# Optimal decomposition by clique separators 

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#### Abstract

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Decompositions of a graph by clique separators are investigated which have the additional property that they do not generate new maximal prime subgraphs. Using such decompositions is preferable in many applications, since they lead to a minimal system of derived subgraphs. The methods used in the proofs are familiar from the investigations of chordal graphs and acyclic hypergraphs and some well-known results for these (hyper-) graphs are shown to be simple special cases of results for maximal prime subgraphs.

Tarjan has described an $\mathrm{O}(\mathrm{nm})$-time algorithm to decompose a graph with $n$ vertices and $m$ edges by means of clique separators. This algorithm is modified, so that no new maximal prime subgraphs are generated, i.e. so that a graph is decomposed exactly into its maximal prime subgraphs which is the unique minimal derived system of prime subgraphs.


## 0. Notations

Throughout this paper $G=(V, E)$ is an undirected graph without loops or multiple edges, where $V$ is called the vertex set and $E$ the edge set of $G$. It is assumed that $n:=|V|$ is finite. Edges are denoted by $\{v, w\} \in E$ with vertices $v, w \in V, v \neq w$. A path in $G$ between vertices $v, w \in V$ is a sequence $v_{0}, v_{1}, \ldots, v_{n} \in V$ with $\{v, w\}=\left\{v_{0}, v_{n}\right\}$ and $\left\{v_{i-1}, v_{i}\right\} \in E$ for $i=1, \ldots, n$.

Let $U$ denote a subset of vertices of a graph $G . G(U):=(U, E(U))$ is the sub-graph of $G$ induced by $U$, where $E(U):=\{\{v, w\} \in E: v, w \in U\}$. All subgraphs in this paper are induced subgraphs. $U$ is a clique, î any two different vertices $v, w \in U$ are adjacent, i.e. $\{v, w\} \in E$. Especially the empty set $\emptyset$ is ? clique. A graph is a clique if its vertex set is a clique. $U$ is called a maximal chigue, if $U$ is maximal w.r.t. inclusion in the set of cliques of $G$. $U \subseteq V$ is a separator for

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$A, B \subseteq V \backslash U$, if every path in $G$ between sume $a \in A$ and $b \in B$ contains a vertex in $U . U$ is a separator for $G$, if there are non-empty sets $A, B \subseteq V \backslash U$ such that $U$ is a separator for $A$ and $B$. Separators that are cliques are called clique separators.

A graph $G$ is reducible if its vertex set contains a clique separator, otherwise $G$ is said to be prime. E.g. $G$ is prime if $G$ is a clique while $G$ is reducible if $G$ is a disconnected graph. A subgraph $G(U)$ is a maximal prime (mp-) subgraph of $G$, if $G(U)$ is prime and $G(X)$ is reducible for all $X$ with $U \subset X \subseteq V$.

Let $A, B, C$ be a vertex partition with $A$ and $B$ non-empty such that $C$ is a clique and separates $A$ and $B$. Then the triple $(A, C, B)$ defines a decomposition of $G$ into the subgraphs $G^{\prime}=G(A \cup C)$ and $G^{\prime \prime}=G(B \cup C)$. If furthermore the mp-subgraphs of $G^{\prime}$ and $G^{\prime \prime}$ are pairwise different and if they are all mpsubgraphs of $G$, then $(A, C, B)$ is called a $P$-decomposition and $C$ is called a $P$-separator. A decomposition $(A, C, B)$ is a $P$-decomposition if and only if the set of mp-subgraphs of $G$ is equal to the disjoint union of the set of mp-subgraphs of $G(A \cup C)$ and $G(B \cup C)$ (see Lemma 2.1(i)). If $G$ is reducible, i.e. if there is a decomposition for $G$, then there also exists a P-decomposition for $G$ (see Corollary 2.7).

The notions of a $D$-ordered sequence of sets, where ' $D$ ' stands for decomposition (Corollary 2.7) or decomposable graph (Theorem 3.3), and of a $D$ numbering of the vertices of a graph (see Definitions 2.3 and 3.1) are also essential for this paper.

## 1. Introduction

If $G$ is a reducible graph, then there is a decomposition $(A, C, B)$ of $G$ into the subgraphs $G^{\prime}=G(A \cup C)$ and $G^{\prime \prime}=G(B \cup C)$. Similarly $G^{\prime}$ and $G^{\prime \prime}$ can be decomposed until all derived subgraphs are prime. Tarjan [24] has given four examples, where such decompositions can be used to solve graph problems for $G$ efficiently. He considered minimizing the fill-in caused by Gaussian elimination, finding a maximum clique, graph coloring, and finding a maximum independent set. The requirement for a decomposition $(A, C, B)$ that $C$ separates $A$ and $B$ guarantees that no structure of the graph is lost when passing from $G$ to $G^{\prime}$ and $G^{\prime \prime}$ and since $C$ is a clique it is possible to combine the solutions of the respective graph problems for $G^{\prime}$ and $G^{\prime \prime}$ to a solution of the graph problem for $G$. Further applications of such decompositions were described by Diestel [6].

Decompositions of a graph are of similar importance in a statistical context. Darroch. I_auritzen and Speed [4] defined graphical models for contingency tables, where every vertex of a graph is associated with a discrete (qualitative) random variable and a missing edge $\{v, w\}$ in the graph corresponds to the conditional independence of the variables associated with $v$ and $w$. If $M(G)$ is a graphical model for a contingency table and $(A, C, B)$ is a decomposition of $G$ into $G^{\prime}$ ard $G^{\prime \prime}$, then the maximum-likelihood estimates for the parameters of the
model $M(G)$ can easily be derived from the simpler, lower-dimensional models $M\left(G^{\prime}\right)$ and $M\left(G^{\prime \prime}\right)$. Similar results hold for continuous (quantitative) random variables and mixtures of discrete and continuous variables ([16, 18], see also Section 6).

Tarjan has pointed out that the derived system of prime subgraphs, when decomposing a graph recursively, is far from being unique. The aims of this paper are

- to show that every graph $G$ can successively be decomposed such that a unique minimal system of prime subgraphs is derived which is the system of mp -subgraphs of $\boldsymbol{G}$,
- to characterize the decompositions involved in that process,
- and to describe an $\mathrm{O}(\mathrm{nm})$-time algorithm to decompose an $n$-vertex, $m$-edge graph into its mp-subgraphs.
The described algorithm is a modification of the algorithm described by Tarjan [24]. Two further aspects are emphasized in this paper
- the generalization of well-known results and techniques for decomposable (ciordal) graphs and decomposable (acyclic) hypergraphs to arbitrary graphs and collectiotis of prime graphs, respectively, and
- the investigation of an arbitrary graph using a closely related decomposable graph which is easier to investigate.
Some uniqueness results for decomposing graphs were previously given by Diestel ( $[5,7-9])$. Throughout this paper we shall use a result that is equivalent to the existence of a reduced simplicial decomposition of a graph into primes [27]. A slightly more general form is proved in Section 2 (Theorem 2.5). Conversely, systems of prime graphs $G_{1}, \ldots, G_{T}$ are characterized which are the mpsubgraphs of their union-graph $G:=G_{1} \cup \cdots \cup G_{T}$ (Theorem 2.10).
In Section 3 some previous ,' known results for decomposable (hyper-)graphs are derived as simple special cases of the general results of Section 2 concerning mp -subgraphs. Decomposable (hyper-)graphs are called like that in [15] and they are also known as chordal, triangulated, or rigid circuit graphs and acyclic hypergraphs, respectively (see e.g. [3] and [10]).
When decomposing a reducible graph recursively into prime subgraphs, the derived system is minimal if and only if ail decompositions involved are P-decompositions (see (2.2)). In Section 4 we characterize $\mathbf{P}$-separators in several ways. For the proof, a technique used by Tarjan [24] is further developed; instead of a given nondecomposable graph $G$, a closely related decomposable graph $G^{*}$ (see (3.8)) is investigated.
Tarjan's algorithm [24] to decompose a graph $G=(V, E)$ recursively into prime subgraphs consists of two steps:
Step 1: Derive a minimal decomposable graph $G_{\pi}=\left(V, E \cup F_{\pi}\right)$.
Step 2: Apply a 'decomposition step' (involving $G$ and $G_{\pi}$ ) to all vertices in $V$.
We shali show in Section 5 that this algorithm can be modified to decompose a graph into its mp-subgraphs. The modification is as follows: In Step 2, the
decomposition step is only applied to certain vertices $v \in V$ which have been (simply) marked in Step 1. The rest of the algorithm remains the same.


## 2. Orderings of maximal prime subgraphs

Assume that a reducible graph $G$ is decomposed by some decomposition $(A, C, B)$ into subgraphs $G^{\prime}=G(A \cup C)$ and $G^{\prime \prime}=G(B \cup C)$ and that $G^{\prime}$ and $G^{\prime \prime}$ are decomposed further until all derived subgraphs are prime. A system of prime subgraphs of $G$ which can be derived in that way, will be called a derived system for short. The elements of derived systems were called atoms by Tarjan [24] and components by Whittaker [28], who derived results similar to those in [27, 14] for components of statistical model formulae.
Every derived system of a graph $G$ contains the mp-subgraphs of $G$ (see (2.2)) and it follows from Corollary 2.7 below that the system of mp-subgraphs of $G$ always is a derived system. Hence it is the unique minimal derived system. The derived system is minimal if and only if all decompositions used are P decompositions (see (2.2)).
The following lemma contains some usefui relations between (maximal) prime subgraphs and decompositions.

Lemma 2.1. Let $G$ be a reducible graph and let $(A, C, B)$ be a decomposition of $G$.
(i) If $G(U)$ is an ( $m$ )p-subgraph of $G$, then $U \subseteq A \cup C$ or $U \subseteq B \cup C$ and $G(U)$ is an $(m) p$-subgraph of $G(A \cup C)$ or $G(B \cup C)$, respectively.
(ii) Every mp-subgraph $G(U)$ of $G(A \cup C)$ or $G(B \cup C)$ with $U \neq C$ is an mp-subgraph of $G$.
(iii) If $G\left(U_{1}\right)$ and $G\left(U_{2}\right)$ are different mp-subgraphs of $G$, then $U_{1} \cap U_{2}$ is a clique.

