & b' \\ b'' & b \end{vmatrix} \\ \begin{vmatrix} b'' & a' \\ b' & b \end{vmatrix} & -\begin{vmatrix} a & b'' \\ b' & b \end{vmatrix} & \begin{vmatrix} a & b'' \\ b'' & a' \end{vmatrix} \end{pmatrix}.$$

Consequently, $|A|$ and A^{ad} are, respectively, the negatives of the determinanten and adjunctam of Gauss.

A few years later, on 30 November 1812, both Binet and Cauchy read before the Institut de France their respective memoirs on the subject of determinants [13, p. 80 and 92]. Both are credited with a useful result, now called the Binet–Cauchy theorem, for evaluating the determinant of a $k \times k$ submatrix of the product $P = AB$ in terms of determinants of $k \times k$ submatrices of A and B. Specifically, in today's notation,

$$|P_{\alpha\beta}| = \sum_{\gamma \in \mathscr{Q}_{k,n}} |A_{\alpha\gamma}|\,|B_{\gamma\beta}|.$$

(See, for example, [9, p. 22] or [10, p. 14].) Here, $\mathscr{Q}_{k,n}$ is the totality of lists $\gamma = (\gamma(1), \ldots, \gamma(k))$ of integers with $1 \leqslant \gamma(1) < \cdots < \gamma(k) \leqslant n$; and, for A an $m \times n$ matrix, $\alpha \in \mathscr{Q}_{k,m}$, and $\gamma \in \mathscr{Q}_{k,n}$, $A_{\alpha\gamma}$ is the $k \times k$ submatrix of A determined by the entries in rows $\alpha(1), \ldots, \alpha(k)$ and columns $\gamma(1), \ldots, \gamma(k)$.

Alternatively, suppose that A is $m \times n$, B is $n \times p$ and $\mathscr{Q}_{k,m}$, $\mathscr{Q}_{k,n}$, and $\mathscr{Q}_{k,p}$ are each ordered, say, lexiographically. Then by defining the kth compound of A to be the $\binom{m}{k} \times \binom{n}{k}$ matrix $C_k(A)$ with (α, β) entry $|A_{\alpha\beta}|$, the Binet–Cauchy theorem may be expressed as

$$C_k(AB) = C_k(A)C_k(B).$$

(See [9, p. 20] or [10, p. 17].)

Cauchy also included in his paper a formula that is presently viewed as an evaluation of the determinant of a matrix via the elements of a row and a column. Specifically, in today's notation, if $A = (a_{ij})$ is $n \times n$, $b = (b_1, \ldots, b_n)^{\mathrm{T}}$, $c = (c_1, \ldots, c_n)$, and d is a scalar, he showed that

$$\left|\begin{pmatrix} A & b \\ c & d \end{pmatrix}\right| = |A|d - \sum_{i=1}^{n} \sum_{j=1}^{n} c_i \left((-1)^{i+j}|A_{\hat{j}\hat{i}}|\right) b_j.$$

He called the system $((-1)^{i+j}|A_{\hat{j}\hat{i}}|)$ *adjoint* to the system (a_{ij}) [13, pp. 104–105]. Since his definition of adjoint agrees with current usage, Cauchy's result may be stated simply as

$$\begin{vmatrix} A & b \\ c & d \end{vmatrix} = |A|d - cA^{\mathrm{ad}}b.$$

The determinantal rank of a matrix is defined to be the size of a largest nonzero minor of the matrix ([9, p. 13] or [10, p. 27]). Thus, if A is $r \times r$ and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is of determinantal rank at most r, since Cauchy's result applies to every $(r+1) \times (r+1)$ submatrix that includes A,

$$0 = |A|D - CA^{\mathrm{ad}}B.$$

In particular, if A is invertible, then $D = CA^{-1}B$.

Cauchy also noted, again in more modern notation, that if A is an $n \times n$ matrix, then $|A^{\mathrm{ad}}| = |A|^{n-1}$ and $(A^{\mathrm{ad}})^{\mathrm{ad}} = |A|^{n-2}A$ [13, p. 110]. These two results generalize the above-mentioned observations of Gauss for the case $n = 3$.

2. Left/right invertibility

Throughout this paper, $\mathscr{R}$ is understood to be a commutative ring with $1 \neq 0$. An $m \times n$ matrix over $\mathscr{R}$ is said to be right invertible if there exists an $n \times m$ matrix B over $\mathscr{R}$ such that $AB = I_m$. In this case, B is called a right inverse of A. Left invertibility is analogously defined. If $m \neq n$, then A cannot be both right and left invertible. In particular, if $m < n$, then $BA = I_n$ implies $n = \operatorname{rank} BA \leqslant \operatorname{rank} A \leqslant m < n$, which is impossible; that is, A cannot be left invertible. However, A may be right invertible.

Several characterizations of right invertibility are available. (See, for example, [8]; compare also [7, p. 141] and [18, p. 932].) By use of the classical adjoint, we provide below a condition that is both necessary and sufficient for the existence of a right inverse. We also show that every right inverse of an $m \times n$ matrix A over $\mathscr{R}$ is expressible as a linear combination of $\binom{n}{m}$ matrices described by the adjoints of the submatrices of A of order m.

Let A be $m \times n$ over $\mathscr{R}$, $m \leqslant n$, and $\beta \in \mathscr{Q}_{m,n}$. Let Q_β be the $n \times m$ matrix with 1 in positions $(\beta(1), 1), \ldots, (\beta(m), m)$ and 0 elsewhere, and let $\mu = (1, \ldots, m)$ be the sole list of $\mathscr{Q}_{m,m}$. Then $AQ_\beta = A_{\mu,\beta}$.

Lemma 2.1. *Let A be an $m \times n$ matrix over $\mathscr{R}$ with $m \leqslant n$. Then A is right invertible iff* 1 *is in the ideal generated by the determinants of the $m \times m$ submatrices of A.*

Proof. Let $AB = I_m$. By the Binet–Cauchy theorem,

$$1 = |I_m| = |AB| = |(AB)_{\mu\mu}| = \sum_{\beta \in \mathscr{Q}_{m,n}} |A_{\mu\beta}|\,|B_{\beta\mu}|$$

and 1 is in the ideal generated by the elements $|A_{\mu\beta}|$, $\beta \in \mathscr{Q}_{m,n}$.

Conversely, suppose that

$$1 = \sum_{\beta \in \mathscr{Q}_{m,n}} b_\beta |A_{\mu\beta}|$$

for some $b_\beta \in \mathscr{R}$. Then

$$\begin{aligned} A\left(\sum_{\beta \in \mathscr{Q}_{m,n}} b_\beta Q_\beta A^{\mathrm{ad}}_{\mu\beta}\right) &= \sum_{\beta \in \mathscr{Q}_{m,n}} b_\beta A Q_\beta A^{\mathrm{ad}}_{\mu\beta} = \sum_{\beta \in \mathscr{Q}_{m,n}} b_\beta A_{\mu\beta} A^{\mathrm{ad}}_{\mu\beta} \\ &= \sum_{\beta \in \mathscr{Q}_{m,n}} b_\beta |A_{\mu\beta}| I_m = \left(\sum_{\beta \in \mathscr{Q}_{m,n}} b_\beta |A_{\mu\beta}|\right) I_m \\ &= 1 I_m = I_m, \end{aligned}$$

and $\sum_{\beta \in \mathscr{Q}_{m,n}} b_\beta Q_\beta A^{\mathrm{ad}}_{\mu\beta}$ is a right inverse of A. □

(For alternative proofs of Lemma 2.1, see [8; 17, pp. 21–22].)

Example 2.1. Let $\mathscr{R}$ be the ring $\mathbb{Z}$ of integers and let

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 4 & 3 & 1 & 3 \end{pmatrix} \in \mathbb{Z}^{2\times 4}.$$

With respect to the notation associated with Lemma 2.1, $m = 2$, $n = 4$, $\mu = (1, 2)$, $\mathscr{Q}_{2,4} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$, and

$$\begin{array}{rccc} A_{\mu\beta}: & \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}, & \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}, & \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}, \\ & \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}, & \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \\ |A_{\mu\beta}|: & 2, & 2, & 2, \\ & 1, & 0, & -1 \\ Q_\beta A^{\mathrm{ad}}_{\mu\beta}: & \begin{pmatrix} 3 & -1 \\ -4 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -4 & 2 \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 3 & -1 \\ 0 & 0 \\ 0 & 0 \\ -4 & 2 \end{pmatrix}, \\ & \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -3 & 1 \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 \\ 3 & -1 \\ 0 & 0 \\ -3 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 3 & -1 \\ -1 & 0 \end{pmatrix}. \end{array}$$

The choice

$$\begin{array}{rccc} b_\beta: & 1, & 1, & 1, \\ & -6, & 2, & -1 \end{array}$$

is such that

$$\sum_{\beta \in \mathcal{Q}_{2,4}} b_\beta |A_{\mu\beta}| = 1 \cdot 2 + 1 \cdot 2 + 1 \cdot 2 + (-6) \cdot 1 + 2 \cdot 0 + (-1)(-1) = 1.$$

Consequently, 1 is in the ideal generated by the $|A_{\mu\beta}|$'s, and

$$B = \sum_{\beta \in \mathcal{Q}_{2,4}} b_\beta Q_\beta A^{\text{ad}}_{\mu\beta} = \begin{pmatrix} 7 & -2 \\ -4 & 0 \\ 11 & -3 \\ -9 & 4 \end{pmatrix}$$

is a right inverse of A. $\square$

The choice of the b_β's in Example 2.1 is clearly not unique. The interested reader may wish to select other appropriate lists, and use them to construct other right inverses of A.

