Rings of Finite Representation Type and Modules of Finite Morley Rank

MIKE PREST*

Department of Mathematical Sciences,
Northern Illinois University,
DeKalb, Illinois 60115

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This paper deals with rings whose modules have very pleasant decomposition properties: the right pure semisimple rings. The category of right modules over such a ring is, in many ways, similar to the category of modules over a semisimple artinian ring. Yet it is still an open question whether or not pure semisimplicity actually is a two-sided property (and hence [30] coincides with being of finite representation type).

Most of the arguments in this paper are model-theoretic-algebraic in nature: it seems that ideas from model theory, and from the model theory of modules, find natural application in the context considered here.

The paper is organized into three sections. The first is introductory and with it I attempt to make the other sections accessible to a reasonably wide readership.

In the second section are gathered together various equivalents to right pure semisimplicity. Their equivalence is given a comparatively short, unified, essentially model-theoretic-algebraic proof.

The main theorem, in the third section, is that a ring is of finite representation type if and only if all its (right) modules have finite Morley rank. A number of related results are developed. Most of these involve some model theory in their statement, but many are, in essence, algebraic.

Since a good deal of information has been compressed into the introductory section, this section should, perhaps, first be read through quickly, and then referred back to as the need arises. This section also is meant to serve as background to a sequel, in which I consider pp formulas and types in terms of the corresponding matrices, and in which the main aim is more purely algebraic.

The global conventions are: $R$ denotes a ring with identity; “module”

* Present address: Mathematics Department, Yale University, New Haven, CT 06520.
means right \( R \)-module; \( \mathbb{M}_R \) denotes the category of (right \( R \)-) modules; \( _R \mathbb{M} \) or \( \mathbb{M}_{R \text{op}} \) denotes the category of left \( R \)-modules; \( R_\omega \) denotes the ringoid of finite rectangular matrices which have entries in \( R \) (with the usual operations).

1.

An embedding \( A \leq B \) of modules is a pure embedding, written \( A \prec _1 B \), if any matrix equation \( \bar{v}H = \bar{a} \) with a solution in \( B \) already has a solution in \( A \); here \( H \in R_\omega \) is a finite rectangular matrix over \( R \), \( \bar{a} \) is a tuple (row vector) from \( A \), and \( \bar{v} \) is a tuple of variables (which matches \( H \)—but I will always take this as implicit when writing down such expressions and equations). That is, \( A \prec _1 B \) if any finite system of linear equations, with coefficients from \( R \) and parameters from \( A \), and with a solution in \( B \), already has a solution in \( A \).

A module \( N \) is pure-injective if it is injective over pure embeddings; and is \( \Sigma \)-pure-injective if \( N^{(I)} \) is pure-injective for all index sets \( I \) (in fact, requiring the condition with \( I = \mathbb{N}_0 \) is enough). Recall [73, Theorem 2] that a module \( N \) is pure-injective if and only if it is algebraically compact: that is, if and only if every (infinite) system of matrix equations, of the form above, which is finitely satisfiable in \( N \), has a simultaneous solution in \( N \). For basic properties of these modules see [27, 37, 50, 55, 73–75].

Pure-injective modules are ubiquitous in as much as every module \( M \) has an essentially unique pure-injective hull \( \tilde{M} \), in which it purely embeds [28, 45, 73]. Moreover, the relationship between \( M \) and \( \tilde{M} \) is, in many ways, closer than that between the module \( M \) and its injective hull \( E(M) \) [64a, Corollaire 1]. Nevertheless, there are very extensive similarities between injective and pure-injective modules. Essentially, this is due to the fact that in certain categories, closely related to \( \mathbb{M}_R \), the pure-injective modules become precisely the injective objects (see [27], [36, 1.2], and also [55]). Although I do not make explicit use of this fact in the present paper, it is worthwhile bearing in mind while reading what follows.

Now I go on to define pp-types. These may be construed in terms of matrices, or in terms of certain kinds of formulas. I will begin by defining them in terms of matrices, but in most proofs I will adopt the more convenient (for present purposes) model-theoretic point of view.

The pp-type ("pp" for "positive primitive"; see below) of an element, or tuple of elements, essentially is a collection of items of information about the tuple—information both on its "internal structure" and on the way it lies in an ambient module. For example, it contains the isomorphism type of the (submodule generated by the) element or tuple; but it also contains information on divisibility, etc. It may well be thought of as a generalised
annihilator (this is noted in [54], and this point of view is adopted also in [6]).