Proof. (i) Assume that $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$. Then ( $A \cap U, C \cap U, B \cap U$ ) is a decomposition of $G(U)$ which contradicts $G(U)$ prime. The second part of $(i)$ is obvious.
(ii) Let $G(U)$ be an mp-subgraph of $G(A \cup C)$ with $U \neq C$ (and similarly for $G(B \cup C)$ ). We cannot have $U \subset C$ since $G(C)$ is prime, hence $U \cap A \neq \emptyset$. There is an mp-subgraph $G(X)$ of $G$ with $U \subseteq X$, i.e. $X \cap A \neq \emptyset$. It follows from part (i) of the lemma that $G(X)$ is an mp-subgraph of $G(A \cup C)$, hence $U=X$.
(iii) $U:=U_{1} \cup U_{2} . G(U)$ is reducible since $G\left(U_{1}\right)$ and $G\left(U_{2}\right)$ are different mp-subgraphs of $G$. Let ( $A^{\prime}, C^{\prime}, B^{\prime}$ ) be a decomposition of $G(U)$. Then $G\left(U_{1}\right)$ and $G\left(U_{2}\right)$ are also maximal prime subgraphs of $G(U)$ and part (i) of the lemma implies $U_{1} \subseteq A^{\prime} \cup C^{\prime}$ and $U_{2} \subseteq B^{\prime} \cup C^{\prime}$ or vice-versa. Hence $U_{1} \cap U_{2} \subseteq C^{\prime}$ which is a clique by definition.


Fig. la. The graph $G$ of Example 1.

The following example will be used throughout in this paper.
Example 1. Let $G=(V, E)$ be the graph shown in Fig. 1a, i.e. $V=\{1,2,3,4,5\}$ and $E=\{\{1,3\},\{1,4\},\{2,3\},\{3,4\},\{4,5\}\}$. The mp-subgraphs of $G$ are the graphs $G_{i}:=G\left(V_{i}\right), i=1,2,3$, with $V_{1}:=\{1,3,4\}, V_{2}:=\{2,3\}$ and $V_{3}:=\{4,5\}$.

Fig. 1b shows a successive decomposition of $G$ into its mp-subgraphs, i.e. into the minimal derived system. First $G$ is decomposed by the triple ( $\{1,2,3\},\{4\},\{5\})$ into $G(\{1,2,3,4\})$ and $G(\{4,5\})=G_{3}$. In a second step $G(\{1,2,3,4\})$ is decomposed by $(\{1,4\},\{3\},\{2\})$ into $G_{1}$ and $G_{2}$. Both decompositions are P-decompositions since $G_{1}$ and $G_{2}$ are the mp-subgraphs of $G(\{1,2,3,4\})$.

Alternatively, the graph $G$ may successively be decomposed as follows:

- decompose $G$ by $\left(A_{1}, C_{1}, B_{1}\right):=(\{2,5\},\{3,4\},\{1\})$ into $G^{\prime}:=$ $G(\{2,3,4,5\})$ and the prime graph $G_{1}=G(\{1,3,4\})$,
- decompose $G^{\prime}:=(\{2,3,4,5\})$ by $\left(A_{2}, C_{2}, B_{2}\right):=(\{4,5\},\{3\},\{2\})$ into $G^{\prime \prime}:=G(\{3,4,5\})$ and the prime graph $G_{2}=G(\{2,3\})$,
- decompose $G^{\prime \prime}:=G(\{3,4,5\})$ by $\left(A_{3}, C_{3}, B_{3}\right):=(\{3\},\{4\},\{5\})$ into the prime graphs $G(\{3,4\})$ and $G_{3}=G(\{4,5\})$.
This decomposition sequence decomposes the graph $G$ into its mp-subgraphs $G_{1}, G_{2}$, and $G_{3}$ and the additional prime subgraph $\sigma(\{3,4\})$ which is a subgraph of $G_{1}$. The graph $G(\{3,4\})$ is an mp-subgraph of $u$ but not of $G$. Hence, the


Fig. 1b. Successive decomposition of $G$ into its maximal prime subgraphs.


Fig. 2. The graph $G$ of Example 2.
decomposition $\left(A_{1}, C_{1}, B_{1}\right)$ of $G$ into $G^{\prime}:=G(\{2,3,4,5\})$ and $G_{1}=G(\{1,3,4\})$ is not a $P$-decomposition.

In the example above all mp-subgraphs of $G$ are cliques. Graphs with this property will be further investigated in Section 3 below.

Example 2. Replacing the clique $G(\{1,3,4\})$ in the graph $G$ of Example 1 by a chordless 4 -cycle with vertices $\{1,3,4,6\}$ we get the graph $G^{\sim}$ shown in Fig. 2. The inp-subgraphs of $G^{\sim}$ are the cliques $G^{\frown}(\{2,3\})$ and $G^{\sim}(\{4,5\})$ and the chordless 4 -cycle $G^{\sim}(\{1,3,4,6\})$ which is not a clique of course. All results derived for $G$ in this section hold for $G^{-}$as well if vertex 6 is added to sets containing vertex 1 . Therefore further details are omitted.
It follows from the lemma that a decomposition $(A, C, B)$ is a P -decomposition if and only if $G(C)$ is not an mp-subgraph of any of the graphs $G(A \cup C)$ and $G(B \cup C)$. Lemma 2.1(i) aiso implies that every derived system of a graph $G$ contains at least the mp-subgraphs of $G$.
If a graph $G$ is recursively decomposed by P-decompositions, then it follows from the definition of $\mathbf{P}$-decompositions that the resulting derived system consists of the pairwise different mp -subgraphs of $G$. Conversely, assume that a decomposition ( $A, C, B$ ) of $G$ into $G^{\prime}=G\left(A \cup C\right.$ ) and $G^{\prime \prime}=G(B \cup C)$ (or similarly any subsequent decomposition) is not a P-decomposition, i.e. $G(C)$ is an mp-subgraph of $G^{\prime}$ or $G^{\prime \prime}$ but not of $G$ (or $G(C)$ is an mp-subgraph of $G^{\prime}$ and $G^{\prime \prime}$ ). According to Lemma 2.1(i) $G(C)$ is contained in the derived system (or occurs twice in the derived system). Hence we can summarize the following.
(2.2) A derived system of a graph (: atas the mp-subgraphs of $G$; it contains exactly these subgraphs (and is :,it?e minimal) if and only if all decompositions involved are P-decomposition:

We are now going to duine a D-ordered sequence of sets which is a central notion in this section. The letter ' $D$ ' stands for decomposition (see Corollary 2.7) or for decomposable graph (see Theorem 3.3). Let $V_{1}, \ldots, V_{N}$ be a sequence of sets, e.g. subsets of the vertex set of a given graph. For $t=!, \ldots, T$ define $R_{t}:=V_{t} \cap\left(V_{1} \cup \cdots \cup V_{t-1}\right)$, the set of clements of $V_{1}$ contained in the remaining sequence $V_{1}, \ldots, V_{t-1}$, and $S_{t}:=V \backslash R_{t}$, the set of elements specific for $V_{t}$ w.r.t. $V_{1}, \ldots, V_{1}$ (especially $R_{1}=\emptyset$ and $S_{1}=V_{1}$ ), i.e. the sequence $V_{1}, \ldots, V_{1}$ is scanned from right to left for this definition.

Below, the symbols $R_{t}$ and $S_{t}$ will always have this meaning. The collection of sets $R_{1}, \ldots, R_{T}$ is called the $R$-system of $V_{1}, \ldots, V_{T}$. The order of the sets in the $R$-system shall not be fixed.

Definition 2.3. The sequence $V_{1}, \ldots, V_{T}$ is said to be $D$-ordered, if for all $t=2, \ldots, T$ there is a $p<t$ with $R_{t} \subseteq V_{p}$.

The sequence $V_{1}, \ldots, V_{T}$ is also said to have the running intersection property ( $[2,3]$ ), if it satisfies the above condition.

Example 1 (cont.). Considei the sequence of sets $V_{1}, V_{2}, V_{3}$ defined above, i.e. the sequence

$$
\{1,3,4\},\{2,3\},\{4,5\} .
$$

Then $R_{1}=\emptyset, R_{2}=\{3\}$, and $R_{3}=\{4\}$. This sequence is D-ordered since $R_{2} \subseteq V_{1}$ and $R_{3} \subseteq V_{1}$. The sets $S_{1}=\{1,3,4\}, S_{2}=\{2\}$, and $S_{3}=\{5\}$ are non-empty, pairwise disjoint and they define a partition of $V=V_{1} \cup V_{2} \cup V_{3}=\{1, \ldots, 5\}$.

Now consider the sequence $V_{2}, V_{3}, V_{1}$, i.e.

$$
\{2,3\},\{4,5\},\{1,3,4\} .
$$

Then $R_{1}=R_{2}=\emptyset$ and $R_{3}=\{3,4\}$. This sequence is not D -ordered since $\{3,4\} \nsubseteq$ $\{2,3\}$ and $\{3,4\} \nsubseteq\{4,5\}$. Similarly the sequence $V_{3}, V_{2}, V_{1}$ is not D-ordered as well.
On the other hand the sequences $V_{1}, V_{3}, V_{2}$ and $V_{2}, V_{1}, V_{3}$ and $V_{3}, V_{1}, V_{2}$ are D -ordered and they all have the same $R$-system as $V_{1}, V_{2}, V_{3}$, i.e. the corresponding sets $R_{1}, R_{2}$, and $R_{3}$ are permutations of the sets $\emptyset,\{3\}$, and $\{4\}$. Note that there are D-orderings with $V_{1}, V_{2}$ and $V_{3}$, respectively, as the first set. These properties generalize (see Proposition 2.4(ii) and (iii)).

The following proposition summarizes some useful properties of D -ordered scts.

Proposition 2.4. Let $V_{1}, \ldots, V_{T}$ be a $D$-ordered sequence of sets.
(i) Let $t, 1 \leqslant t \leqslant T$, be fixed. If there is an $s \neq t$, $s$ minimal, such that $V_{t} \subseteq V_{s}$, then:
(i.1) $V_{1}, \ldots, V_{t-1}, V_{t+1}, \ldots, V_{T}$ is $D$-ordered if $s<t$;
(i.2) $V_{1}, \ldots, V_{t-1}, V_{s}, V_{t+1}, \ldots, V_{s-1}, V_{s+1}, \ldots, V_{T}$ is $D$-ordered if $s>t$.
(ii) Every permutation of $V_{1}, \ldots, V_{T}$ which is also $D$-ordered, has the same $R$-system.
(iii) For every $t \in\{1, \ldots, T\}$ there is a permutation $\sigma:\{1, \ldots, T\} \rightarrow$ $\{1, \ldots, T\}$ with $\sigma(1)=t$ such that $V_{\alpha(1)}, \ldots, V_{\sigma(T)}$ is $D$-ordered.

Proof: (i) (i.1) is obvious. (i.2) The minimality of $s$ and the D-c:dering of $V_{1}, \ldots, V_{T}$ imply $\left(V_{s} \backslash V_{t}\right) \cap V_{j}=\emptyset$ for $j<s$. (i.2) follows directly from this property.
(ii) The proof is by induction on $T$. The case $T=1$ is trivial. For the general case $T \geqslant 2$ let $V_{1}, \ldots, V_{T}$ be D -ordered and let $V_{\sigma(1)}, \ldots, V_{\sigma(T)}$ be a D-ordered permutation of that sequence. Define $R_{t}:=V_{t} \cap\left(V_{1} \cup \cdots \cup V_{t-1}\right)$ for $t=$ $1, \ldots, T, V:=V_{\sigma(T)}, R:=V_{\sigma(T)} \cap\left(V_{\sigma(1)} \cup \cdots \cup V_{\sigma(T-1)}\right)$, and $S:=V \backslash R$. Let $t:=\sigma(T)$ denote the position of $V$ in $V_{1}, \ldots, V_{T} . S \cap V_{q}=\emptyset$ for $q \neq t$ and there is a $p \neq t, p$ minimal, such that $R \subseteq V_{p}$. We have to distinguish two cases.