A continuation of Example 2.1 provides an illustration of the next lemma. Specifically, the right inverse B of the example is such that

$$|B_{\beta\mu}|: \quad -8, \quad 1, \quad 10, \quad 12, \quad -16, \quad 17.$$

Direct calculation gives

$$\sum_{\beta \in \mathcal{Q}_{2,4}} |A_{\mu\beta}||B_{\beta\mu}| = 1 \quad \text{and} \quad B = \sum_{\beta \in \mathcal{Q}_{2,4}} |B_{\beta\mu}| Q_\beta A^{\text{ad}}_{\mu\beta}.$$

That is, the right inverse B is expressible as a linear combination of the matrices $Q_\beta A^{\text{ad}}_{\mu\beta}$ with coefficients $|B_{\beta\mu}|$. This observation is now shown to hold in general.

Lemma 2.2. *Let A be $m \times n$, B be $n \times m$, $1 < m \leqslant n$, and $AB = I_m$. Then*

$$B = \sum_{\beta \in \mathcal{Q}_{m,n}} |B_{\beta\mu}| Q_\beta A^{\text{ad}}_{\mu\beta}.$$

Proof. For convenience, let $\sum_{\beta \in \mathcal{Q}_{m,n}}$ be abbreviated as simply $\sum_\beta$. Also, let $i \in \beta$ mean that $i = \beta(k)$ for some index k; in that case, let this unique index be denoted by $k = i(\beta)$. Finally, let $\beta\backslash i$ denote the list of $\mathcal{Q}_{m-1,n}$ obtained by the deletion of i from the list $\beta \in \mathcal{Q}_{m,n}$.

Let $B = (b_{ij})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. Since $(Q_\beta A^{\text{ad}}_{\mu\beta})_{ij}$ is 0 if $i \notin \beta$ and is $(A^{\text{ad}}_{\mu\beta})_{i(\beta),j}$ if $i \in \beta$, the (i, j) entry of the right-hand side of the desired equality is

$$\begin{aligned} \left(\sum_\beta |B_{\beta\mu}| Q_\beta A^{\text{ad}}_{\mu\beta} \right)_{ij} &= \sum_{\substack{\beta \\ i \in \beta}} |B_{\beta\mu}| (A^{\text{ad}}_{\mu\beta})_{i(\beta),j} \\ &= \sum_{\substack{\beta \\ i \in \beta}} \left(\sum_{k=1}^{m} (-1)^{i(\beta)+k} b_{ik} |B_{\beta\backslash i, \hat{k}}| \right) \\ &\quad \times (-1)^{i(\beta)+j} |(A_{\mu\beta})_{\hat{j}, \beta\backslash i}|. \end{aligned}$$

Also, since there is a natural bijection between the $\beta \in \mathcal{Q}_{m,n}$ with $i \in \beta$ and the $\beta' \in \mathcal{Q}_{m-1,n}$ with $i \notin \beta$, and since $(A_{\mu\beta})_{\hat{j}\beta'} = A_{\hat{j}\beta'}$ whenever β corresponds to β' under this bijection, the preceding sum is

$$\sum_{\substack{\beta' \\ i\notin\beta'}} \left(\sum_{k=1}^{m} (-1)^k b_{ik} |B_{\beta',\hat{k}}| \right) (-1)^j |A_{\hat{j}\beta'}|.$$

But, for every $\gamma \in \mathcal{Q}_{m-1,n}$, if $i \in \gamma$, then $\sum_{k=1}^{m} (-1)^k b_{ik} |B_{\gamma\hat{k}}| = 0$. Thus, the preceding sum is

$$\begin{aligned}
&\sum_{\gamma\in\mathcal{Q}_{m-1,n}} \left(\sum_{k=1}^{m} (-1)^k b_{ik} |B_{\gamma\hat{k}}| \right) (-1)^j |A_{\hat{j}\gamma}| \\
&\quad = \sum_{k=1}^{m} b_{ik} (-1)^{j+k} \left(\sum_{\gamma\in\mathcal{Q}_{m-1,n}} |A_{\hat{j}\gamma}|\, |B_{\gamma\hat{k}}| \right) \\
&\quad = \sum_{k=1}^{m} b_{ik} (-1)^{j+k} |(AB)_{\hat{j}\hat{k}}| = \sum_{k=1}^{m} b_{ik} (-1)^{j+k} |(I_m)_{\hat{j}\hat{k}}| \\
&\quad = \sum_{k=1}^{m} b_{ik} (I_m^{\mathrm{ad}})_{kj} = \sum_{k=1}^{m} b_{ik} (I_m)_{kj} = b_{ij}. \qquad \square
\end{aligned}$$

If $m = n$, then the statement of Lemma 2.1 reduces to the familiar result that A is right invertible iff $|A|$ is invertible. Moreover, in that case, if $AB = I_m$, then $|A||B| = 1$ and $B = A^{-1} = |A|^{-1} A^{\mathrm{ad}} = |B| A^{\mathrm{ad}}$. This latter observation may be viewed as a special case of Lemma 2.2.

We conclude this section by combining the preceding two lemmas into a single theorem. (Compare [17, pp. 22–23].)

Theorem 2.1. *Let A be $m \times n$, B be $n \times m$, $m \leqslant n$, and $\mu = (1, \ldots, m) \in \mathcal{Q}_{m,m}$. Then $AB = I_m$ iff*

$$|AB| = 1 \quad \textit{and} \quad B = \sum_{\beta\in\mathcal{Q}_{m,n}} |B_{\beta\mu}| Q_\beta A^{\mathrm{ad}}_{\mu\beta}.$$

Proof. If $AB = I_m$, then $|AB| = |I_m| = 1$ and, for $m > 1$, by Lemma 2.2, B is as expressed. Also, if $m = 1$, since $\mathcal{Q}_{1,n} = \{(1), \ldots, (n)\}$, $Q_{(s)}$ is the $n \times 1$ column with 1 in position s and 0 elsewhere, $\mu = (1)$, and the adjoint of a 1×1 matrix is the 1×1 identity, then

$$\sum_{\beta\in\mathcal{Q}_{1,n}} |B_{\beta(1)}| Q_\beta A^{\mathrm{ad}}_{\mu\beta} = \sum_{s=1}^{n} |B_{(s)(1)}| Q_{(s)} = \sum_{s=1}^{n} b_{s1} Q_{(s)} = B.$$

Conversely, if the conditions are satisfied, then

$$
\begin{aligned}
AB &= A\left(\sum_{\beta} |B_{\beta\mu}| Q_\beta A^{\mathrm{ad}}_{\mu\beta}\right) \\
&= \sum_{\beta} |B_{\beta\mu}|(AQ_\beta)A^{\mathrm{ad}}_{\mu\beta} = \sum_{\beta} |B_{\beta\mu}| A_{\mu\beta} A^{\mathrm{ad}}_{\mu\beta} \\
&= \sum_{\beta} |B_{\beta\mu}|(|A_{\mu\beta}| I_m) = \left(\sum_{\beta} |A_{\mu\beta}|\,|B_{\beta\mu}|\right) I_m \\
&= |(AB)_{\mu\mu}| I_m = |AB| I_m = 1 I_m = I_m. \qquad \square
\end{aligned}
$$

The results of this section have emphasized the consideration of right inverses. Analogous results hold for left inverses. In particular, for $\alpha \in \mathcal{Q}_{m,n}$, if P_α is the $m \times n$ matrix with 1 in positions $(1, \alpha(1)), \ldots, (m, \alpha(m))$ and 0 elsewhere, then by arguments similar to the preceding, $AB = I_m$ iff

$$
|AB| = 1 \quad \text{and} \quad A = \sum_{\alpha \in \mathcal{Q}_{m,n}} |A_{\mu\alpha}| B^{\mathrm{ad}}_{\alpha\mu} P_\alpha.
$$

3. Regular matrices

An element a of a ring is said to be *regular* in the sense of von Neumann if there is an element b of the ring such that $aba = a$. More generally, a matrix A is said to be *regular* in the sense of von Neumann if there is a matrix G such that $AGA = A$. In that case, G is said to be an inner inverse of A. Extensive studies have been made of this notion. In particular, von Neumann noted in 1936 that a (square) matrix over a ring in which every element is regular must also be regular ([14, p. 713]; see also [6; 17, p.19]). In this section we prove this result for matrices over a commmutative ring by use of the classical adjoint.

Lemma 3.1. *Let A be $m \times n$ of determinantal rank $r \geqslant 1$. If $\alpha \in \mathcal{Q}_{r,m}$, $\beta \in \mathcal{Q}_{r,n}$ with P_α, Q_β described as above, then*

$$
AQ_\beta(A^{\mathrm{ad}}_{\alpha\beta})P_\alpha A = |A_{\alpha\beta}|A.
$$

Proof. Since A is of rank r,

$$
\begin{pmatrix} P_\alpha A Q_\beta & P_\alpha A \\ AQ_\beta & A \end{pmatrix} = \begin{pmatrix} P_\alpha & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A & A \\ A & A \end{pmatrix} \begin{pmatrix} Q_\beta & 0 \\ 0 & I_n \end{pmatrix}
$$

is of rank at most r. Since $A_{\alpha\beta} = P_\alpha A Q_\beta$, the conclusion is a consequence of the application of Cauchy's result discussed in Section 1 with $B = P_\alpha A$, $C = AQ_\beta$, and $D = A$. $\square$

A ring $\mathscr{R}$ is said to be regular if every element of $\mathscr{R}$ is regular. It is known that every finitely generated ideal of a regular ring is principal and generated by an idempotent. (See, for example, [5, Corollary 1, p. 213].) For the sake of completeness, we establish this fact here for commutative rings.