Let $M$ be a module, and let $\bar{a}$ be a tuple of elements from $M$ (I will say $a$ is in $M$, and write $\bar{a} \in M$ if no ambiguity should arise). Set $tp^M(\bar{a})^+ = \{H : H \in R_\omega, \text{ and there is some } \bar{b} \in M \text{ such that } (\bar{a}\bar{b})H = 0\}$. Thus, for example, the set of $1 \times 1$ matrices in $tp^M(a)^+$ is just the right ideal of the annihilator of $a$; and, for this restricted part, $M$ is of course irrelevant. Indeed, it may be shown that $tp^M(\bar{a})^+$, although not in general closed under ordinary matrix addition, is rather close to being a kind of right ideal of the ringoid $R_\omega$. Where the context allows, I will shorten the notation to $tp(\bar{a})^+$.

Now note that if $H \in tp^M(\bar{a})^+$ then this condition may be expressed by the statement that there exists a tuple (of appropriate length) such that a certain finite collection of $R$-linear equations (involving this tuple and $\bar{a}$) holds. More precisely, suppose that $\bar{a} = (a_1, \ldots, a_n)$ and $H$ is $(n + 1) \times m$. Write $H$ as $(\xi)$ with $R_m = (r_{ij})$ and $S_m = (s_{kj})$. Then the condition that $H$ should be in $tp^M(\bar{a})^+$ may be expressed by the formal statement:

$$\exists w_1, \ldots, w_l \bigwedge_{j=1}^{m} \sum_{i=1}^{n} a_i r_{ij} + \sum_{k=1}^{l} w_k s_{kj} = 0$$

(here $\bigwedge$ means "and").

Write $\phi^a_H(\bar{v})$, $\phi_H$ or just $\phi$ (where the length of $\bar{v}$, $l(\bar{v})$ equals $n$) for the formula

$$\exists w_1, \ldots, w_l \bigwedge_{j=1}^{m} \sum_{i=1}^{n} v_i r_{ij} + \sum_{k=1}^{l} w_k s_{kj} = 0.$$  

Then we have $H \in tp^M(\bar{a})^+$ if and only if $M$ satisfies the corresponding sentence $\phi(\bar{a})$ (that is, $\phi(\bar{v})$ with $\bar{a}$ replacing $\bar{v}$ in the obvious way), or for short, $M \models \phi(\bar{a})$. A formula (logically equivalent to one) of the above form $\phi(\bar{v})$, that is, a formula with only existential quantifiers prefixing a conjunction of atomic formulas (= $R$-linear equations in our context), is termed a positive primitive, or pp, formula.

Conversely, given any pp formula $\phi(\bar{v})$ with the length $l(\bar{v})$ of $\bar{v}$ equal to $n$, it should be clear how to write down an associated matrix $H = H_\bar{a}$ in $R_\omega$, such that $\phi(\bar{v}) = \phi^a_H(\bar{v})$: that is, such that $\phi(\bar{v})$ says "$\exists w(\bar{v}\bar{w}) H_\bar{a} = 0$"; that is, such that for any tuple $\bar{a}$ (of length $n$), $M \models \phi(\bar{a})$ if and only if there is $\bar{b}$ in $M$ with $(\bar{a}\bar{b})H_\bar{a} = 0$.

So we see that $tp^M(\bar{a})^+$, the pp-type of $\bar{a}$ in $M$, may be construed as the collection of all pp formulas satisfied by $\bar{a}$ in $M$, and indeed this is how I will normally regard it.

Let us reverse our viewpoint for a while, and note that for any $n \in \omega$, any
pp formula \( \phi = \phi(\vec{v}) \), with \( l(\vec{v}) = n \), gives rise to a subset of any module of the form \( M^n \), namely:

\[
M^\phi = \{ \vec{a} \in M^n : M \models \phi(\vec{a}) \}
\]

—the collection of tuples of \( M^n \) which satisfy \( \phi \).

Indeed, this definition makes sense for any formula, but note that if \( \phi \) is pp then, by linearity of such formulas, \( M^\phi \) is an abelian subgroup of \( M^n \), and is said to be a pp-definable subgroup of \( M^n \). In the terminology of [35] this is a “subgroup of finite \( R \)-definition,” and pp formulas are essentially “\( p \)-functors”; in the terminology of [76] it is an “endlichmatriziellen Untergruppe.” It is an easy exercise to check that such subgroups form a lattice under (finite) addition and intersection, with \( M^\phi \cap M^\psi = M^{\phi \land \psi} \) (“\( \land \)” means “and”), and \( M^\phi + M^\psi = M^{\phi + \psi} \), where \( (\phi + \psi)(\vec{v}) \) is \( \exists \vec{v}_1, \vec{v}_2 (\phi(\vec{v}_1) \land \psi(\vec{v}_2) \land \vec{v} = \vec{v}_1 + \vec{v}_2) \).