Case 1: $p<t$.
Then we have

$$
V_{1}, \ldots,(R \subseteq) V_{p}, \ldots,(V=) V_{t}, \ldots, V_{T}
$$

with $R_{t}=R$. The subsequence $V_{1} \ldots, V_{t-1}, V_{t+1}, \ldots, v_{T}$ is $D$-ordered and $V_{\sigma(1)}, \ldots, V_{\sigma(T-1)}$ is a D -ordered permutation of this subsequence. The $R$ systems of the subsequences are the same as for the full sequences, except that the set $R$ is omitted once, and they are identical by the induction hypothesis. Hence the $R$-systems of the full sequences are also identical in this case.
Case 2: $p>t$.
Defining $W:=V_{p} \backslash R$ we get

$$
V_{1}, \ldots V_{t}(=R \cup S), \ldots, V_{p}(=R \cup W), \ldots, V_{r}
$$

where $W^{\prime} \cap S=\emptyset$ and $R \subseteq R_{p}$. Using the minimality of $p$ and the D-ordering of $V_{i}, \ldots, V_{T}$ we get $R_{p}=R$. Interchanging $V_{t}$ and $V_{p}$ in $V_{1}, \ldots, V_{T}$ does not violate the D -ordering of the sequence and its does not affect the $R$-system of the sequence. But then we are in Case 1 and the proof of (ii) is completed.
(iii) We can assume without loss of generality $t=T$, since otherwise we can consider the subsequence $V_{1}, \ldots, V_{t}$ first and define $\sigma(s):=s$ for $s>t$. So assume that $t=T$. The proof is by induction on $T$. The case $T=1$ is trivial. Let $V_{1}, \ldots, V_{T}$ be D -ordered for some $T \geqslant 2$. There is a $p<T$ such that $R_{T} \subseteq V_{p}$. Using the induction hypothesis there is a permutation $\sigma^{\prime}:\{1, \ldots, T-1\} \rightarrow$ $\{1, \ldots, T-1\}$ with $\sigma^{\prime}(1)=p$ such that $V_{\sigma^{\prime}(1)}, \ldots, V_{\sigma^{\prime}(T-1)}$ is D-ordered. Defining $\sigma(1):=T$ and $\delta(s):=\sigma^{\prime}(s-1)$ for $s>1$ we get the desired permutation.

In Example i we considered a graph $G$ with mp-subgraphs $G\left(V_{1}\right), \ldots, G\left(V_{T}\right)$, $T=3$. We heve seen that for every $t \in\{1, \ldots, T\}$ there is a D-ordering of the sets $V_{1}, \ldots, V_{T}$ containing $V_{t}$ as the first set in the sequence. Theorem 2.5 shows that this property generalizes.

Theorem 2.5. For every mp-subgraph $G(U)$ of $G$, there is an ordering of the different mp-subgraphs $\left.G\left(V_{1}\right), \ldots, G_{( } V_{T}\right)$ of $G$, such that:
(i) $V_{1}, \ldots, V_{T}$ is $D$-ordered and
(ii) $V_{1}=U$.

Part (i) of the theorem is almost equivalent to the existence of a so-called reduced simplicial decomposition of a graph, proved by Wagner and Halin [27], while the property of a simplicial decomposition corresponding to (ii) was stated in [11]. Our proof of Theorem 2.5 copies a proof of Haberman ([13], Lemma 5.10; see also Theorem 3 in Andersen [1]) and we think that the proof given here is more direct than that in [27]. Haberman proved what we might call the existence of a D-ordering of the maximal cliques (hyperedges) of a decomposable (hyper-)graph (see Theorem 3.3 and 3.6 of the following section). It is surprising that Haberman's proof works exactly the same way for the mp-subgraphs of an arbitrary graph.

Proof of Theorem 2.5. The existence of a D-ordering is proved by induction on $n=|V|$. Then the existence of a D-ordering satisfying (ii) follows from Proposition 2.4(iii). The case $n=1$ is trivial. Let $G=(V, E)$ with $n \geqslant 2$ and assume that the theorem is true for all graphs with less than $n$ vertices. If $G$ is prime, there is nothing to prove. Otherwise let ( $A, C, B$ ) be a decomposition of $\boldsymbol{G}$. By the induction hypothesis there is a D -ordering $A_{1}, \ldots, A_{a}$ of the vertex sets of the mp -subgraphs of $G(A \cup C)$. Using Proposition 2.4(iii), there is also a D-ordering $B_{1}, \ldots, B_{b}$ of the vertex sets of the mp-subgraphs of $G(B \cup C)$ with $C \subseteq B_{1}$, since $G(C)$ is prime. It is easy to see that the joint sequence

$$
\begin{equation*}
A_{1}, \ldots, A_{a}, B_{1}, \ldots, B_{b} \tag{2.6}
\end{equation*}
$$

is also D -ordered. If $(A, C, B)$ is a P -decomposition, then (2.6) is the desired D-ordering. Otherwise $G(C)$ is an mp-subgraph of $G(A \cup C)$ or/and $G(B \cup C)$, i.e.:
( $\alpha$ ) $C$ occurs exactly once in the sequence (2.6) and it is strictly contained in another set of the sequence or
( $\beta$ ) $C$ occurs exactly twice in the sequence (2.6).
In both cases, we get the desired sequence after omitting $C$ once in the sequence (2.6) according to Proposition 2.4(i).

Corollary 2.7. (i) If $V_{1}, \ldots, V_{T}$ are the $D$-ordered vertex sets of the mp-subgraphs of a reducible graph, then

$$
(A, C, B):=\left(\bigcup_{t=1}^{T-1} V_{t} \backslash R_{T}, R_{T}, S_{T}\right)
$$

is a $P$-decomposition of $G$ into

$$
G^{\prime}=G(A \cup C)=G\left(\bigcup_{t=1}^{T-1} V_{t}\right)
$$

and the prime graph $G^{\prime \prime}=G(B \cup C)=G\left(V_{T}\right), G\left(V_{1}\right), \ldots, G\left(V_{T-1}\right)$ are the mp-subgraphs of $G^{\prime}$ and the sequence $V_{1}, \ldots, V_{r_{-1}}$ is $D$-ordered.
(ii) If there exists a decomposition for $G$, then there also exists a $P$ decomposition for $G$.

Proof. (ii) follows directly from (i). (i) $C=R_{T}$ is a clique by Lemma 2.1(iii). Using Lemma 2.1(i), (ii) it remains to show that $(A, C, B)$ is a decomposition of $G$ and that $G(C)$ is not an mp-subgraph of $G^{\prime}: V_{i} \nsubseteq V_{j}$ for $i \neq j$ implies $A \neq \emptyset$ and $B \neq \emptyset$. If $\{b, v\} \in E$ for some $b \in B, v \in V \backslash B$, then there is an mp-subgraph $G(U)$ of $G$ with $\{b, v\} \subseteq U$, and we have $U=V_{T}=B \cup C$ by the definition of $B$, hence $v \in C$. This shows that $(A, C, B)$ is a decomposition of $G$. Furthermore, $C=R_{T} \subset V_{p}$ for some $p<T$, hence $G(C)$ is not maximal prime w.r.t. $G^{\prime}$.

Corollary 2.7(i) can be applied recursively: Since the sequence $V_{1}, \ldots, V_{T-1}$ is D-ordered, $G^{\prime}$ can be decomposed in the same way as $G$ (if $T>2$ ) and so on, until finally $G\left(V_{1} \cup V_{2}\right)$ is decomposed into $G\left(V_{1}\right)$ and $G\left(V_{2}\right)$. The desired system then consists of exactly one copy of each of the mp-subgraphs of $G$. Hence,
with every D-ordering of the vertex sets of the mp-subgraphs of a graph $G$, there is associated a recursive decomposition of $G$ into its mp-subgraphs.
In Example 1 the sequence $V_{1}, V_{2}, V_{3}$, i.e. the sequence $\{1,3,4\},\{2,3\},\{4,5\}$, is a $D$-ordering of the vertex sets of the mp-subgraphs of $G$. The recursive decomposition of $G$ into its mp-subgraphs shown in Fig. 1b corresponds to that sequence: $G$ is first decomposed by $\left(\left(V_{1} \cup V_{2}\right) \backslash R_{3}, R_{3}, S_{3}\right)=(\{1,2,3\},\{4\},\{5\})$ into $G(\{1,2,3,4\})=\boldsymbol{G}\left(V_{1} \cup V_{2}\right)$ and the prime graph $G(\{4,5\})=G\left(V_{3}\right)$. The graphs $G\left(V_{1}\right)$ and $G\left(V_{2}\right)$ are the mp-subgraphs of $G\left(V_{1} \cup V_{2}\right)$ and the sequence $V_{1}, V_{2}$ is D-ordered with $R_{2}=\{1,3,4\} \cap\{2,3\}=\{3\}$ and $S_{2}=\{2\}$. According to Corollary 2.7(i) $\left(V_{1} \backslash R_{2}, R_{2}, S_{2}\right)=(\{1,4\},\{3\},\{2\})$ is a P-decomposition of $G\left(V_{1} \cup V_{2}\right)$ into $G\left(V_{1}\right)$ and $G\left(V_{2}\right)$.

The algorithm described in Section 5 which decomposes a graph into its mp-subgraphs, generates a D-ordering of the vertex sets of these subgraphs as well.

Generalizing a notation used in [15] and [26] we define the following.
Definition 2.8. An mp-subgraph $G(U)$ of a reducible graph $G$ is called extremal, if there is an mp-subgraph $G\left(U^{*}\right)$ with $U^{*} \neq U$, such that for every mp-subgraph $G\left(U^{\prime}\right)$ with $U^{\prime} \neq U$ we have $U^{\prime} \cap U \subseteq U^{*} \cap U$.

In the graph shown in Fig. 1a there are two extremal mp-subgraphs, namely $G_{2}=G(\{2,3\})$ and $G_{3}=G(\{4,5\})$ while $G_{1}=G(\{1,3,4\})$ is not extremal.

Corollary 2.9. For every reducible graph $G$ there exist at least two extremal mp-subgraphs.