Lemma 3.2. *Let $\mathscr{I}$ be a finitely generated ideal in a regular commutative ring $\mathscr{R}$. Then $\mathscr{I} = e\mathscr{R}$ for some idempotent $e \in \mathscr{R}$.*

Proof. The proof is by induction on the number of generators of $\mathscr{I}$. First, if $\mathscr{I} = a\mathscr{R}$, since $aba = a$ for some $b \in \mathscr{R}$, then $a\mathscr{R} = ab\mathscr{R}$ with ab idempotent. Second, suppose that the conclusion is valid for all ideals generated by fewer than $n > 1$ elements, and let $\mathscr{I} = a_1\mathscr{R} + \cdots + a_n\mathscr{R}$. By the induction hypothesis, $a_1\mathscr{R} + \cdots + a_{n-1}\mathscr{R} = e\mathscr{R}$ for some idempotent e, and $a_n\mathscr{R} = f\mathscr{R}$ for some idempotent f. Let $g = e + f - ef$. Since $e^2 = e$, $f^2 = f$, and $\mathscr{R}$ is commutative, $ge = e$, $gf = f$, and g is idempotent. Since $g \in \mathscr{I}$, then $g\mathscr{R} \subseteq \mathscr{I}$, and since both e and f are generated by g, $\mathscr{I} \subseteq g\mathscr{R}$. Consequently, $\mathscr{I} = g\mathscr{R}$ for an idempotent g. $\square$

For A $m \times n$ and $1 \leqslant s \leqslant \min\{m, n\}$, let $\mathscr{I}_s(A)$ be the ideal of $\mathscr{R}$ generated by the determinants of the $s \times s$ submatrices of A. By Lemma 3.2, if $\mathscr{R}$ is regular, then $\mathscr{I}_s(A) = e\mathscr{R}$ for some idempotent e; in particular, e is the identity element of $\mathscr{I}_s(A)$.

Theorem 3.1. *Let $\mathscr{R}$ be a regular commutative ring with 1. Then every matrix over $\mathscr{R}$ is regular.*

Proof. The proof is by induction on the determinantal rank of the matrices over $\mathscr{R}$. First, zero matrices are clearly regular. Second, let $r \geqslant 1$ be an integer and suppose that all matrices over $\mathscr{R}$ of rank less than r are regular. Let A be an $m \times n$ matrix over $\mathscr{R}$ of rank r.

Since $\mathscr{R}$ is regular, $\mathscr{I}_r(A) = e\mathscr{R}$ for some idempotent e. In particular, since $e \in \mathscr{I}_r(A)$,

$$e = \sum_{\alpha\in\mathscr{Q}_{r,m}} \sum_{\beta\in\mathscr{Q}_{r,n}} b_{\beta\alpha}|A_{\alpha\beta}|$$

for some $b_{\beta\alpha} \in \mathscr{R}$. For convenience, let $B = (b_{\beta\alpha}) \in \mathscr{R}^{\binom{n}{r}\times\binom{m}{r}}$ and define

$$A_B = \sum_{\alpha\in\mathscr{Q}_{r,m}} \sum_{\beta\in\mathscr{Q}_{r,n}} b_{\beta\alpha}(Q_\beta A^{\mathrm{ad}}_{\alpha\beta} P_\alpha).$$

By Lemma 3.1,

$$\begin{aligned} AA_BA &= \sum_{\alpha\in\mathscr{Q}_{r,m}} \sum_{\beta\in\mathscr{Q}_{r,n}} b_{\beta\alpha}(AQ_\beta A^{\mathrm{ad}}_{\alpha\beta} P_\alpha A) \\ &= \sum_{\alpha\in\mathscr{Q}_{r,m}} \sum_{\beta\in\mathscr{Q}_{r,n}} b_{\beta\alpha}|A_{\alpha\beta}|A = eA. \end{aligned}$$

Since e is the identity element of $\mathscr{I}_r(A)$, $e|A_{\alpha\beta}| = |A_{\alpha\beta}| = 1|A_{\alpha\beta}|$ for every $\alpha \in \mathscr{Q}_{r,m}$, $\beta \in \mathscr{Q}_{r,n}$. Also, since e is idempotent, $1 - e$ is idempotent. Therefore

$$|((1-e)A)_{\alpha\beta}| = (1-e)^r|A_{\alpha\beta}| = (1-e)|A_{\alpha\beta}| = 0$$

and $(1-e)A$ is of determinantal rank less than r. By the induction hypothesis, $(1-e)A$ is regular; that is, $((1-e)A)H((1-e)A) = (1-e)A$ for some $n \times m$ H. Consequently,

$$\begin{aligned} A(eA_B + (1-e)H)A &= (eA + (1-e)A)(eA_B + (1-e)H)(eA + (1-e)A) \\ &= e(AA_BA) + ((1-e)A)H((1-e)A) \\ &= e(eA) + (1-e)A = eA + (1-e)A = A \end{aligned}$$

and A is regular. (See also [17, pp. 118–119].) $\square$

For $A \in \mathscr{R}^{m\times n}$ of determinantal rank r, the mapping

$$\mathscr{R}^{\binom{n}{r}\times\binom{m}{r}} \to \mathscr{R}^{n\times m};\ B \mapsto A_B = \sum_{\alpha\in\mathscr{Q}_{r,m}} \sum_{\beta\in\mathscr{Q}_{r,n}} b_{\beta\alpha} Q_\beta A^{\mathrm{ad}}_{\alpha\beta} P_\alpha$$

with $B = (b_{\beta\alpha})$, is a linear transformation called the adjoint mapping of A. (See [20, p. 144].) Conceptually, $Q_\beta A^{\mathrm{ad}}_{\alpha\beta} P_\alpha$ is the $n \times m$ matrix with all zero entries except for the distribution of the entries of the classical adjoint of $A_{\alpha\beta}$ in its β rows and α columns. A_B is the linear combination of these matrices with coefficients given by the entries of B. In particular, in case $r = 1$, since the adjoint of a 1×1 matrix is the 1×1 identity matrix, then $A_B = B$. Also, the expressions for right and left inverses given above in Section 2 may be viewed as special cases of adjoint mappings where either $m = r$ or $n = r$ and part of the mapping is rendered vacuous. Later, in Section 4 below, the concept is used to express Moore–Penrose inverses of matrices. More immediately, we now use the adjoint mapping to provide a characterization of regular matrices over a commutative ring.

Lemma 3.3. *If A is $s \times s$, D is $r \times r$, and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is of determinantal rank $\leqslant r$, then $\left|\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}\right| = |-A||D|$. In particular, if $s = r$, then $|B||C| = |A||D|$.*

Proof. Let A_i and B_{s-i} denote the matrices consisting of the first i rows of A and the last $s - i$ rows of B, respectively. Let $M_0 = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$, $M_s = \begin{pmatrix} -A & 0 \\ C & D \end{pmatrix}$, and, for $0 < i < s$, let

$$M_i = \begin{pmatrix} -A_i & 0 \\ 0 & B_{s-i} \\ C & D \end{pmatrix}.$$

If a_i and b_i are the ith rows of A and B, respectively, then

$$M_i = \begin{pmatrix} -A_{i-1} & 0 \\ -a_i & 0 \\ 0 & B_{s-i} \\ C & D \end{pmatrix}, \quad M_{i-1} = \begin{pmatrix} -A_{i-1} & 0 \\ 0 & b_i \\ 0 & B_{s-i} \\ C & D \end{pmatrix},$$

and, since $(a_i, b_i) = -(-a_i, 0) + (0, b_i)$,

$$\left| \begin{pmatrix} -A_{i-1} & 0 \\ a_i & b_i \\ 0 & B_{s-i} \\ C & D \end{pmatrix} \right| = -|M_i| + |M_{i-1}|.$$

Since the rows $i, s+1, \ldots, s+r$ of the matrix on the left are $r+1$ rows of the given matrix of rank at most r, then, by the Laplace expansion by these rows, the determinant on the left is zero. That is, $|M_i| = |M_{i-1}|$ for $i = 1, \ldots, s$ and

$$\left| \begin{pmatrix} 0 & B \\ C & D \end{pmatrix} \right| = |M_0| = |M_s| = \left| \begin{pmatrix} -A & 0 \\ C & D \end{pmatrix} \right| = |-A|\,|D|.$$

In particular, if $s = r$, then

$$\begin{aligned} |B|\,|C| &= \left| \begin{pmatrix} B & 0 \\ D & C \end{pmatrix} \right| = (-1)^r \left| \begin{pmatrix} 0 & B \\ C & D \end{pmatrix} \right| \\ &= (-1)^r |-A|\,|D| = |A|\,|D|. \qquad \square \end{aligned}$$