The notation \( \text{tp}^M(\vec{a}) \) is used for the set of all (first-order) formulas \( \chi(\vec{v}) \), with \( l(\vec{v}) = l(\vec{a}) \), which are satisfied by \( \vec{a} \) in \( M \): \( \text{tp}^M(\vec{a}) = \{ \chi(\vec{v}) : M \models \chi(\vec{a}) \} \).

Here the language is the one usual for \( R \)-modules (see, say, [21], [31] or [54]). Thus \( \chi(\vec{v}) \) is built up from \( R \)-linear equations using \( \neg \) (not), \( \land \) (and), \( \lor \) (or), \( \rightarrow \) (implies), \( \exists \) (there exists), \( \forall \) (for all). It is a remarkable fact [11] that \( \text{tp}^M(\vec{a})^+ \) determines the (full) type \( \text{tp}^M(\vec{a}) \), in the following sense.

Let \( \text{tp}^M(\vec{a})^- = \{ \neg \phi(\vec{v}) : l(\vec{v}) = l(\vec{a}), \phi(\vec{v}) \text{ is pp, but } \phi(\vec{v}) \notin \text{tp}^M(\vec{a})^+ \} \). Then any formula in \( \text{tp}^M(\vec{a}) \) is a formal consequence, in the theory of \( M \), of formulas in \( \text{tp}^M(\vec{a})^+ \cup \text{tp}(\vec{a})^- \). In particular, if \( \text{tp}^M(\vec{a})^+ = \text{tp}^{M'}(\vec{b})^+ \), where \( M \) and \( M' \) have the same theory (see below), in particular if \( M = M' \), then \( \text{tp}^M(\vec{a}) = \text{tp}^{M'}(\vec{b}) \). Typical notation is \( p \) for \( \text{tp}^M(\vec{a}) \), \( p^+ \) for \( \text{tp}^M(\vec{a})^+ \), and \( p^- \) for \( \text{tp}^M(\vec{a})^- \). If there is no ambiguity, the “\( M \)” will be omitted.

For any fixed \( n \in \omega \), the collection of all possible pp-types \( \text{tp}^M(\vec{a})^+ \), as \( \vec{a} \in M \in M_n \), vary with \( l(\vec{a}) = n \), is naturally ordered by inclusion. Thus we obtain the lattice (this is easily checked) \( P_n^R \), or just \( P_n \), of pp-(\( n \))-types. By Baur’s result [11] this induces an order, as opposed to a pre-order, on \( n \) types (i.e., types in \( n \) free variables) by setting \( p \geq q, p \) is above \( q \), if \( p^+ \supseteq q^+ \) in \( P_n \).

Note that there is a natural projection \( \pi : P_1 \rightarrow L_R \), where \( L_R \) denotes the lattice of all right ideals of \( R \). This is given by \( p^+ \mapsto p^+ \cap R \), where \( p^+ \) is regarded as a collection of matrices, and elements of \( R \) are regarded as \( 1 \times 1 \) matrices. There are two natural embeddings going the other way: \( i^R : L_R \hookrightarrow P_1 \) given by \( I \mapsto \text{tp}^{E(R/I)}(1 + I)^+ \), where \( E(M) \) denotes the injective hull of \( M \), and \( i^L : L_R \hookrightarrow P_1 \) given by \( I \mapsto \text{tp}^{R/I}(1 + I)^+ \); being, respectively, the right and left adjoints of \( \pi \). There is a very real sense, then, in which pp-types are generalized annihilators.

Given a right ideal \( I \) of \( R \), we may form \( E_I = E(R/I) \)—the injective hull of any element whose annihilator is precisely \( I \). Similarly we may, given a pp-
type \( p^+ \in P_n \), form \( N_p \) as the "hull" of any \( n \)-tuple whose pp-type is precisely \( p^+ \). This hull has the following properties:

(i) \( N_p \) is a pure-injective module.

(ii) If \( N \) is pure-injective, and \( \bar{a} \in N \) is such that \( \text{tp}^N(\bar{a})^+ = p^+ \), then there is a copy of \( N_p \) which contains \( \bar{a} \) and is purely embedded in \( N \), so is a direct summand of \( N \).