Proof. Let $V_{1}, \ldots, V_{T}$ be a D-ordering of the vertex sets of the mp-subgraphs of $G$, then $T>1$ and $G\left(V_{T}\right)$ is extremal (see Theorem 2.5). There also exists a
permutation $\sigma$ such that $V_{\sigma(1)}, \ldots, V_{\sigma(T)}$ is D-ordered and $\sigma(1)=T$, hence $\sigma(T) \neq T$. Then $G\left(V_{\sigma(T)}\right)$ is an extremal mp-subgraph of $G$ which is different from $G\left(V_{T}\right)$.

Corollary 2.9 generalizes Dirac's result [10] that for every rigid circuit graph, there exist at least two non-adjacent simplicial vertices (see Proposition 3.5.). It also generalizes a related result for hypergraphs (see [15] and [26]) that for every decomposable hypergraph there exist at least two extremal hyperedges (see Corollary 3.7).

The following Theorem 2.10 reverses the result of Theorem 2.5. It is equivalent to the uniqueness of a simplicial decomposition proved in [27]. Assuric that $G_{1}=\left(V_{1}, E_{1}\right), \ldots, G_{T}=\left(V_{T}, E_{T}\right)$ are prime graphs with the following propertics:
( $\alpha$ ) $V_{i} \nsubseteq V_{j}$ for $i \neq j, i, j=1, \ldots, T$;
( $\beta$ ) $V_{i} \cap V_{j}$ is a clique of $G_{i}$ and $G_{j}$ for $i \neq j, i, j=1, \ldots, T$;
( $\gamma$ ) there is a D-ordering of the sets $V_{1}, \ldots, V_{T}$.
Define

$$
G:=\bigcup_{t=1}^{T} G_{t}:=\left(\bigcup_{t=1}^{T} V_{t}, \bigcup_{t=1}^{T} E_{t}\right)
$$

as the union of the $G_{t}$ 's.
Theorem 2.10. $G_{1}, \ldots, G_{T}$ are the different $m p$-subgraphs of $G$.
Proof. The proof is by induction on $T$. The case $T=1$ is trivial. Let $G_{1}, \ldots, G_{T}$ satisfy $(\alpha)-(\gamma)$, assume $V_{1}, \ldots, V_{T}$ is D-ordered, and

$$
G:=\bigcup_{t=1}^{T} G_{t}, \quad A:=\bigcup_{t=1}^{T-1} V_{t} \backslash R_{T}, \quad B:=S_{T}, \quad \text { and } \quad C:=R_{T},
$$

where $R_{T}$ and $S_{T}$ are defined as usual. ( $\beta$ ) implies $R_{T}$ is a clique. As in the above proof of Corollary $2.7(\mathrm{i})(A, C, B)$ is a decomposition for $G$. ( $\beta$ ) implies $G(A \cup C)=G_{1} \cup \cdots \cup G_{T-1}$ and $G(B \cup C)=G_{T}$. Using the induction hypothesis $G_{t}=G\left(V_{t}\right), t=1, \ldots, T-1$, and $G_{T}$ are the mp-subgraphs of $G(A \cup C)$ and $G(B \cup C)$, respectively. It follows from Lemma 2.1(i) and (ii) that they are the mp-subgraphs of $G$, since $V_{t} \neq C$ for all $t=1, \ldots, T$.

Note that the conditions $(\alpha)-(\gamma)$ are also necessary for Theorem 2.10 by Lemma 2.1(iii) and Theorem 2.5. In Theorem 3.6 we shall derive a well-known characterization of decomposable hypergraphs as a simple special case of Theorem 2.10.

Remark 2.11. Let $S$ be a sct of arbitrary graphs and

$$
S^{*}:=\left\{\bigcup_{t=1}^{T} G_{t}: T \in N, G_{t} \in S \text { for all } t \text { and } G_{1}, \ldots, G_{T} \text { satisfy }(\alpha)-(\gamma)\right\} .
$$

Then $\left(S^{*}\right)^{*}=S$. The proof of this relation is similar to the proof of Theorem 2.5.

## 3. Decomposable graphs and hypergraphs

In this section we are going to define decca!pusable (hyper-)graphs and to summarize some of their properties. We also show that for every graph $G$ with mp-subgraphs $G\left(V_{1}\right), \ldots, G\left(V_{T}\right)$, theie is a decomposable graph $G^{*}$, such that $V_{1}, \ldots, V_{T}$ are the maximal cliques of $G^{*}$. These properties will be important for the following sections.

Following [15] a graph $G$ is called decomposable if and only if $G$ is a clique or there exists a decomposition $(A, C, B)$ of $G$ into decomposable subgraphs $G(A \cup C)$ and $G(B \cup C)$. This recursive definition makes sense, since the number of vertices in $A \cup C$ and $B \cup C$ is less than in $V$. Dirac [10] has shown that a graph $G$ is decomposable in the above sense if and only if $G$ contains no chordless $n$-cycle for $n \geqslant 4$. Graphs with this property are called triangulated, rioid circuit, or chordal graphs.

We need some more notation for a graph $G=(V, E)$ :

$$
\operatorname{Adj}_{G}(U):=\{w \in V \backslash U: \text { there is an } u \in U \text { with }\{u, w\} \in E\}
$$

is called the adjacency set of $U \subseteq V$ w.r.t. $G$. We write $\operatorname{Adj}_{G}(v)$ for short if $U=\{v\}$ and omit the index $G$ if there is no confusion possible. A vertex $v \in V$ is called simplicial if $\operatorname{Adj}(v)$ is a clique.

A map $\pi: V \rightarrow\{1, \ldots, n\}$ is called a numbering of the vertices. $\pi$ is perfect for $G$ if the monotone adjacency sets $\operatorname{MAdj}(v):=\operatorname{Adj}(v) \cap\{w: \pi(w)>\pi(v)\}$ are cliques for all $v \in V$. The numberings of the vertices of a graph characterized in the following definition are important for the remaining sections, especially for the algorithm described in Section 5.

Definition 3.1. A numbering $\pi$ of the vertices of a graph $G$ is called a $D$-numbering if there is a $D$-ordering $C_{1}, \ldots, C_{T}$ of the maximal cliques of $G$ such that

$$
\pi\left(S_{1}\right)=\left\{n, n-1, \ldots, n-\left|S_{1}\right|+1\right\}, \ldots, \pi\left(S_{T}\right)=\left\{\left|S_{T}\right|, \ldots, 2,1\right\}
$$

where $S_{t}=C_{t} \backslash\left(C_{1} \cup \cdots \cup C_{t-1}\right)$ as usual.
It would be more natural and often more convenient to require $\pi\left(S_{1}\right)=$ $\left\{1, \ldots,\left|S_{1}\right|\right\}$ etc., but we would then have to reverse the inequality in the definition of a perfect numbering (such a numbering is called reducible by some authors) to get (3.2). We have not done so in consideration of [23] and [24], since we use their results in the following sections. With the above definition we get the following.

## (3.2) Every D-numbering is perfect.

This follows since for every $v \in V$, we have $v \in S_{t}$ for some (unique) $t$ and $\operatorname{MAdj}(v) \subseteq V_{t}$. The following example shows that the converse of (3.2) is not true.

Example 1 (cont.). Again we consider the graph $G$ shown in Fig. 1a. The graph $G$ is decomposable since $G$ can successively be decomposed into cliques (see Fig. 1b).

The sequence $\{1,3,4\},\{2,3\},\{4,5\}$ is a D -ordering of the maximal cliques of $G$ with $S_{1}=\{1,3,4\}, S_{2}=\{2\}$ and $S_{3}=\{5\}$. A corresponding D-numbering is defined by $\pi(1)=5, \pi(3)=3, \pi(4)=4, \pi(2)=2$, and $\pi(5)=1$. It follows from (3.2) that this numbering is perfect for $G$.

Now consider the numbering defined by the identity map id: $V \rightarrow V$ and the corresponding monotone adjacency sets:

| $\operatorname{MAdj}(1)=\{3,4\}$ | $\cap\{2,3,4,5\}$ | $=\{3,4\}$ |  |
| :--- | :--- | :--- | :--- |
| $\operatorname{MAdj}(2)$ | $=\{3\}$ | $\cap\{3,4,5\}$ | $=\{3\}$ |
| $\operatorname{MAdj}(3)$ | $=\{1,2,4\}$ | $\cap\{4,5\}$ | $=\{4\}$ |
| $\operatorname{MAdj}(4)$ | $=\{1,3,5\}$ | $\cap\{5\}$ | $=\{5\}$ |
| $\operatorname{MAdj}(5)$ | $=\{4\}$ | $\cap \emptyset$ | $=\emptyset$ |

Hence $\pi=$ id is another perfect numbering for $G$ but this is not a D-numbering. Hence the converse of (3.2) is not true. To see that the identity map is not a D-numbering note that for every D-ordering of the maximal cliques of $G$ we get $S_{3}=\{2\}$ or $S_{3}=\{5\}$. This implies that either vertex 2 or vertex 5 is numbered as 1 by a D -numbering but not vertex 1 .
If $\pi$ is a D-numbering of the vertices of a graph $G$, then the D-ordering $C_{1}, \ldots, C_{T}$ of the maximal cliques of $G$ in Definition 3.1 is unique (it will be called the D -ordering associated with $\pi$ ): If $\pi$ is a D -numbering and $\pi(w)=$ $\pi(v)+1$, then $v$ and $w$ are contained in a joint set $S_{t}$ if and only if $\operatorname{MAdj}(v)=\operatorname{MAdj}(w) \cup\{w\}$. Hence the partition of $V=S_{1} \cup \cdots \cup S_{T}$ is determined by $\pi$. Furthermore $C_{t}=\operatorname{MAdj}\left(s_{t}\right) \cup\left\{s_{t}\right\}$, if $s_{t} \in S_{t}$ satisfies $\pi\left(s_{t}\right)=$ $\min \left\{\pi(s): s \in S_{t}\right\}$. This proves the uniqueness of the D -ordering associated with $\pi$.

Theorem 3.3. The following properties are equivalent:
(i) $G$ is decomposable;
(ii) every mp-subgraph of $G$ is a clique;
(iii) there is a $D$-ordering $C_{1}, \ldots, C_{T}$ of the maximal cliques of $G$;
(iv) there is a perfect numbering $\pi$ of the vertices of $G$.

The equivalence of (i)-(iv) is more or lesss standard, see e.g. [10, 12, 15, 17-$19,22-24,28]$ for these and other characterizations. The implications (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) can be proved by a simple induction on $n=|V|$, (iii) $\Rightarrow$ (iv) follows from (3.2) and (ii) $\Rightarrow$ (iii) is now an immediate consequence of the general result in Theorem 2.5 concerning mp-subgraphs of an arbitrary graph. Using Proposition 2.4(iii), (3.2) and Theorem 3.3 we also get:
(3.4) If $G$ is decomposable, then every maximal clique $C$ can be chosen as $C_{1}$ for a $D$-ordering $C_{1}, \ldots, C_{T}$ of the maximal cliques of $G$ and there is a perfect ( $D$-)numbering $\pi$ with $\pi(C)=\{n, n-1, \ldots, n-|C|+1\}$.