Lemma 3.4. *Let $A \in \mathscr{R}^{m\times n}$ be of determinantal rank $r \geqslant 1$. If A is regular, then $\mathscr{I}_r(A) = e\mathscr{R}$ for some idempotent e. Specifically, if $AGA = A$, then $\operatorname{tr} C_r(GA)$ is the identity element of $\mathscr{I}_r(A)$.*

Proof. Let A be of rank $r \geqslant 1$, and let $AGA = A$. For every $\alpha, \gamma \in \mathscr{Q}_{r,m}$, $\beta, \delta \in \mathscr{Q}_{r,n}$,

$$\begin{pmatrix} A_{\alpha\beta} & A_{\alpha\delta} \\ A_{\gamma\beta} & A_{\gamma\delta} \end{pmatrix}$$

is of rank at most r. Thus, by Lemma 3.3, $|A_{\alpha\beta}||A_{\gamma\delta}| = |A_{\alpha\delta}||A_{\gamma\beta}|$. By the Binet–Cauchy theorem,

$$\begin{aligned} |A_{\alpha\beta}| = |(AGA)_{\alpha\beta}| &= \sum_{\gamma,\delta} |A_{\alpha\delta}||G_{\delta\gamma}||A_{\gamma\beta}| \\ &= \sum_{\gamma,\delta} |G_{\delta\gamma}||A_{\gamma\delta}||A_{\alpha\beta}| = \left(\sum_{\gamma,\delta} |G_{\delta\gamma}||A_{\gamma\delta}| \right) |A_{\alpha\beta}|. \end{aligned}$$

Therefore, the element $e = \sum_{\gamma,\delta} |G_{\delta\gamma}||A_{\gamma\delta}| = \operatorname{tr} C_r(GA)$ is the identity element of $\mathscr{I}_r(A)$; in particular, e is idempotent and $\mathscr{I}_r(A) = e\mathscr{R}$. $\square$

Theorem 3.2. *Let $\mathscr{R}$ be a commutative ring with 1, and let A be a matrix over $\mathscr{R}$. Then A is regular iff there exists a list $(e_1, \ldots, e_t)$ of idempotents of $\mathscr{R}$ such that*

1° $e_1 + \cdots + e_t = 1$,
2° $e_i e_j = 0$ *whenever* $i \neq j$,
3° *if* $r_i = \operatorname{rank}(e_i A) \neq 0$, *then* $\mathscr{I}_{r_i}(e_i A) = e_i \mathscr{R}$.

Proof. The necessity of the idempotents is established by induction on rank. First, for matrices of rank 0, the conditions are satisfied by $t = 1$ and $e_1 = 1$. Second, assume the existence of such a list of idempotents for each regular matrix of rank less than $r > 0$, and let A be regular of rank r. Since A is regular, by Lemma 3.4, $\mathscr{I}_r(A) = e\mathscr{R}$ for an idempotent e. For $\alpha \in \mathscr{Q}_{r,m}$, $\beta \in \mathscr{Q}_{r,n}$,

$$|(eA)_{\alpha\beta}| = e^r |A_{\alpha\beta}| = e|A_{\alpha\beta}| = |A_{\alpha\beta}|$$

and $\mathscr{I}_r(eA) = \mathscr{I}_r(A)$. Also, as in the proof of Theorem 3.1, $\operatorname{rank} eA = r > \operatorname{rank}((1-e)A)$. Since A is regular, $(1-e)A$ is regular, and by the induction hypothesis, there is a list $(f_1, \ldots, f_s)$ of idempotents satisfying 1°, 2°, and 3° with A replaced by $(1-e)A$. Then

$$(e, f_1(1-e), \ldots, f_s(1-e))$$

is a list of idempotents, whose pairwise products are zero,

$$e + f_1(1-e) + \cdots + f_s(1-e) = e + 1(1-e) = 1$$

with $\mathscr{I}_r(eA) = e\mathscr{R}$. Finally, let $r_j = \operatorname{rank}((f_j(1-e))A) \neq 0$. Since every element of $\mathscr{I}_{r_j}(f_j(1-e)A))$ is a multiple of the idempotent $1-e$ and $f_j = f_j 1 \in f_j\mathscr{R} = \mathscr{I}_{r_j}(f_j((1-e)A))$, then $f_j = g_j(1-e)$ for some g_j and $f_j(1-e) = g_j(1-e)^2 = g_j(1-e) = f_j$. Consequently, $\mathscr{I}_{r_j}((f_j(1-e))A) = \mathscr{I}_{r_j}(f_j((1-e)A) = f_j\mathscr{R} = (f_j(1-e))\mathscr{R}$. That is, the above list of $s+1$ idempotents satisfy the three conditions for A.

Conversely, suppose that idempotents exist satisfying conditions 1°, 2°, and 3° for the matrix A. Let $r_i = \operatorname{rank}(e_i A)$. If $r_i = 0$, then $e_i A = 0$. So

$$A = 1A = (e_1 + \cdots + e_t)A = \sum_{r_i \neq 0} e_i A + 0.$$

Whenever $r_i \neq 0$, by 3°, $e_i = e_i \cdot 1 \in e_i\mathscr{R} = \mathscr{I}_{r_i}(e_i A)$ and we can express

$$e_i = \sum_{\alpha \in \mathscr{Q}_{r_i,m}} \sum_{\beta \in \mathscr{Q}_{r_i,n}} b^{(i)}_{\beta\alpha} |(e_i A)_{\alpha\beta}|$$

with $b^{(i)}_{\beta\alpha} \in \mathscr{R}$. Let $(e_i A)_{B_i}$ be the image of the adjoint mapping of $e_i A$ under $B_i = (b^{(i)}_{\beta\alpha})$. By Lemma 3.1, $(e_i A)(e_i A)_{B_i}(e_i A) = e_i(e_i A) = e_i A$. Consequently, by properties 1° and 2°, $\sum_{r_i \neq 0}(e_i A)_{B_i} + 0$ is an inner inverse of $\sum_{r_i \neq 0} e_i A + 0$, and A is regular. □

Theorem 3.2 may be extended to provide a *unique* decomposition of a regular matrix A into components $e_i A$ with $\mathscr{I}_{r_i}(e_i A) = e_i\mathscr{R}$ whenever $r_i > 0$. (See [17, pp. 114–117; 19, p. 85].)

4. The Moore–Penrose inverse

In this section we assume that $\mathscr{R}$ possesses an involution $^-$. That is, $^-$ is a mapping $a \mapsto \bar{a}$ on $\mathscr{R}$ such that for every a and b in $\mathscr{R}$,

$$\overline{a+b} = \bar{a} + \bar{b},\ \overline{ab} = \bar{b}\bar{a},\ \bar{\bar{a}} = a.$$

In particular, $\bar{0} = 0$ and $\bar{1} = 1$. Since $\mathscr{R}$ is commutative, one possible involution is the identity mapping with $\bar{a} = a$. Also, the category of matrices over $\mathscr{R}$ is assumed to have the involution $(a_{ij}) \mapsto (a_{ij})^* = (\bar{a}_{ji})$ induced by $^-$. In particular,

$$(A+B)^* = A^* + B^*,\ (AB)^* = B^*A^*,\ A^{**} = A,$$

whenever the compatibility conditions on the sizes of the matrices are satisfied.

Given an $m \times n$ matrix A of determinantal rank r over such an $\mathscr{R}$, the expression

$$\sum_{\alpha \in \mathscr{Q}_{r,m}} \sum_{\beta \in \mathscr{Q}_{r,n}} |A_{\alpha\beta}|\,|\overline{A_{\alpha\beta}}|,$$

where $|\overline{A_{\alpha\beta}}|$ is the involution in $\mathscr{R}$ of the determinant of $A_{\alpha\beta}$, is called the square of the volume of A, and is denoted by $\mathrm{vol}^2 A$. Equivalently, since $|\overline{A_{\alpha\beta}}| = |A^*_{\beta\alpha}|$, where $A^*_{\beta\alpha} = (A^*)_{\beta\alpha}$, $\mathrm{vol}^2 A = \mathrm{tr}\, C_r(AA^*) = \mathrm{tr}\, C_r(A^*A)$. (See, for example, [2, pp. 89–90].)

A matrix A over $\mathscr{R}$ is said to have a Moore–Penrose inverse provided that there is a matrix $A^\dagger$ over $\mathscr{R}$ such that

$$AA^\dagger A = A, \quad A^\dagger A A^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

If such an $A^\dagger$ exists, then it is unique, and is called the Moore–Penrose inverse of A.

The four equations defining $A^\dagger$ were first given for complex matrices in 1955 by Penrose [15]. In 1956, Rado [16] noted that an equivalent notion had actually been introduced in 1920 by Moore [12]. A recent review of Moore's life and mathematical contributions, including detailed comments on Refs. [11,12], is given in [3].

If $\mathscr{R}$ is the ring of complex numbers, then there are several methods available to investigate the Moore–Penrose inverse of a matrix A. For example, by use of the singular value decomposition of A, it is a simple exercise to establish the existence and uniqueness of $A^\dagger$ [9, p. 421]. Alternatively, a construction of $A^\dagger$, which is generally credited to MacDuffee, may be easily obtained by use of a full rank factorization of A [4, pp. 23–24]. Unfortunately, these techniques are not generally available for matrices over an arbitrary commutative ring. However, whenever a given matrix A possesses a full rank factorization, by the results of Section 2, we may show the following.