(iii) \( N_p \) is essentially unique (with properties (i) and (ii)). So in (ii) we may reasonably write \( N_{\bar{a}} \) for an appropriate copy of \( N_p \). Any two such copies are \( \bar{a} \)-isomorphic.

(iv) In (ii), \( N_{\bar{a}} \) as there is a minimal direct summand of \( N \) containing \( \bar{a} \). It is also a maximal "pp-essential extension" of \( \bar{a} \) (see [27; 55, 2.51]). In particular, if \( b \in N_{\bar{a}} \) and \( b \neq 0 \), then there is a pp formula \( \phi \) linking \( \bar{a} \) and \( b \); that is, \( \phi(b, \bar{a}) \) holds but \( \phi(0, \bar{a}) \) does not hold.

(v) If \( \bar{a} \in N' \prec N \) then \( \text{tp}^N(\bar{a})^+ = \text{tp}^N(\bar{a})^+ \). In particular, if \( \bar{a} \in M \) then the pp-type of \( \bar{a} \) is the same, whether \( \bar{a} \) is regarded as lying in \( M, M', \) or \( N_{\bar{a}} \).

More information on these hulls is contained in [27, 55, 75].

A property of, in fact a characterisation of, pure-injective modules which I will often use is the following [27; 55, 0.4 and 2.9; 75, 3.31.

**Lemma 1.1.** Let \( \bar{a}, \bar{b} \) be sequences of the same length, with \( \bar{a} \in A, \bar{b} \in N \), where \( N \) is pure-injective. Let \( p = \text{tp}^A(\bar{a}), q = \text{tp}^N(\bar{b}), \) and suppose that \( q \geq p \). Then there is an \( R \)-homomorphism \( f: A \rightarrow N \) with \( f\bar{a} = \bar{b} \). If, further, \( q = p \) and \( A \) is pure-injective, then \( f \) restricted to any hull \( N_{\bar{a}} \) of \( \bar{a} \) in \( A \) is a pure embedding.

Note that conversely, given \( f: A \rightarrow B \) and \( \bar{a} \in A \), then \( \text{tp}^A(\bar{a}) \leq \text{tp}^B(f\bar{a}) \), since linear relations, so pp sentences, are preserved by homomorphisms.

Some general model-theoretic background will be required. I will specialise some general definitions to modules for concreteness.

If \( M \) is a module, its (complete) theory, \( \text{Th}(M) \) or \( T_M \), is the set of all sentences (in the language of \( R \)-modules) which hold true in \( M \). The module \( M' \) is elementarily equivalent to \( M \), \( M' \equiv M \), if \( T_{M'} = T_M \). This happens precisely when \( M' \) is a model of \( T_M, M' \models T_M \), that is, precisely when \( M' \) satisfies all the sentences in \( T_M \). \( M \) is an elementary substructure of \( M' \), or \( M' \) is an elementary extension of \( M \), written \( M < M' \), if for any formula \( \chi(\bar{v}) \) and tuple \( \bar{a} \) in \( M, M \models \chi(\bar{a}) \) if and only if \( M' \models \chi(\bar{a}) \). Clearly, if \( M < M' \) then \( M < \downarrow M' \) and \( M \equiv M' \). For modules the converse also is true [64b, Théorème 2].

A complete theory \( T \) is simply a theory of the form \( T_M \) for some \( M \): "complete" since, for every sentence \( \chi \), either \( \chi \in T_M \) or \( \neg \chi \in T_M \). Typically
the common theory of all $R$-modules is far from complete: that is, there are many non-elementarily equivalent $R$-modules ([63] considers this).

Normally I will consider models of a fixed complete theory $T$. When doing this it is convenient to work within a very saturated model $\bar{M}$ of $T$: essentially $\bar{M}$ is a very large model of $T$ in which all "small" situations occur (I will be more precise below).

If $M$ is a module and $T = T_M$, set $T^n = Th(M^n)$, for $n \in \omega$, and $T^{\aleph_0} = Th(M^{\aleph_0})$. This does define $T^n$ and $T^{\aleph_0}$ uniquely since [24] if $M_i \equiv N_i$ ($i \in I$), then $\prod_i M_i \equiv \prod_i N_i$, and in fact $\bigoplus_i M_i \equiv \bigoplus_i N_i \equiv \prod_i N_i$. It is the case (for example, see [31; 64a, Corollaire 2] that for all infinite cardinals $\kappa$ and modules $M$, $M^{(\aleph_0)} = M^{\aleph_0} \equiv M^{(\kappa)}$).