Finally we derive an important property of decomposable graphs from the general results of the previous section. Proposition 3.5 was first proved in [10], see also [19].

Proposition 3.5. If $G$ is decomposable and not a clique, then there exist at least two non-adjacent simplicial vertices of $G$.

Proof. Using Corollary 2.9 and Theorem 3.3 we get: $G$ is reducible since $G$ is not a clique, there exist at least two different extremal mp-subgraphs $G\left(U_{1}\right)$ and $G\left(U_{2}\right)$ of $G, U_{1}$ and $U_{2}$ are cliques, $U_{1} \backslash U_{2} \neq \emptyset$, and $U_{2} \backslash U_{1} \neq \emptyset$. Then any two vertices $v \in \boldsymbol{L}_{1} \backslash U_{.}$and $w \in U_{2} \backslash U_{1}$ are simplicial and non-adjacent.

The simplicial vertices of the graph $G$ shown in Fig. la are the vertices 1,2 and 5. They are pairwise non-adjacent.

A hypergraph $H$ is a pair $H=\left(V,\left\{V_{1}, \ldots, V_{T}\right\}\right)$, where $V$ is a (here finite) set of vertices and $V_{1}, \ldots, V_{T}$ are subsets of $V$, called the edges or hyperedges of $\boldsymbol{H}$. $H$ is reduced if $V_{i} \nsubseteq V_{j}$ for $i \neq j$. For convenience we consider only reduced hypergraphs, but the following results do not depend on this restriction since subsets in a D-ordering can always be eliminated using Proposition 2.4(i).

The graph $G(H):=\left(V^{\prime}, E(H)\right)$ is called the (2-section) graph of the hypergraph $H=\left(V,\left\{V_{1}, \ldots, V_{T}\right\}\right)$, where an edge $\{e, f\}$ belongs to $E(H)$ if and only if $e \neq f$ and $\{e, f\} \subseteq V_{t}$ for some $t$. Note that $G(H)=G_{1} \cup \cdots \cup G_{T}$, where $G_{t}$ is the clique graph with vertex set $V_{t} . H=\left(V,\left\{V_{1}, \ldots, V_{T}\right\}\right)$ is conformal, if $V_{1}, \ldots, V_{T}$ are the maximal cliques of $G(H)$. A hyperedge $V_{i} \in\left\{V_{1}, \ldots, V_{T}\right\}$ is called extremai, if there is a hyperedge $V^{*} \neq V_{i}$ such that every hyperedge $V^{\prime} \neq V_{i}$ we have $V^{\prime} \cap V_{i} \subseteq V^{*} \cap V_{i}$.

Theorem 3.6. For a hypergrayh $H=\left(V,\left\{V_{1}, \ldots, V_{T}\right\}\right)$ the following properties are equivalent:
(i) There is a $D$-ordering of the sets $V_{1}, \ldots, V_{T}$;
(ii) $H$ is a conformal hypergraph and $G(H)$ is a decomposable graph.
$H$ is said to be a decomposable hypergraph, if it satisfies the above properties (i). (ii). This definition is equivalent to the recursive definition given in [15] (see their Theorem 1). The equivalence of (i) and (ii), together with some ten other characterications (e.g. $H$ is acyclic) can be found in [3], see also [2]. But Theorem 3.6 is also a special case of Theorem 2.10: If (i) is satisfied, then the clique graphs $G_{1}, \ldots, G_{T}$ are the mp-subgraphs of $G(H)=G_{1} \cup \cdots \cup G_{T}$ by Theorem 2.10. Hence $V_{1}, \ldots, V_{T}$ are the maximal cliques of $G(H)$ and $G(H)$ is decomposable
$\mathrm{L}_{\mathrm{j}}$ Theorem 3.3. The implication (ii) $\Rightarrow$ (i) follows from the definition of a coniormal hypergraph and Theorem 3.3.

We get the following corollary (see also [15] and [26]):
Corollary 3.7. If $H=\left(V,\left\{V_{1}, \ldots, V_{T}\right\}\right)$ is a decomposable hypergraph and $T>1$, then there exist at least two extremal hyperedges of $H$.

Proof. Using Thegrem 3.6 we can assume that $V_{1}, \ldots, V_{T}$ is D -ordered. Proposition 2.4(iii) implies the existence of a permutation $\sigma$ with $\sigma(1)=T$, i.e. $\sigma(T) \neq T$, such that $V_{\sigma(1)}, \ldots, V_{\sigma(T)}$ is D-ordered. Then $V_{T}$ and $V_{\sigma(T)}$ are two different extremal hyperedges of $H$.

Theorems 2.5 and 3.6 nave a useful corollary: Let $G\left(V_{1}\right) \ldots, G\left(V_{T}\right)$ be the mp-subgraphs of a graph $G=(V, E)$ and $H:=\left(V,\left\{V_{1}, \ldots, V_{T} j\right)\right.$. Then:
(3.8) $G^{*}:=\left(V, E^{*}\right):=G(H)$ is a decomposable graph and $V_{1}, \ldots, V_{T}$ are the maximal cliques of $G^{*}$.
$G^{*}=\left(V, E^{*}\right)$ may be constructed from $G=(V, E)$ by adding all edges $\{e, f\} \notin E$ with $e \neq f$ and $\{e, f\} \subseteq V$, for some $t$; especially $E \subseteq \dot{E}^{*}$.
Decomposable graphs are usually much easier to investigate than arbitrary graphs. Especially the existence of a perfect numbering of the vertices of a decomposable graph (see Theorem 3.3) is essential for many algorithms (see e.g. [24, 25]). As far as clique separators of a graph $G$ are concerned, the following lemma allows to investigate the (simpler) decomposable graph $G^{*}$ of (3.8) instead of $G$. This technique is illustrated in the following two sections.

Lemma 3.9. Let $G, G^{*}$ be as in (3.8) and let $C$ be a clique' of $G$. Then:
$C$ is a $(P-)$ separator for $G=(V, E)$ if and only if
$C$ is a $(P-)$ separator for $G^{*}=\left(V, E^{*}\right)$.
Proof. If $C$ is a separator for $G$, there is a decomposition $(A, C, B)$ for $G$. Assume $\{a, b\}$ is an edge of $G^{*}$ for some $a \in A$ and $b \in B$. Then there is a maximal clique $V_{t}$ of $G^{*}$ containing $a$ and $b$, but $G\left(V_{t}\right)$ is an mp-subgraph of $G$ which contradicts Lemma 2.1(i). This proves that every separator for $G$ is also a separator for $G^{*}$. The converse is trivial since $E \subseteq E^{*}$. The equivalence for P -decompositions is also obvious, since the vertex sets of the mp-subgraphs of $G$ and $G^{*}$ are identical.

Example 2 (cont.). To simplify notation let us now denote the graph shown in Fig. 2 by $G$ instead of $G^{\sim}$. The mp-subgraphs of $G$ are the cliques $G(\{2,3\})$ and $G(\{4,5\})$ and the chordless 4-cycle $G(\{1,3,4,6\})$. The graph $G^{*}$ is derived from $G$ by adding the chords of the 4 -cycle, i.e. the edges $\{1,4\}$ and $\{3,6\}$. The
maximal cliques of $G^{*}$ are $\{1,3,4,6\},\{2,3\}$, and $\{4,5\}$ and this sequerce is D-ordered. Hence $G^{*}$ is decomposable (Theorem 3.3(iii)) as asserted in (3.8).

The sets $\{3\}$ and $\{4\}$ are $P$-separators for $G$ and $G^{*}$, while e.g. $\{3,4\}$ or $\{1,3\}$ are separators but not P -separators for $G$ and $G^{*}$. The set $C=\{1,3,6\}$ is a separator for $G^{*}$ but not for $G$. Since $C$ is not a clique of $G$ this does not controdict Lemma 3.9.

The above Lemma 3.9 remains true if $G^{*}=\left(V, E^{*}\right)$ is replaced by any other graph $G^{\prime}=(V, E \cup F)$ with $F \subseteq E^{*} \backslash E$, while it becomes false for every $F \nsubseteq E^{*} \backslash$ $E$. Especially every minimal fill-in graph $G_{\pi}=\left(V, E \cup F_{\pi}\right)$ for $G$ (see Section 5) satisfies $\boldsymbol{F}_{\pi} \subseteq E^{*} \backslash E$. These considerations generalize Lemma 1 of [24] (see also Lemma 5.1(ii) below).

## 4. Characterizations of P-separators

P-separators are characterized in several ways in this section. The equivalence of the characterizations (iii) and (iv) in Theorem 4.1 is essential for the correctness of the algorithm described in the following section.

Similar as in [27], a clique separator $C \subseteq V$ is called admissible for $G=(V, E)$ if there are at least two different connected components $X$ and $Y$ of $G(V \backslash C)$ with $\operatorname{Adj}(X)=C=\operatorname{Adj}(Y) . C \subseteq V$ is a minimal separator for $v, w \in V$, if $C$, but no proper subset of $C$, separates $\{v\}$ and $\{w\}$ in $G$. $C$ is a relative minimal separator for $G$. if there are vertices $v, w \in V$ such that $C$ is a minimal separator for $v$ and $w$.

Let $G=\left(V^{\prime}, E\right)$ be a graph with the mp-subgraphs $G\left(V_{1}\right), \ldots, G\left(V_{T}\right)$. We may assume (Theorem 2.5) that the sequence $V_{1}, \ldots, V_{T}$ is D -ordered and $R_{t}:=V_{t} \cap$ ( $V_{1} \cup \cdots \cup V_{t-1}$ ) as usual. Note that the R -system of $V_{1}, \ldots, V_{T}$ is the same for every D-ordered permutation of this sequence (see Proposition 2.4(ii)).

Theorem 4.1. The following properties are equivalent for $C \subseteq V$ :
(i) $C$ is a clique and a relative minimal separator for $G$;
(ii) $C$ is an admissible separator for $G$;
(iii) $C$ is a $P$-separator for $G$;
(iv) $C \in\left\{R_{2}, \ldots, R_{T}\right\}$.

Proof. We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).
(i) $\Rightarrow$ (ii): is obvious.
(ii) $\Rightarrow$ (iii): Assume first that $G$ is decomposable. Let $X, Y$ be two different connected components of $G(V \backslash C)$ with $\operatorname{Adj}(X)=C=\operatorname{Adj}(Y)$ and let $(A, C, B)$ be a decomposition of $G$ with $X \subseteq A$ and $Y \subseteq B$. To show that $(A, C, B)$ is a P -decomposition of $G$ it is sufficient to prove (see Lemma 2.1(ii) and Theorem 3.3(ii)) that $C$ is not a maximal clique of $G(X \cup C)$ or $G(Y \cup C)$.
$G(X \cup C)$ (and similarly $G(Y \cup C)$ ) is decomposable and $C$ is a clique, hence (see (3.4)) there is a perfect numbering $\pi$ of the vertices of $G(X \cup C)$ with $\pi(C)=\{k, k-1, \ldots, k-|C|+1\}$, where $k:=|X \cup C|$. Let $x$ be the vertex with $\pi(x)=k-|C|$. The facts that $G(X)$ is connected, $\pi$ is perfect for $G(X \cup C)$, and $\operatorname{Adj}(X)=C$ imply $C \subseteq \operatorname{Adj}(x)$ (see also [22], Theorem 3, or [24], Proof of Lemma $2)$ and hence $C \cup\{x\}$ is a clique. This proves (ii) $\Rightarrow$ (iii), if $G$ is decomposable.