Lemma 4.1. *Suppose that $A \in \mathscr{R}^{m\times n}$ of determinantal rank $r \geqslant 1$ possesses a full rank factorization. If $\mathrm{vol}^2 A$ is invertible in $\mathscr{R}$, then A possesses a Moore–Penrose inverse and*

$$A^\dagger = (\mathrm{vol}^2 A)^{-1} \sum_{\alpha \in \mathscr{Q}_{r,m}} \sum_{\beta \in \mathscr{Q}_{r,n}} |\overline{A_{\alpha\beta}}|\, Q_\beta A^{\mathrm{ad}}_{\alpha\beta} P_\alpha.$$

Proof. Let $A \in \mathscr{R}^{m\times n}$ be of determinantal rank $r \geqslant 1$ with $\mathrm{vol}^2 A$ invertible. By hypothesis, suppose that $A = MN$ with $M \in \mathscr{R}^{m\times r}$, $N \in \mathscr{R}^{r\times n}$. Since M^*MNN^* is $r \times r$, its rth compound is the scalar $|M^*M||NN^*| = |M^*MNN^*| = C_r(M^*MNN^*) = \mathrm{tr}\, C_r(M^*MNN^*) = \mathrm{tr}\, C_r(MNN^*M^*) = \mathrm{tr}\, C_r(AA^*) = \mathrm{vol}^2 A$. Since $\mathrm{vol}^2 A$ is invertible, then both $|M^*M|$ and $|NN^*|$ are invertible in $\mathscr{R}$. Consequently, both $M^*M, NN^* \in \mathscr{R}^{r\times r}$ are invertible. A straightforward exercise shows that

$$A^\dagger = N^*(NN^*)^{-1}(M^*M)^{-1}M^*$$

satisfies the Penrose equations.

Also, since $N^*(NN^*)^{-1}$ is a right inverse of N, by Theorem 2.1 above with $\rho = (1, \ldots, r) \in \mathscr{Q}_{r,r}$,

$$\begin{aligned} N^*(NN^*)^{-1} &= \sum_{\beta\in\mathscr{Q}_{r,n}} |(N^*(NN^*)^{-1})_{\beta\rho}|Q_\beta N^{\mathrm{ad}}_{\rho\beta} \\ &= \sum_{\beta\in\mathscr{Q}_{r,n}} |N^*_{\beta\rho}(NN^*)^{-1}|Q_\beta N^{\mathrm{ad}}_{\rho\beta} = |(NN^*)^{-1}| \sum_{\beta\in\mathscr{Q}_{r,n}} |N^*_{\beta\rho}|Q_\beta N^{\mathrm{ad}}_{\rho\beta}. \end{aligned}$$

Similarly,

$$(M^*M)^{-1}M^* = |(M^*M)^{-1}| \sum_{\alpha\in\mathscr{Q}_{r,m}} |M^*_{\rho\alpha}|M^{\mathrm{ad}}_{\alpha\rho} P_\alpha.$$

Consequently,

$$A^\dagger = |(NN^*)^{-1}||(M^*M)^{-1}| \sum_{\alpha\in\mathscr{Q}_{r,m}} \sum_{\beta\in\mathscr{Q}_{r,n}} |N^*_{\beta\rho}||M^*_{\rho\alpha}|Q_\beta N^{\mathrm{ad}}_{\rho\beta} M^{\mathrm{ad}}_{\alpha\rho} P_\alpha.$$

Clearly, $|(NN^*)^{-1}||(M^*M)^{-1}| = |M^*MNN^*|^{-1} = (\mathrm{vol}^2 A)^{-1}$. Also, since the adjoint of the product of two square matrices is the product of their adjoints in reverse order and $M_{\alpha\rho}N_{\rho\beta} = (MN)_{\alpha\beta} = A_{\alpha\beta}$, then $N^{\mathrm{ad}}_{\rho\beta}M^{\mathrm{ad}}_{\alpha\rho} = A^{\mathrm{ad}}_{\alpha\beta}$; and since $(N^*_{\beta\rho})(M^*_{\rho\alpha}) = (N^*M^*)_{\beta\alpha} = (MN)^*_{\beta\alpha} = A^*_{\beta\alpha}$, then $|N^*_{\beta\rho}||M^*_{\rho\alpha}| = |A^*_{\beta\alpha}| = |\overline{A_{\alpha\beta}}|$. That is,

$$A^\dagger = (\mathrm{vol}^2 A)^{-1} \sum_{\alpha\in\mathscr{Q}_{r,m}} \sum_{\beta\in\mathscr{Q}_{r,n}} |\overline{A_{\alpha\beta}}|Q_\beta A^{\mathrm{ad}}_{\alpha\beta} P_\alpha. \qquad \square$$

Lemma 4.1 provides a formula for the Moore–Penrose inverse of a matrix that consists of a linear combination of matrices described in terms of adjoints. We illustrate this formulation with an example.

Example 4.1. Let $\mathscr{R}$ be the ring $\mathbb{Z}_5$ of integers modulo 5 with the identity involution $a \mapsto \bar{a} = a$, and let

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 1 & 0 \\ 0 & 4 & 4 \end{pmatrix} \in \mathbb{Z}_5^{4\times 3}.$$

A is of rank 2, $\mathrm{vol}^2 A = 1$, and

$$M = \begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

provides a full rank factorization MN of A. Since $\mathrm{vol}^2 A = 1$,

$$A^\dagger = \begin{pmatrix} 4 & 2 & 3 & 3 \\ 3 & 0 & 4 & 2 \\ 4 & 3 & 1 & 4 \end{pmatrix}$$

is the sum of the eighteen matrices displayed in Table 1.

The formula for $A^\dagger$ given in Lemma 4.1 may be expressed as an image of the adjoint mapping of A, which was defined in Section 3. Specifically, since the entries of $C_r(A^*)$ are $|A^*_{\beta\alpha}| = |\overline{A_{\alpha\beta}}|$,

$$A^\dagger = (\mathrm{vol}^2 A)^{-1} A_{C_r(A^*)}.$$

In case $r = 1$, since $A_{C_1(A^*)} = C_1(A^*) = A^*$, then $A^\dagger = (\mathrm{vol}^2 A)^{-1} A^*$.

The formula for $A^\dagger$ given in Lemma 4.1 may be viewed as a natural generalization of the adjoint representation of the inverse of an invertible matrix. Specifically, if $A \in \mathscr{R}^{m\times n}$ of rank r is invertible in the usual sense, then $m = n = r$, $\mathrm{vol}^2 A = |A||\overline{A}|$, and

$$A^{-1} = |A|^{-1} A^{\mathrm{ad}} = (|A||\overline{A}|)^{-1} |\overline{A}| A^{\mathrm{ad}} = (\mathrm{vol}^2 A)^{-1} |\overline{A}| A^{\mathrm{ad}}.$$

Even though the proof of Lemma 4.1 depends upon the existence of a full rank factorization of the given matrix, the resulting formula for $A^\dagger$ depends only on the entries of A. This observation suggests the possibility of a stronger result. Indeed, as we now show, the conclusion of Lemma 4.1 is valid without the full rank factorization hypothesis.

Theorem 4.1. *Let $A \in \mathscr{R}^{m\times n}$ be of determinantal rank $r \geqslant 1$. If $\mathrm{vol}^2 A$ is invertible in $\mathscr{R}$, then A possesses a Moore–Penrose inverse and*

$$A^\dagger = (\mathrm{vol}^2 A)^{-1} \sum_{\alpha\in\mathscr{Q}_{r,m}} \sum_{\beta\in\mathscr{Q}_{r,n}} |\overline{A_{\alpha\beta}}| Q_\beta A^{\mathrm{ad}}_{\alpha\beta} P_\alpha.$$

Proof. If $r = 1$, then the formula reduces to $A^\dagger = (\mathrm{vol}^2 A)^{-1} A^*$. In this case, since $A = (a_{ij})$ is of rank 1, then for all possible subscripts, $a_{ik}a_{hj} = a_{ij}a_{hk}$. Therefore,

$$(AA^*A)_{ij} = \sum_{k=1}^{n}\sum_{h=1}^{m} a_{ik}\bar{a}_{hk}a_{hj} = \sum_{h=1}^{m}\sum_{k=1}^{n} a_{hk}\bar{a}_{hk}a_{ij} = (\mathrm{tr}\, AA^*)a_{ij}$$

and $AA^*A = (\mathrm{vol}^2 A)A$. Since $\overline{\mathrm{vol}^2 A} = \mathrm{vol}^2 A$, the Penrose equations are clearly satisfied by $(\mathrm{vol}^2 A)^{-1} A^*$. Thus, we assume that $r > 1$, and proceed to argue that $(\mathrm{vol}^2 A)^{-1} A_{C_r(A^*)}$ satisfies the Penrose equations.

Table 1 The matrices $|\overline{A_{\alpha\beta}}|Q_\beta A^{\mathrm{ad}}_{\alpha\beta}P_\alpha,\ \alpha \in \mathcal{Q}_{2,4}, \beta \in \mathcal{Q}_{2,3}$ associated with the A of Example 4.1

$$4\begin{pmatrix}3&1&0&0\\4&1&0&0\\0&0&0&0\end{pmatrix}\quad 4\begin{pmatrix}2&2&0&0\\0&0&0&0\\4&1&0&0\end{pmatrix}\quad 4\begin{pmatrix}0&0&0&0\\2&2&0&0\\2&4&0&0\end{pmatrix}$$

$$2\begin{pmatrix}1&0&1&0\\4&0&1&0\\0&0&0&0\end{pmatrix}\quad 2\begin{pmatrix}0&0&2&0\\0&0&0&0\\4&0&1&0\end{pmatrix}\quad 2\begin{pmatrix}0&0&0&0\\0&0&2&0\\4&0&4&0\end{pmatrix}$$

$$4\begin{pmatrix}4&0&0&1\\0&0&0&1\\0&0&0&0\end{pmatrix}\quad 4\begin{pmatrix}4&0&0&2\\0&0&0&0\\0&0&0&1\end{pmatrix}\quad 4\begin{pmatrix}0&0&0&0\\4&0&0&2\\1&0&0&4\end{pmatrix}$$

$$3\begin{pmatrix}0&1&2&0\\0&4&1&0\\0&0&0&0\end{pmatrix}\quad 3\begin{pmatrix}0&0&3&0\\0&0&0&0\\0&4&1&0\end{pmatrix}\quad 3\begin{pmatrix}0&0&0&0\\0&0&3&0\\0&4&3&0\end{pmatrix}$$

$$4\begin{pmatrix}0&4&0&2\\0&0&0&1\\0&0&0&0\end{pmatrix}\quad 4\begin{pmatrix}0&4&0&3\\0&0&0&0\\0&0&0&1\end{pmatrix}\quad 4\begin{pmatrix}0&0&0&0\\0&4&0&3\\0&1&0&3\end{pmatrix}$$

$$4\begin{pmatrix}0&0&4&4\\0&0&0&1\\0&0&0&0\end{pmatrix}\quad 4\begin{pmatrix}0&0&4&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\quad 4\begin{pmatrix}0&0&0&0\\0&0&4&0\\0&0&1&1\end{pmatrix}$$

First, by Lemma 3.1, with $\alpha \in \mathcal{Q}_{r,m}$, $\beta \in \mathcal{Q}_{r,n}$,

$$\begin{aligned} AA_{C_r(A^*)}A &= A\left(\sum_\alpha\sum_\beta |\overline{A_{\alpha\beta}}|Q_\beta A^{\mathrm{ad}}_{\alpha\beta}P_\alpha\right)A \\ &= \sum_\alpha\sum_\beta |\overline{A_{\alpha\beta}}||A_{\alpha\beta}|A = (\mathrm{vol}^2 A)A. \end{aligned}$$

Second, let $v \in \gamma \in \mathcal{Q}_{r,m}$ and $u \in \delta \in \mathcal{Q}_{r,n}$. Since the $((r-1)+r) \times ((r-1)+r)$ matrix

$$\begin{pmatrix} A_{\gamma\backslash v,\delta\backslash u} & A_{\gamma\backslash v,\beta} \\ A_{\alpha,\delta\backslash u} & A_{\alpha\beta} \end{pmatrix}$$

is of rank at most r, by Lemma 3.3,

$$(-1)^{r-1}|A_{\gamma\backslash v,\delta\backslash u}||A_{\alpha\beta}| = |-A_{\gamma\backslash v,\delta\backslash u}||A_{\alpha\beta}| = \left|\begin{pmatrix} 0 & A_{\gamma\backslash v,\beta} \\ A_{\alpha,\delta\backslash u} & A_{\alpha\beta} \end{pmatrix}\right|.$$

The Laplace expansion of the latter determinant by its first $r-1$ rows gives

$$\sum_{j=1}^{r}(-1)^{1+2+\cdots+(r-1)+r+(r+1)+\cdots+(r+(r-1))-(r+(j-1))}|A_{\gamma\backslash v,\beta\backslash\beta(j)}||A_{\gamma,\delta\backslash u}, A_{\alpha\beta(j)}|,$$

where $A_{\alpha\beta(j)}$ is the jth column of $A_{\alpha\beta}$. (See, for example [10, p. 14].) By an expansion on the last column of $|A_{\gamma,\delta\backslash u}, A_{\alpha\beta(j)}|$, the preceding sum becomes

$$\sum_{j=1}^{r}(-1)^{2r^2-r+(r-1+j)}|A_{\gamma\backslash v,\beta\backslash\beta(j)}|\sum_{i=1}^{r}(-1)^{r+i}a_{\alpha(i),\beta(j)}|A_{\alpha\backslash\alpha(i),\delta\backslash u}|$$
$$=(-1)^{r-1}\sum_{i,j=1}^{r}(-1)^{i+j}|A_{\alpha\backslash\alpha(i),\delta\backslash u}|(A_{\alpha\beta})_{ij}|A_{\gamma\backslash v,\beta\backslash\beta(j)}|.$$

Therefore, if $u=\delta(s)$ and $v=\gamma(t)$, then

$$\begin{aligned}|A_{\alpha\beta}|(A^{\mathrm{ad}}_{\gamma\delta})_{st}&=|A_{\alpha\beta}|(-1)^{s+t}|A_{\gamma\backslash v,\delta\backslash u}|\\&=\sum_{i,j=1}^{r}(-1)^{i+s}|A_{\alpha\backslash\alpha(i),\delta\backslash u}|(A_{\alpha\beta})_{ij}(-1)^{j+t}|A_{\gamma\backslash v,\beta\backslash\beta(j)}|\\&=\sum_{i,j=1}^{r}(A^{\mathrm{ad}}_{\alpha\delta})_{si}(A_{\alpha\beta})_{ij}(A^{\mathrm{ad}}_{\gamma\beta})_{jt},\end{aligned}$$

and $|A_{\alpha\beta}|A^{\mathrm{ad}}_{\gamma\delta}=A^{\mathrm{ad}}_{\alpha\delta}A_{\alpha\beta}A^{\mathrm{ad}}_{\gamma\beta}$. By Lemma 3.3, $|A_{\alpha\beta}||A_{\gamma\delta}|=|A_{\alpha\delta}||A_{\gamma\beta}|$. Thus,

$$\begin{aligned}A_{C_r(A^*)}AA_{C_r(A^*)}&=\left(\sum_{\alpha}\sum_{\beta}|\overline{A_{\alpha\beta}}|Q_\beta A^{\mathrm{ad}}_{\alpha\beta}P_\alpha\right)A\left(\sum_{\gamma}\sum_{\delta}|\overline{A_{\gamma\delta}}|Q_\delta A^{\mathrm{ad}}_{\gamma\delta}P_\gamma\right)\\&=\sum_{\alpha}\sum_{\beta}\sum_{\gamma}\sum_{\delta}|\overline{A_{\alpha\beta}}||\overline{A_{\gamma\delta}}|Q_\beta A^{\mathrm{ad}}_{\alpha\beta}A_{\alpha\delta}A^{\mathrm{ad}}_{\gamma\delta}P_\gamma\\&=\sum_{\alpha}\sum_{\beta}\sum_{\gamma}\sum_{\delta}|\overline{A_{\alpha\delta}}||\overline{A_{\gamma\beta}}||A_{\alpha\delta}|Q_\beta A^{\mathrm{ad}}_{\gamma\beta}P_\gamma\\&=\sum_{\beta}\sum_{\gamma}\left(\sum_{\alpha}\sum_{\delta}|A_{\alpha\delta}||\overline{A_{\alpha\delta}}|\right)|\overline{A_{\gamma\beta}}|Q_\beta A^{\mathrm{ad}}_{\gamma\beta}P_\gamma\\&=(\mathrm{vol}^2A)A_{C_r(A^*)}.\end{aligned}$$

(Compare [1, p. 99].)

Third, since $(Q_\beta A^{\mathrm{ad}}_{\alpha\beta}P_\alpha)_{ij}$ is zero whenever $j\notin\alpha$ or $i\notin\beta$ and is $(A^{\mathrm{ad}}_{\alpha\beta})_{i(\beta),j(\alpha)}$ whenever $j\in\alpha$ and $i\in\beta$,

$$\begin{aligned}(AA_{C_r(A^*)})_{kj}&=\sum_{i=1}^{n}a_{ki}\left(\sum_{\substack{\alpha\\ j\in\alpha}}\sum_{\substack{\beta\\ i\in\beta}}|\overline{A_{\alpha\beta}}|(A^{\mathrm{ad}}_{\alpha\beta})_{i(\beta),j(a)}\right)\\&=\sum_{\substack{\alpha\\ j\in\alpha}}\sum_{\beta}|\overline{A_{\alpha\beta}}|\sum_{\substack{i\\ i\in\beta}}(-1)^{j(\alpha)+i(\beta)}a_{ki}|A_{\alpha\backslash j,\beta\backslash i}|\\&=\sum_{\substack{\alpha\\ j\in\alpha}}\sum_{\beta}|\overline{A_{\alpha\beta}}||A_{\alpha(j\leftarrow k),\beta}|,\end{aligned}$$

where $A_{\alpha(j\leftarrow k),\beta}$ is the same matrix as $A_{\alpha\beta}$ except that the elements $A_{j\beta(s)}$ of row j have been replaced by the corresponding elements $A_{k\beta(s)}$. If $k = j$, then $A_{\alpha(j\leftarrow k),\beta} = A_{\alpha\beta}$ and

$$(AA_{C_r(A^*)})_{jj} = \sum_{\substack{\alpha \\ j\in\alpha}}\sum_{\beta} |\overline{A_{\alpha\beta}}||A_{\alpha\beta}| = \left(\overline{AA_{C_r(A^*)}}\right)_{jj}.$$