If $C$ is an admissible separator for an arbitrary graph $G$ then $C$ is a separator for the decomposable Graph $G^{*}$, defined in (3.8) (see Lemma 3.9) and $C$ is admissible w.r.t. $G^{*}$ too, since $G \subseteq G^{*}$. Hence $C$ is a P-separator for $G^{*}$, as we have just proved. This implies that $C$ is also a P -separator for $G$ (Lemma 3.9).
(Note that we could have also used (3.1) of [14] which is proved in a different manner.)
(iii) $\Rightarrow$ (iv): Let $(A, C, B)$ be a P-decomposition of $G, G\left(A_{1}\right), \ldots, G\left(A_{a}\right)$ and $G\left(B_{1}\right), \ldots, G\left(B_{h}\right)$ the mp-subgraphs of $G(A \cup C)$ and $G(B \cup C)$, respectively, and assume that $A_{1}, \ldots, A_{a}$ and $B_{1}, \ldots, B_{b}$ are D-ordered with $C \subseteq B_{1}$ (see Theorem 2.5). Since ( $A, C, B$ ) is a P-decomposition $A_{1}, \ldots, A_{a}, B_{1}, \ldots, B_{b}$ are the vertex sets of the mp-subgraphs of $G$ and this sequence is obviously D-ordered with $B_{1} \cap\left(A_{1} \cup \cdots \cup A_{a}\right)=C$, hence $R_{a+1}=C$ for this D -ordering. Using Proposition 2.4(ii) we get the desired result.
(iv) $\Rightarrow$ (i); Let $V_{1}, \ldots, V_{T}$ be the D-ordered vertex sets of the mp-subgraphs of $G$ and $C=R_{t}$ for some $t \geqslant 2$. There is a $p<t$ with $R_{t}=V_{p} \cap V_{t}$ and $R_{t}$ is a clique by Lemma 2.1(iii). Let $A \subseteq V$ denote the connected component of $G(V \backslash C)$ containing $V_{t} \backslash C$ and $B:=V \backslash(A \cup C)$. It follows by an induction on $T-t$ that $\left(V_{1} \backslash C\right) \cup \cdots \cup\left(V_{t-1} \backslash C\right) \subseteq B$, similar as (1.1) in [14]. So $(A, C, B)$ is a decomposition of $G$ and it is also a P-decomposition by Lemma 2.1(i) and (ii), since $C \subset V_{p} \subseteq B \cup C$ and $C \subset V_{t} \subseteq A \cup C$. Hence we have shown the following.
(4.2) $(A, C, B)$ is a $P$-decomposition of $G$ with $\bigcup_{s=1}^{\prime-1} V_{s} \backslash C \subseteq B, V_{\uparrow} \backslash C \subseteq A$, and $C=R_{t}$.
$G\left(V_{p}\right), G\left(V_{t}\right)$ are prime and $C$ a clique, hence $V_{p} \backslash C, V_{t} \backslash C$ are connected and $C \subseteq \operatorname{Adj}\left(V_{p} \backslash C\right), C \subseteq \operatorname{Adj}\left(V_{t} \backslash C\right)$. Therefore $C$ is a minimal separator for every $v \in V_{P} \backslash C$ and $w \in V_{t} \backslash C$ and $C$ is a clique.

Example 1 (cont.). In the graph $G$ of Fig. 1a every separator $C$ for the vertices 1 and 5 contains vertex and $C=\{4\}$ is a separator for these vertices. Hence $C=\{4\}$ is the (unique) minimal separator tor vertices 1 and 5 and it is a relative minimal separator for $G$. The connected components of $G(V \backslash C)=G(\{1,2,3,5\})$ are $X=\{1,2,3\}$ and $Y=\{5\}$ with $\operatorname{Adj}(X)=\operatorname{Adj}(Y)=\{4\}=C$.

We have already shown that $C=\{4\}$ is a $P$-separator for $G$ (see Fig. 1b) and that the R-system corresponding to the vertex sets of the mp-subgraphs of $G$ consists of $\emptyset,\{3\}$, and $\{4\}$. Hence $C=\{3\}$ satisfies conditions (i)-(iv) of Theorem 4.1. On the other hand, the set $\{3,4\}$ e.g. is a separator for $G$ which does not satisfy any of these conditions.

Theorem 4.1(i) shows that P -separators have a minimality property. In contrast, Tarjan [24] considered maximal clique separators. Especially, if $G$ is decomposable then no P -separator is a maximal clique separaior.

## 5. An algcrithm for optimal decomposition by clique separators

We are going to describe an $\mathbf{O}(\mathrm{nm})$-time algorithm to decompose an $n$-vertex, $m$-edge ( $m \geqslant 1$ ) graph $G$ optimally, i.e. such that the derived system of prime subgraphs consists of exactly one copy of each mp-subgraph of $G$. It was shown in Section 2 that this is the unique minimal derived system. Our algorithm will be a modification of an algorithm described by Tarjan [24].
The algorithm uses a minimal numbering of the vertices of a graph $G=(V, E)$ : for every numbering $\pi$ of the vertices of $G$, there is a unique minimal set $F_{\pi}$ of edges, such that $\pi$ is a perfect numbering for $G_{\pi}:=\left(V, E \cup F_{\pi}\right)$. The graph $G_{\pi}$ is called the fill-in graph of $\pi$ and $G_{\pi}$ is decomposable by Theorem 3.3 (see e.g. [21,23,24]). The numbering $\pi$ is called minimal, if there is no numbering $\sigma$ with $F_{\sigma} \subset F_{\pi}$ and $G_{\pi}$ is then called a minimal fill-in graph. Our modification of Tarjan's algorithm uses the fact that for every minimal numbering $\pi$, there exists a numbering $\pi^{\prime}$ with $F_{\pi}=F_{\pi^{\prime}}$, such that $\pi^{\prime}$ is a D-numbering (see Definition 3.1) w.r.t. the decomposable graph $G_{\pi^{\prime}}=\boldsymbol{G}_{\boldsymbol{\pi}}$. This follows from Theorem 3.3 and (3.2).

If $\pi$ is a D -numbering w.r.t. $G_{\pi}$, then the D -ordering $C_{1}, \ldots, C_{T}$ of the maximal cliques of $G_{\pi}$ associated with $\pi$ is unique (see Section 3). Hence we can define $F(\pi):=\left(f_{2}, \ldots, f_{T}\right)$, where $f_{t} \in S_{t}=C_{t} \backslash\left(C_{1} \cup \cdots \cup C_{t-1}\right)$ is determined by $\pi\left(f_{t}\right)=\max \left\{\pi(s): s \in S_{t}\right\}$ for $t=2, \ldots, T$. (If the vertices are numbered backwards as $n, n-1, \ldots, 1$, then $f_{t}$ is the first vertex of $S_{t}$ which is numbered.) The vector $F(\pi)$ is important for the algorithm below, since $\operatorname{MAdj}\left(f_{t}\right)=R_{t}=C_{t} \backslash S_{t}$ and a set $C$ is $P$-separator for $G_{\pi}$ if and only if $C \in\left\{R_{2}, \ldots, R_{T}\right\}$ (see Theorem 4.1).

Example 2 (cont.). Let $G=(V, E)$ denote the graph shown in Fig. 2 and lei $\pi$ denote the numbering of $G$ defined by $\pi(4)=6, \pi(6)=4$, and $\pi(i)=i$ otherwise. Then $G_{\pi}=(V, E \cup\{3,6\})$, while e.g. $G_{\mathrm{id}}=(V, E \cup\{\{3,6\},\{5,6\}\})$. Since $G$ itself is not decomposable, we get that $\pi$ (but not id) is a minimal numbering for G.

The cliques of $G_{\pi}$ are $\{4,5\},\{3,4,6\},\{2,3\},\{1,3,6\}$. This sequence is D-ordered with $S_{1}=\{4,5\}, S_{2}=\{3,6\}, S_{3}=\{2\}$, and $S_{4}=\{1\}$ and this is the D-ordering associated with $\pi$. It follows that $F(\pi)=\left(f_{2}, f_{3}, f_{4}\right)=(6,2,1)$ since $\max \left\{\pi(s): s \in S_{2}\right\}=4=\pi(6)$.
Our algorithm to decompose a graph $G=(V, E)$ into its mp-subgraphs consists of the following two steps.

Step 1: Find a minimal numbering $\pi$ for $G$ such that $\pi$ is a $D$-numbering w.r.t. $G_{\pi}$, determine the vector $F(\pi)=\left(f_{2}, \ldots, f_{T}\right)$, and compute $C\left(f_{t}\right):=\operatorname{MAdj}_{G_{T}}\left(f_{t}\right)$ for $t=2, \ldots, T$.

Step 2: For $t=T, T-1, \ldots, 2$ (in that order) apply the following decomposition step:

Decomposition step: Let $A$ be the vertex set of the connected component of $G\left(V \backslash C\left(f_{t}\right)\right)$ containing $f_{t}$ and $B:=V \backslash\left(A \cup C\left(f_{t}\right)\right)$. If $C\left(f_{t}\right)$ is a clique of $G$ we call the decomposition step successful and decompose $G$ into $G^{\prime}=G\left(A \cup C\left(f_{t}\right)\right)$ and $G^{\prime \prime}=G\left(B \cup C\left(f_{t}\right)\right)$, separated by $C\left(f_{t}\right)$. Replace $G$ by $G^{\prime \prime}$.

Hence our algorithm is the same as Tarjan's ([24, pp. 224/225]), except that:

- we require in Step 1 that $\pi$ is a D-numbering w.r.t. $G_{\pi}$ and we have to determine $F(\pi)$;
- the decomposition step is applied only for $f_{T}, f_{T-1}, \ldots, f_{2}$ instead of all $v \in V$;
- the case $B=\emptyset$ cannot occur in a successful decomposition step of our algorithm (see the proof of (5.4)).

Example 2 (cont.). Let $G$ and $\pi$ as before in this section. We showed that then $F(\pi)=\left(f_{22}, f_{3}, f_{4}\right)=(6,2,1)$.

$$
C\left(f_{2}\right)=\operatorname{MAdj}_{c_{i_{T}}}(6)=\operatorname{Adj}_{c_{i_{\pi}}}(6) \cap\{w: \pi(w)>\pi(6)\}=\{1,3,4\} \cap\{4,5\}=\{4\},
$$

$C\left(f_{3}\right)=\{3\}$, and $C\left(f_{4}\right)=\{3,6\}$. Hence $T=4$ and Step 2 is carried out in the following way:
$t=4: C\left(f_{4}\right)=\{3,6\}$ is not a clique of $G$. Hence this decomposition step is not successful.
$t=3: C\left(f_{3}\right)=\{3\}$ is a clique of $G$ and $G$ is decomposed by the triple ( $\{2\},\{3\},\{1,4,5,6\}$ ) into the (prime) graph $G(\{2,3\})$ and $G(\{1,3,4,5,6\})$.
$t=2: C\left(f_{2}\right)=\{4\}$ is a clique of $G(\{1,3,4,5,6\})$ and this graph is decompused by the triple ( $\{1,3,6\},\{4\},\{5\}$ ) into the (prime) graphs $G(\{1,3,4,6\}$ ) and $G(\{4,5\})$.
Hence $G$ has been decomposed into its mp-subgraphs $G(\{2,3\})$, $G(\{1,3,4,6\})$, and $G(\{4,5\})$.
Rose, Tarjan and Lueker [23] described an $\mathrm{O}(n m)$-time algorithm to find a minimal numbering of an $n$-vertex, $m$-edge graph. We shall show in the Appendix that this algorithm always generates a numbering $\pi$ which is a D-numbering w.r.t. $G_{\pi}$, and that it is also easy to derive the vector $F(\pi)=$ $\left(f_{2}, \ldots, f_{T}\right)$ with that algorithm. It follows that the total time required for our algorithm is $\mathrm{O}(\mathrm{nm})$ as for Tarjan's algorithm. (The algorithms to find a minimal numbering described in [20] and [21] do not produce a D-numbering in general.)

The relations between $G$ and $G_{\pi}$ listed in the following lemma will be useful in the proof of the correctness of our algorithm (Theorem 5.3).

Lemma 5.1. Let $C$ be a clique of $G$ and $\pi$ a minimal numbering for $G$.
(i) The connected components of $G(V \backslash C)$ and $G_{\pi}(V \backslash C)$ are the same.
(ii) $C$ is a (relative minimal) separator for $G$ if and only if $C$ is a (relative minimal) separator for $G_{\pi}$.
(iii) If $(A, C, B)$ is a $P$-decomposition of $G_{\pi}$, then $(A, C, B)$ is a $P$ decomposition of $G$.

Proof. (i) is just Lemma 1 in [24]. (ii) follows from (i) and from $G \subseteq G_{\pi}$. (iii) is implied by the following property.
(5.2) If $U \subseteq V$ and $U$ is a clique of $G_{\pi}$, then there is an mp-subgraph $G\left(U^{\prime}\right)$ of $G$ with $U \subseteq U^{\prime}$.

To prove (5.2) let $V_{i}, \ldots, V_{T}$ be a D-ordering of the mp-subgraphs of $G$. There is a $t$, $t$ minimal, such that $U \subseteq V_{1} \cup \cdots \cup V_{t}$. If $t=1$ set $U^{\prime}:=V_{1}$. Otherwise $U \subseteq V_{t}$ follows from (4.2) and part (i) of this lemma.

Note that the inverse implication to that in Lemma 5.1(iii) is also true and can be proved similarly as Theorem 4.1 (ii) $\Rightarrow$ (iii).

Theorem 5.3. The decomposition algorithm consisting of Step 1 and Step 2 described above is correct, i.e. the algorithm produces a decomposition of a given graph $G$ into exactly one copy of each of the mp-subgraphs of $G$.

Proof. Let $\pi$ be a minimal numbering of a graph $G=(V, E)$ which is associated with the D-ordering $C_{1}, \ldots, C_{T}$ of the maximal cliques of $G_{\pi}$ with $C_{t}=R_{t} \cup S_{t}$ as usual and $F(\pi)=\left(f_{2}, \ldots, f_{T}\right)$. Then $C\left(f_{t}\right)=R_{t}$ for all $t \geqslant 2$ and $R_{1}=\emptyset$. Define

$$
t^{*}:=\max \left\{t: 1 \leqslant t \leqslant T \text { and } R_{t} \text { is a clique of } G\right\}
$$

i.e. $t^{*}$ corresponds to the first successful decomposition step or $t^{*}=1$. Define $C:=R_{r}, A$ as the connected component of $G(V \backslash C)$ containing $f_{i}$, $B:=$ $V \backslash(A \cup C)$, and $I_{B}:=\left\{t: 1 \leqslant t \leqslant T\right.$ and $\left.C_{t} \subseteq B \cup C\right\}$. Theorem 5.3 is proved by showing the following.
(5.4) The first successful decomposition step is correct, i.e.: If $t^{*}=1$, then $G$ is prime. Otherwise $(A, C, B)$ is a $P$-decomposition of $G$ with $G(A \cup C)$ prime.
(5.5) After the first successful tecomposition step we have a situation as in the beginning, i.e. if $t^{*}>1$, then:
(i) For $t \in I_{B}$

$$
C_{t} \cap \bigcup_{s=1}^{t-1} C_{s}=C_{t} \cap \bigcup_{s=1, s \in I_{H}}^{t-1} C_{s},
$$

i.e. the partitions $C_{t}=R_{t} \cup S_{t}$ (and hence the $f_{t}$ 's) are the same w.r.t. the subsequence $\left(C_{t}: t \in I_{B}\right)$ as w.r.t. the orig.nal sequence $C_{1}, \ldots, C_{T}$;
(ii) $\left(C_{1}: t \in I_{B}\right)$ is a $D$-ordering of the maximal cliques of $G_{\pi}(B \cup C)$;
(iii) the restriction of $\pi$ to $B \cup C$ is minimal for $G(B \cup C)$ and a D -numbering w.r.t. $G_{\pi}(b \cup C)$;
(iv) $1,2, \ldots, t^{*}-1 \in I_{B}$ and $t^{*} \notin I_{B}$.

Note that (i) implies that we do not have to apply the decomposition step again for any $t \in I_{B}$ with $t \geqslant t^{*}$ (but all $t<t^{*}$ ) after the first successful decomposition step (for $t=t^{*}$ ), since the sets $R_{t}=C\left(f_{t}\right)$ remain the same for $t \in I_{B}, t \geqslant t^{*}$, and it was already checked before (without success). whether they are cliques.
To prove (5.4) and (5.5) assume $G$ is reducible. Then there is a clique $C$ which is a relative minimal separating set for $G$. Lemma 5.1 (ii) and Theorem 4.1 (i) $\Rightarrow$ (iv) applied to $G_{\pi}$ imply $t^{*} \geqslant 2$. Hence $G$ is prime if $t^{*}=1$.

Assume $t^{*} \geqslant 2$ from now on. The set $A$ is the connected component of $G(V \backslash C)$ (by definition) and $G_{\pi}(V \backslash C)$ (by Lemma 5.1(i)) which contains $f_{r}$. Applying (4.2) to $G_{\pi}$ we get:

- $(A, C, B)$ is a P-decomposition for $G_{\pi}$;
- $C_{1}, \ldots . C_{t^{*-1}} \subseteq B \cup C$ and $C_{r^{*}} \subseteq A \cup C$, i.e. $1,2, \ldots, t^{*}-1 \in I_{B}$ and $t^{*} \notin I_{B}$.
$(A, C, B)$ is a P -decomposition for $G$ by Lemma 5.1(iii), especially $B \neq \emptyset$. Before proving that $G(A \cup C)$ is prime, we note that the remaining parts of (5.5) are rather obvious; we omit the details of the proofs.

Now let $I_{A}:=\{1,2, \ldots, T\} \backslash I_{B}$. Then $t^{*} \in I_{A} \subseteq\left\{t^{*}, t^{*}+1, \ldots, T\right\}$ and $\left(C_{t}: t \in\right.$ $I_{A}$ ) are the D-ordered maximal cliques of $G_{\pi}(A \cup C)$ etc., as in (5.5) for $G(B \cup C)$. It follows from the definition of $t^{*}$ w.r.t. $G$ that $t^{*}$ defined w.r.t. $G(A \cup C)$ corresponds to the case ' $t^{*}=1$ ', i.e. $G(A \cup C)$ is prime, as we have shown ai, ve.

## 6. Generalizations

The results of this paper can be directly generalized to so-called strong decompositions of graphs with marked vertices ([18]). Strong decompositions are important e.g. in the context of graphical models for mixtures of discrete and continuous random variables ( $[16,17]$ ), where the two types of vertices (marked and unmarked) correspond to discrete and continuous randen variables, respectively.

Assume that $G=(V, E)$ is a graph with two types of vertices, given by a partition $V=\Gamma \cup \Delta$ of the vertex set. A decomposition $(A, C, B)$ of $G$ is called strong, if any of the three conditions $A \subseteq \Gamma, B \subseteq \Gamma$ or $C \subseteq \Delta$ holds. $G$ is $S$-prime, if there is no strong decomposition $(A, C, B)$ of $G$.

It is possible to show that every graph with two types of vertices can be recursively decomposed by strong decompositions into its maximal $S$-prime subgraphs. The algorithm of Section 5 can be modified for this purpose.

## Appendix

We are going to show that every minimal numbering $\pi$ of the vertices of a graph $G=(V, E)$ which is generated by the algorithm described in [23] (henceforth called the $R T L$-algorithm), is a D-numbering w.r.t. $G_{\pi}=\left(V, E \cup F_{\pi}\right)$ and we shall show how the vector $F(\pi)=\left(f_{2}, \ldots, f_{T}\right)$ can be determined by the RTL-algorithm. A numbering $\pi$ generated by the RTL-algorithm is called a lexicographic numbering. Assume that such a numbering $\pi$ is fixed and to simplify the notations, assume that the vertices in $V$ are labeled as $1,2, \ldots, n$ such that $\pi$ is the identity map.