Thus, consider $j \neq k$. Since, by duplication of rows, $k \in \alpha$ implies $|A_{\alpha(j\leftarrow k),\beta}| = 0$, then

$$(AA_{C_r(A^*)})_{kj} = \sum_{\substack{\alpha \\ j\in\alpha \\ k\notin\alpha}}\sum_{\beta} |\overline{A_{\alpha\beta}}||A_{\alpha(j\leftarrow k),\beta}|.$$

For $j \in \alpha$ and $k \notin \alpha$, let α_{jk} be the list of $\mathscr{Q}_{r,m}$ obtained from α by the deletion of j and the inclusion of k. In particular, $\alpha \mapsto \alpha_{jk}$ provides a bijection of the lists of $\mathscr{Q}_{r,m}$ which contain j but not k to the lists of $\mathscr{Q}_{r,m}$ which contain k but not j, and

$$|A_{\alpha(j\leftarrow k),\beta}| = (-1)^{j(\alpha)+k(\alpha_{jk})}|A_{\alpha_{jk},\beta}|.$$

Therefore, by a change in the index of summation via $\gamma = \alpha_{jk}$ and $\gamma_{kj} = \alpha$,

$$\begin{aligned}(\overline{AA_{C_r(A^*)}})_{kj} &= \sum_{\substack{\alpha \\ j\in\alpha \\ k\notin\alpha}}\sum_{\beta} |A_{\alpha\beta}|(-1)^{j(\alpha)+k(\alpha_{jk})}|\overline{A_{\alpha_{jk},\beta}}| \\ &= \sum_{\substack{\gamma \\ k\in\gamma \\ j\notin\gamma}}\sum_{\beta} |A_{\gamma_{kj},\beta}|(-1)^{j(\gamma_{kj})+k(\gamma)}|\overline{A_{\gamma\beta}}| \\ &= \sum_{\substack{\gamma \\ k\in\gamma \\ j\notin\gamma}}\sum_{\beta} |\overline{A_{\gamma,\beta}}||A_{\gamma(k\leftarrow j),\beta}| = (AA_{C_r(A^*)})_{jk}.\end{aligned}$$

Since an analogous result also holds for the product in reverse order,

$$(AA_{C_r(A^*)})^* = AA_{C_r(A^*)}, \quad (A_{C_r(A^*)}A)^* = A_{C_r(A^*)}A.$$

Consequently, since $\overline{\mathrm{vol}^2 A} = \mathrm{vol}^2 A$, under the assumption that $\mathrm{vol}^2 A$ is invertible, then $(\mathrm{vol}^2 A)^{-1}\ A_{C_r(A^*)}$ satisfies the Penrose equations for A. $\square$

Theorem 4.1 enables one to explicitly calculate the entries of $A^\dagger$ in terms of the entries of A. In particular, it provides an immediate proof of Moore's 1920 formula specifying the entries of $A^\dagger$ in terms of determinants. Specifically, the following corollary is a restatement of Moore's formula in [12].

Corollary 4.1. *Let $A \in \mathscr{R}^{m\times n}$ be of determinantal rank $r \geqslant 1$. If $\mathrm{vol}^2 A$ is invertible, then*

$$(A^\dagger)_{ij} = (\mathrm{vol}^2 A)^{-1} \sum_{\gamma\in\mathscr{Q}_{r-1,m}} \sum_{\delta\in\mathscr{Q}_{r-1,n}} |\overline{A_{(j,\gamma),(i,\delta)}}||A_{\gamma\delta}|,$$

where $(j,\gamma) = (j,\gamma(1),\ldots,\gamma(r-1))$ and $(i,\delta) = (i,\delta(1),\ldots,\delta(r-1))$.

Proof. If $r = 1$, then $A^\dagger = (\mathrm{vol}^2 A)^{-1} A^*$ and $(A^\dagger)_{ij} = (\mathrm{vol}^2 A)^{-1} \cdot |\overline{A_{ji}}| \cdot 1$.

If $r > 1$, then

$$\begin{aligned}(A^\dagger)_{ij} &= (\mathrm{vol}^2 A)^{-1} \sum_{\substack{\alpha \in \mathcal{Q}_{r,m} \\ j \in \alpha}} \sum_{\substack{\beta \in \mathcal{Q}_{r,n} \\ i \in \beta}} |\overline{A_{\alpha\beta}}|(A^{\mathrm{ad}}_{\alpha\beta})_{i(\beta),j(\alpha)} \\ &= (\mathrm{vol}^2 A)^{-1} \sum_{\substack{\alpha \in \mathcal{Q}_{r,m} \\ j \in \alpha}} \sum_{\substack{\beta \in \mathcal{Q}_{r,n} \\ i \in \beta}} |\overline{A_{\alpha\beta}}|(-1)^{i(\beta)+j(\alpha)}|A_{\alpha\backslash j,\beta\backslash i}| \\ &= (\mathrm{vol}^2 A)^{-1} \sum_{\substack{\alpha' \in \mathcal{Q}_{r-1,m} \\ j \notin \alpha'}} \sum_{\substack{\beta' \in \mathcal{Q}_{r-1,n} \\ i \notin \beta'}} |\overline{A_{(j,\alpha'),(i,\beta')}}||A_{\alpha',\beta'}|,\end{aligned}$$

where α' corresponds to α under the natural bijection between the $\alpha' \in \mathcal{Q}_{r-1,m}$ with $j \notin \alpha$ and the $\alpha \in \mathcal{Q}_{r,m}$ with $j \in \alpha$; similarly for β'. Finally, since $|A_{(j,\gamma),(i,\delta)}| = 0$ whenever $j \in \gamma \in \mathcal{Q}_{r-1,m}$ or $i \in \delta \in \mathcal{Q}_{r-1,n}$, the preceding sum is as in the statement of the corollary. $\square$

Lemma 4.2. *Let A be a matrix of rank r over a commutative ring with involution $\bar{\ }$. If $A^\dagger$ satisfies the Penrose equations for A with respect to $*$ induced by $\bar{\ }$, then*

$$C_r(A^*) = (\mathrm{vol}^2 A)C_r(A^\dagger), \quad \mathrm{tr}\, C_r(A^\dagger A) = (\mathrm{vol}^2 A)(\mathrm{vol}^2 A^\dagger).$$

Proof. By the Penrose equations,

$$\begin{aligned}A^* &= (AA^\dagger A)^* = A^* A^{\dagger *} A^* = A^*(AA^\dagger)^* = A^* A A^\dagger \\ &= A^* A A^\dagger A A^\dagger = A^* A (A^\dagger A)^* A^\dagger = A^* A A^* A^{\dagger *} A^\dagger.\end{aligned}$$

Since A^* is of rank r, by Lemma 3.3, $C_r(A^*)$ is of rank 1. By an argument similar to the one used in the first paragraph of the proof of Theorem 4.1,

$$C_r(A^*)C_r(A)C_r(A^*) = (\mathrm{tr}\, C_r(A^*)C_r(A))C_r(A^*).$$

Therefore,

$$\begin{aligned}C_r(A^*) &= C_r(A^*)C_r(A)C_r(A^*)C_r(A^{\dagger *}A^\dagger) \\ &= (\mathrm{tr}\, C_r(A^*)C_r(A))C_r(A^*)C_r(A^{\dagger *}A^\dagger) \\ &= (\mathrm{tr}\, C_r(A^*A))C_r(A^*A^{\dagger *}A^\dagger) \\ &= (\mathrm{vol}^2 A)C_r((A^\dagger A)^* A^\dagger) \\ &= (\mathrm{vol}^2 A)C_r(A^\dagger A A^\dagger) \\ &= (\mathrm{vol}^2 A)C_r(A^\dagger).\end{aligned}$$

Also,

$$\begin{aligned}C_r(A^\dagger A) &= C_r((A^\dagger A)^*) = C_r(A^* A^{\dagger *}) = C_r(A^*)C_r(A^{\dagger *}) \\ &= (\mathrm{vol}^2 A)C_r(A^\dagger)C_r(A^{\dagger *}) = (\mathrm{vol}^2 A)C_r(A^\dagger A^{\dagger *})\end{aligned}$$

and

$$\operatorname{tr} C_r(A^\dagger A) = (\mathrm{vol}^2 A)\operatorname{tr} C_r(A^\dagger A^{\dagger *}) = (\mathrm{vol}^2 A)(\mathrm{vol}^2 A^\dagger). \qquad \square$$

Corollary 4.2. *Let A be a matrix over $\mathscr{R}$ of determinantal rank $r \geqslant 1$. Suppose that $1 \in \mathscr{I}_r(A)$. Then $A^\dagger$ exists iff $\mathrm{vol}^2 A$ is invertible in $\mathscr{R}$. In that case,*

$$A^\dagger = (\mathrm{vol}^2 A)^{-1} A_{C_r(A^*)}.$$

Proof. Suppose that $A^\dagger$ exists. Since $AA^\dagger A = A$, by Lemma 3.4, $\operatorname{tr} C_r(A^\dagger A)$ is the identity element of $\mathscr{I}_r(A)$. By assumption, $1 \in \mathscr{I}_r(A)$. Thus, $1 = \operatorname{tr} C_r(A^\dagger A) = (\mathrm{vol}^2 A)(\mathrm{vol}^2 A^\dagger)$ and $\mathrm{vol}^2 A$ is invertible.