The RTL-algorithm numbers the vertices in decreasing order from $n$ to 1. Assume the vertices $n, n-1, \ldots, i+1$ are already numbered (for some $i \in$ $\{n, n-1, \ldots, 1\}$ ), then there is a label $L_{i}(j)$ associated with every unnumbered vertex $j$. The labels which are defined in terms of the graph $G$ to carry out the algorithm, have the following property w.r.t. $G_{\boldsymbol{r}}$ ([23, Lemma 7]).

$$
\begin{equation*}
L_{i}(j)=\operatorname{MAdj}_{\pi}(j) \cap\{n, n-1, \ldots, i+1\} \tag{A.1}
\end{equation*}
$$

where MAdj $_{\pi}$ denotes the monotone adjacency set w.r.t. $G_{\pi}$. The $\cdots \Gamma L$-algorithm is defined by the condition

$$
\begin{equation*}
L_{i}(j) \leqslant L_{i}(i) \quad \text { for all } j \leqslant i \text { and } i=n, n-1, \ldots, 2 \tag{A.2}
\end{equation*}
$$

i.e. the vertex numbered next after $n, n-1, \ldots, i+1$ has a maximal label, where the ordering ' $<$ ', called lexicographic ordering, is defined as follows: for $A=\left\{a_{1}, \ldots, a_{j}\right\} \subseteq V, B=\left\{b_{i}, \ldots, b_{k}\right\} \subseteq V$ with $a_{1}>\cdots>a_{j}$ and $b_{1}>\cdots>b_{k}$ define $A<B$ if and only if

- there is a $p$ such that $a_{q}=b_{q}$ for $q=1, \ldots, p-1$ and $a_{p}<b_{p}$, or
$\bullet j<k$ and $a_{q}=b_{q}$ for $q=1, \ldots, j$.
To motivate some necessary definitions, assume that we have already shown that $\pi$ is a D -numbering w.r.t. $G_{\pi}$. Let $C_{1}, \ldots, C_{T}$ be the D-ordering of the maximal cliques of $G_{\pi}$ which is associated with $\pi$, and $S_{t}:=C_{t} \backslash\left(C_{1} \cup \cdots \cup C_{t-1}\right)$ as usual. Note that for every $i \in S_{t}(t=1, \ldots, T)$

$$
\begin{equation*}
L_{i}(i)=\operatorname{MAdj}_{\pi}(i)=C_{t} \cap\{n, n-1, \ldots, i+1\} \tag{A.3}
\end{equation*}
$$

Assume that $i+1$ and $i$ are two consecutive numbered vertices with $i+1 \in S_{t}$ and $i \in S_{t^{\prime}}$. Then we have to distinguish two cases:

Case 1: $t^{\prime}=t$, i.e. $i+1, i \in S_{t}$. It follows from (A.1) and (A.3) that $L_{i}(i)=$ $L_{i+1}(i+1) \cup\{i+1\}$.

Case 2: $t^{\prime}=t+1$ hence $i \notin C_{r}$. Then $L_{i}(i) \neq L_{i+1}(i+1) \cup\{i+1\}$, since otherwise $L_{i+1}(i+1) \cup\{i+1\}=C_{:}($see $(\mathrm{A} .3))$ implies $C_{t} \cup\{i\}$ is a clique which gives a contradiction, since $C_{\text {}}$ is a maximal clique.
Since we assumed that $\pi$ is a D-numbering which is associated with $C_{1}, \ldots, C_{T}$, we can say that the vertices in $S_{1}\left(=C_{1}\right)$ are numbered first (as $\left.n, n-1, \ldots, n-\left|S_{1}\right|+1\right)$, then the vertices in $S_{2}$, etc.

The above considerations show that we have just finished to number the vertices in some $S_{t}$ and switch to $S_{t+1}$ if and only if the vertex $i$ to be numbered next satisfies

$$
\begin{equation*}
L_{i}(i) \neq L_{i+1}(i+1) \cup\{i+1\} \tag{A.4}
\end{equation*}
$$

Therefore, the components of $F(\pi)=\left(f_{2}, \ldots, f_{T}\right)$ satisfy

$$
\begin{align*}
& f_{2}>\cdots>f_{T} \text { and }\left\{f_{2}, \ldots, f_{T}\right\}=\left\{i: n-1 \geqslant i \geqslant 1 \text { and } L_{i}(i)\right. \\
& \quad \neq L_{i+1}(i+1) \cup\{i+1\}, \text { furthermore } f_{1}:=n . \tag{A.5}
\end{align*}
$$

Using (A.2), the inequality (A.4) can equivalently be expressed as

$$
\begin{equation*}
L_{i}(i)<L_{i+1}(i+1) \tag{A.6}
\end{equation*}
$$

Hence $f_{2}, \ldots, f_{T}$ can be simply derived from the RTL-algorithm as the first, second, etc. numbered vertex which has a label that is not strictly greater than the label of the previously-numbered vertex.

While $f_{2}, \ldots, f_{T}$ are the vertices of $S_{2}, \ldots, S_{T}$ which are numbered first by the RTL-algorithm, the vertices

$$
\begin{equation*}
e_{t}:=f_{t+1}+1 \quad \text { for } t=1,2, \ldots, T-1 \text { and } e_{T}:=1 \tag{A.7}
\end{equation*}
$$

are the vertices numbered last by the algorithm in $S_{1}, \ldots, S_{T}$. Using (A.1) and (A.3) we get

$$
\begin{equation*}
C_{t}=L_{e_{1}}\left(e_{t}\right) \cup\left\{e_{t}\right\} \quad \text { for } t=1, \ldots, T \tag{A.8}
\end{equation*}
$$

It remains to show that every lexicographic numbering $\pi$ is a $D$-numbering w.r.t. $G_{\pi}$. Let $\pi$ be a lexicographic numbering and assume for convenience that $\pi$ is the identity-map.

Theorem A.9. If $T, f_{t}, e_{t}$ and $C_{t}$ are defined by (A.5), (A.7), and (A.8) then:
(i) $C_{1}, \ldots, C_{T}$ are the maximal cliques of $G_{\pi}$;
(ii) the sequence $C_{1}, \ldots, C_{T}$ is $D$-ordered;
(iii) $S_{t}=\left\{i: f_{t} \geqslant i \geqslant e_{t}\right\}$ for all $t=1, \ldots, T$ (where $S_{t}=C_{t} \backslash\left(C_{1} \cup \cdots \cup C_{t-1}\right)$ as usual), i.e. $\pi$ is a D-numbering which is associated with $C_{1}, \ldots, C_{T}$, and $f_{t}=\max \left\{i: i \in S_{t}\right\}$.

Proof. Note that $\pi$ satisfies (A.2) and is perfect for $G_{\pi}$ by definition. Furthermore, (A.1) is satisfied by Lemma 7 in [23]. Therefore $C_{t}=\operatorname{MAdj}_{\pi}\left(e_{t}\right) \cup\left\{e_{t}\right\}$ is a clique of $G_{\pi}$. Assume that $C_{t} \cup\{j\}$ is also a clique of $G_{\pi}$ for some $j \in V \backslash C_{r}$. Then $j<e_{t}$ and we have

$$
L_{e_{t}-1}(j)=\operatorname{MAdj}_{\pi}(j) \cap\left\{n, n-1, \ldots, e_{t}\right\} \geqslant C_{t}=L_{e_{t}}\left(e_{t}\right) \cup\left\{e_{t}\right\}
$$

This together with (A.6) implies

$$
L_{e_{t}-1}(j) \geqslant L_{e_{t}}\left(e_{t}\right) \cup\left\{e_{t}\right\}>L_{e_{t}}\left(e_{t}\right) \geqslant L_{e_{t}-1}\left(e_{t}-1\right)
$$

which contradicts (A.2). Hence $C_{1}, \ldots C_{T}$ are maximal cliques of $G_{\pi}$. Next we show that $S_{t}=\left\{f_{t}, f_{t}-1, \ldots, e_{t}\right\}$ : It follows from the definitions (A.5), (A.7), and (A.8) that for $s \in S$,

$$
\begin{equation*}
L_{s}(s) \cup\left\{s, s-1, \ldots, e_{t}\right\}=L_{r_{t}}\left(e_{t}\right) \cup\left\{e_{t}\right\}=C_{t} \tag{A.10}
\end{equation*}
$$

Using (A.10) for $s=f_{t}$ we get $\left\{f_{t}, f_{t}-1, \ldots, e_{t}\right\} \subseteq C_{t}$ for all $t$ and $C_{t} \subseteq\{n, n-1$, $\left.\ldots, e_{t}\right\}$ by definition. Hence $C_{1} \cup \cdots \cup C_{t-1}=\left\{n, n-1, \ldots, e_{t-1}\right\}$ and $S_{t}=$ $C_{t} \backslash\left(C_{1} \cup \cdots \cup C_{t-1}\right)=\left\{f_{t}, f_{t}-1, \ldots, e_{t}\right\}$ which proves (iii). Furthermore, $S_{1} \cup$ $\cdots \cup S_{T}$ is a partition of $V$ with $C_{1} \cup \cdots \cup C_{t}=S_{1} \cup \cdots \cup S_{t}$ for all $t$.
Now let $C$ be a clique of $G_{\pi}$ and $c:=\min \{i: i \in C\}$. As we have just shown, there is a $t$ with $c \in S$, and using (A.10)

$$
C \subseteq \operatorname{MAdj}_{x}(c) \cup\{c\}=L_{c}(c) \cup\{c\} \subseteq L_{e_{t}}\left(e_{t}\right) \cup\left\{e_{t}\right\}=C_{t}
$$

Hence there are no other maximal cliques of $G_{\pi}$ than $C_{1}, \ldots, C_{T}$.
It remains to show that $C_{1}, \ldots, C_{T}$ is $D$-ordered. Let $t \geqslant 2$ be fixed and $\left\{i_{1}, \ldots, i_{k}\right\}:=R_{t}=C_{t} \cap\left(C_{1} \cup \cdots \cup C_{t-1}\right) \subseteq C_{1} \cup \cdots \cup C_{t-1}=S_{1} \cup \cdots \cup S_{t-1}$. We have to show $R_{t} \subseteq C_{p}$, for some $p<t$. Assume $i_{1}>\cdots>i_{k}$. There is a $p<t$ with $i_{k} \in S_{p}$. The set $R_{t}$ is a clique of $G_{\pi}$ as a subset of $C_{r}$. Using (A.10) we get

$$
R_{t} \subseteq \operatorname{MAdj}\left(i_{k}\right) \cup\left\{i_{k}\right\}=L_{i_{k}}\left(i_{k}\right) \cup\left\{i_{k}\right\} \subseteq C_{p}
$$

As it was mentioned in Section 5, the algorithms described in [20] and [21] do not generally produce a minimal numbering $\pi$ which is a D-numbering w.r.t. $G_{\pi}$. But we could nevertheless use these algorithms as an alternative to the RTL-algorithm as follows:

- First generate any minimal numbering $\pi$ for $G$ (e.g. using the algorithms described in [20] or [21]) and compute $G_{\pi}$;
- then apply maximum cardinality search (MCS) (see [25]) to the (decomposable) graph $G_{\pi}$.

For decomposable graphs MCS works similarly as the RTL-algorithm but (A.2) is replaced by the simpler condition $\left|L_{i}(j)\right| \leqslant\left|L_{i}(i)\right|$ i.e. the next vertex to be numbered (as $i$ ) is adjacent to the largest number of previously numbered vertices.

It was shown in [17], Theorem 4.3.1 that every numbering $\pi$ which is generated by MCS w.i.t. a decomposable graph is a D-numbering for this graph. The proof is similar to the above proof for the RTL-algorithm, we mainly have to replace the inequality $L_{i}(i) \neq L_{i+1}(i+1) \cup\{i+1\}$ in (A.5) by $\left|L_{i}(i)\right| \neq$ $\left|L_{i+1}(i+1)\right|+1$.

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