Conversely, if $\mathrm{vol}^2 A$ is invertible, then, by Theorem 4.1, $A^\dagger$ exists and is equal to $(\mathrm{vol}^2 A)^{-1} A_{C_r(A^*)}$. $\square$

Example 4.2. Let $\mathscr{R}$ be the ring $\mathbb{Z}_6$ of integers modulo 6 with the identity involution $a \mapsto \bar{a} = a$. We consider three matrices in $\mathbb{Z}_6^{3\times 3}$.

(i) $$A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 4 & 3 \\ 2 & 4 & 3 \end{pmatrix}$$

implies rank $A = 2$,

$$C_2(A) = \begin{pmatrix} 4 & 3 & 0 \\ 2 & 3 & 0 \\ 2 & 3 & 0 \end{pmatrix},$$

$1 = 3 + 4 \in \mathscr{I}_2(A)$, and $\mathrm{vol}^2 A = 4^2 + 2^2 + 2^2 + 3^2 + 3^2 + 3^2 = 3$. Since $1 \in \mathscr{I}_2(A)$ and $\mathrm{vol}^2 A$ is not invertible in $\mathscr{R}$, A does not possess a Moore–Penrose inverse.

(ii) $$A = \begin{pmatrix} 5 & 3 & 4 \\ 0 & 3 & 5 \\ 4 & 0 & 2 \end{pmatrix}$$

implies rank $A = 2$,

$$C_2(A) = \begin{pmatrix} 3 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix},$$

$1 \in \mathscr{I}_2(A)$, and $\mathrm{vol}^2 A = 3^2 + 1^2 + 3^2 + 4^2 = 5$, which is invertible with inverse 5. Thus, $A^\dagger$ exists and

$$\begin{aligned} A^\dagger &= 5\left[3\begin{pmatrix} 3 & 3 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 5 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 & 0 \\ 5 & 2 & 0 \\ 3 & 3 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} 4 & 5 & 2 \\ 3 & 3 & 0 \\ 3 & 2 & 0 \end{pmatrix}. \end{aligned}$$

(iii) $A = \begin{pmatrix} 3 & 5 & 3 \\ 2 & 3 & 3 \\ 2 & 0 & 3 \end{pmatrix}$

implies rank $A = 3$, $C_3(A) = (3)$, and $1 \notin \mathscr{I}_3(A)$. Thus, the hypothesis of Corollary 4.2 is not satisfied. However, by means of an appropriate additive decomposition of A and by an application of Corollary 4.2 to each of the summands, we demonstrate below that this A does possess a Moore–Penrose inverse. In particular, since $\mathrm{vol}^2 A = 3$ is not invertible in $\mathbb{Z}_6$, this example shows that the converse of Theorem 4.1 fails to hold. □

More generally, by a combination of Corollary 4.2 and an extension of Theorem 3.2, we now characterize the existence and structure of $A^\dagger$ for any given matrix A in terms of an additive decomposition of A, specified in terms of a decomposition of the identity element 1 into a sum of idempotents.

Theorem 4.2. *Let $A \in \mathscr{R}^{m\times n}$. Then $A^\dagger$ exists iff there exists a list $(e_1, \ldots, e_t)$ of idempotents in $\mathscr{R}$ satisfying*

1° $e_1 + \cdots + e_t = 1$,
2° $e_i e_j = 0$ *whenever* $i \neq j$,
$3^{\circ'}$ *if* $r_i = $ *rank* $(e_i A) \neq 0$, *then* $e_i = \bar{e}_i$ *and* $\mathrm{vol}^2(e_i A)$ *is invertible in* $e_i\mathscr{R}$.

In that case,

$$A^\dagger = \sum_{r_i \neq 0} (\mathrm{vol}^2 e_i A)^{-1} (e_i A)_{C_{r_i}(e_i A^*)} + 0,$$

where $(\mathrm{vol}^2(e_i A))^{-1}$ *is the inverse of* $\mathrm{vol}^2(e_i A)$ *in* $e_i\mathscr{R}$.

Proof. Let A be such that $A^\dagger$ exists. If $A = 0$, then conditions 1°, 2°, and $3^{\circ'}$ are satisfied by $t = 1$ and $e_t = 1$.

Thus, suppose that $A \neq 0$. Since $AA^\dagger A = A$, then A is regular and $e = \mathrm{tr}(A^\dagger A) = \mathrm{tr}(A^\dagger A)^* = \bar{e}$. A slightly modified repeat of the induction proof of Theorem 3.2 provides a construction of a list $(e_1, \ldots, e_t)$ of idempotents that satisfy 1°, 2°, and 3° with $e_i = \bar{e}_i$ for each i. In particular, if $r_i = \mathrm{rank}(e_i A) \neq 0$ then $\mathscr{I}_{r_i}(e_i A) = e_i\mathscr{R}$ with identity element e_i. Since $e_i \in \mathscr{I}_{r_i}(e_i A)$ and $(e_i A)^\dagger = e_i A^\dagger$ exists, by Corollary 4.2 with 1 replaced by e_i and A by $e_i A$, $\mathrm{vol}^2(e_i A)$ is invertible in $e_i\mathscr{R}$. Consequently, conditions 1°, 2°, and $3^{\circ'}$ are satisfied.

Conversely, let $(e_1, \ldots, e_t)$ be a list of idempotents satisfying 1°, 2°, and $3^{\circ'}$ and let $r_i = \mathrm{rank}(e_i A)$. Then

$$A = e_1 A + \cdots + e_t A = \sum_{r_i \neq 0} e_i A + \sum_{r_i = 0} e_i A = \sum_{r_i \neq 0} e_i A + 0.$$

If $r_i \neq 0$ for every i, then $A = 0$ and $A^\dagger = 0$ exists. Thus, we assume that $r_i \neq 0$ for some i. For every such i, e_i is clearly the identity element of $e_i\mathscr{R}$, and we show that $\mathscr{I}_{r_i}(e_iA) = e_i\mathscr{R}$. First,

$$\mathrm{vol}^2(e_iA) = \sum_{\alpha\in\mathscr{Q}_{r_i,m}} \sum_{\beta\in\mathscr{Q}_{r_i,n}} |(e_iA)_{\alpha\beta}||\overline{(e_iA)_{\alpha\beta}}| \in \mathscr{I}_{r_i}(e_iA).$$

By $3^{\circ'}$, e_i is a multiple of $\mathrm{vol}^2(e_iA)$, which means that $e_i \in \mathscr{I}_{r_i}(e_iA)$. Hence, $e_i\mathscr{R} \subseteq \mathscr{I}_{r_i}(e_iA)$. On the other hand, since every element of $\mathscr{I}_{r_i}(e_iA)$ is a multiple of e_i, then $\mathscr{I}_{r_i}(e_iA) \subseteq e_i\mathscr{R}$. Consequently, $\mathscr{I}_{r_i}(e_iA) = e_i\mathscr{R}$ with identity element e_i.

By Corollary 4.2, with 1 replaced by e_i and A by e_iA, since by $3^{\circ'}$ $\mathrm{vol}^2(e_iA)$ is invertible in $e_i\mathscr{R}$, $(e_iA)^\dagger$ exists. Furthermore, since $(e_iA)^* = \bar{e}_iA^* = e_iA^*$,

$$(e_iA)^\dagger = (\mathrm{vol}^2(e_iA))^{-1}(e_iA)_{C_{r_i}((e_iA)^*)} = (\mathrm{vol}^2(e_iA))^{-1}(e_iA)_{C_{r_i}(e_iA^*)}.$$

Finally, since $\sum_{r_i\neq 0}(e_iA)^\dagger$ satisfies the Penrose equation for $A = \sum_{r_i\neq 0} e_iA$, $A^\dagger$ exists and $A^\dagger = \sum_{r_i\neq 0}(e_iA)^\dagger$. (Compare [17, p. 123].) $\square$

Example 4.3. Consider again

$$\text{(iii)} \quad A = \begin{pmatrix} 3 & 5 & 3 \\ 2 & 3 & 3 \\ 2 & 0 & 3 \end{pmatrix}$$

of Example 4.2. As noted above, the hypothesis of Corollary 4.2 is not satisfied. However, 3 and 4 are idempotents in $\mathscr{R} = \mathbb{Z}_6$ with $3 + 4 = 1$, $3 \cdot 4 = 0$, and

$$3A = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 3 & 3, \\ 0 & 0 & 3 \end{pmatrix} \quad 4A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

First, $\mathrm{rank}(3A) = 3$, $\mathrm{vol}^2(3A) = |3A|^2 = 3$, which is invertible with inverse 3 in $3\mathscr{R}$, and

$$3(3A)_{C_3(3A^*)} = 3(3(3A)^{\mathrm{ad}}) = (3A)^{\mathrm{ad}} = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{pmatrix}.$$

Second, $\mathrm{rank}(4A) = 2$,

$$C_2(4A) = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\mathrm{vol}^2(4A) = 2^2 + 2^2 = 2$, which is invertible with inverse 2 in $4\mathscr{R}$, and

$$2(4A)_{C_2(4A^*)} = 2\left[2\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 4 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 4 & 4 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Theorem 4.2, $A^\dagger$ exists and

$$A^\dagger = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 4 & 4 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 3 & 3 \\ 0 & 0 & 3 \end{pmatrix}. \qquad \square$$

The image A_B of the adjoint mapping may also be used to describe other generalized inverses of A besides the Moore–Penrose. (See, for example, [17,19–21].) Thus, the classical adjoint has not only played, but continues to play, a significant role in the study of generalized invertibility of matrices.

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