Fix $n \in \omega$ and $T$ (always a complete theory). Then $S_n(T)$ is the set of all types of the form $tp^M(\bar{a})$, where $M \models T$, $\bar{a} \in M'$, and $l(\bar{a}) = n$. Note that this really is a set, since a type is just a set of formulas in the language for $R$-modules.

More generally, let $A \subseteq M \models T$, where $T$ is complete. Then $S_n^T(A)$ is the set of all $n$-types over $A$: a typical such type has the form $tp^M(\bar{c}/A) = \langle \chi(\bar{v}, \bar{a}) : \bar{a} \in A \text{ and } M' \models \chi(\bar{c}, \bar{a}) \rangle$, where $A \subseteq M' \models T$, and $\bar{c} \in M'$ with $l(\bar{c}) = n$. That is, $tp^M(\bar{c}/A)$ is the set of all formulas, with parameters from $A$, which are satisfied by $\bar{c}$ in $M'$. It is a fundamental fact that any set of formulas in the free variables $\bar{v}$, with parameters from $A$, and which is formally consistent with $T$, may be extended to at least one type in $S_n^T(A)$. In particular, such a set of formulas has some realisation $\bar{c} = (c_1, ..., c_n)$ in some model $M'$ of $T$.

Let $A \subseteq M \models T$, $p \in S_n^T(A)$. Then $M$ realises $p$ if there is $\bar{c} \in M$ with $tp^M(\bar{c}/A) = p$. One writes $M \models p(\bar{c})$. $M$ is saturated over a subset $A$ if $M$ realises all types in $S_n^T(A)$, for each $n \in \omega$. $M$ is $\kappa$-saturated, where $\kappa$ is an infinite cardinal, if $M$ is saturated over all its subsets of cardinality strictly less than $\kappa$. For any $M$ and $\kappa \geq \aleph_0$, $M$ has a $\kappa$ saturated elementary extension. The notation $\bar{M}$ will be reserved for a $\kappa$-saturated model, where $\kappa$ is some cardinal larger than the cardinality of any set we may wish to consider.

Proofs of and more detail on the "classical" model theory above may be found, for example, in [12, 15, 66], also see [8b]. I now give some definitions and results from stability theory, for which see [15, 51] (also see [7, 39, 42, 44, 67], though these go well beyond what I need here).

A complete theory $T$ is $\kappa$-stable ($\kappa$ being an infinite cardinal) if $|S_n^T(A)| \leq \kappa$ whenever $A \subseteq M \models T$ with $|A| \leq \kappa$. $T$ is stable if $T$ is $\kappa$-stable for some $\kappa$. Every complete theory of modules is stable [10, 26].

For stable theories there are essentially three possibilities.

(i) $T$ is totally transcendental, t.t., if the restriction of $T$ to any countable sublanguage (subring) is $\omega$-stable (i.e., $\aleph_0$-stable). Any $\omega$-stable theory is $\kappa$-stable for each infinite $\kappa$.
(ii) \( T \) is **superstable** if there is some cardinal \( \kappa_0 \) such that \( T \) is \( \kappa \)-stable for all \( \kappa \geq \kappa_0 \).

(iii) \( T \) is \( \kappa \)-stable for \( \kappa \geq \kappa_0 \) (some fixed \( \kappa_0 \)) provided \( \kappa^{\aleph_0} = \kappa \) (note, for example, that \( \aleph_n^{\aleph_0} = \aleph_n \) for \( n \in \omega \), but \( \aleph_{\omega}^{\aleph_0} > \aleph_{\omega} \)).

Then [67] (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii), and (iii) is equivalent to stability of \( T \).

I have stated already that all modules are stable. The following characterise totally transcendental and superstable modules, where one says, for example, that a module \( M \) is superstable if its complete theory \( T_M \) is superstable.

**Theorem 1.2** (Garavaglia [31, Lemma 5] and Macintyre, [43, Lemma 3]). *A module \( M \) is totally transcendental if and only if \( M \) has the descending chain condition on pp-definable subgroups.*

It follows easily from this that if \( T \) is totally transcendental, then every pp-type is equivalent, in \( T \), to a single pp formula.

**Theorem 1.3** (Garavaglia [31, Lemma 7]). *A module \( M \) is superstable if and only if for all descending chains \( M^{*0} \supseteq M^{*1} \supseteq \cdots \supseteq M^{*n} \supseteq \cdots \) of pp-definable subgroups of \( M \), all but finitely many of the factor groups \( M^{*n}/M^{*n-1} \) are finite.*

Note that in both these cases the property has to be checked only for \( M \); it then follows for all \( M' \in M \).

The proofs of these two results are sketched in the course of proving 2.1 below.

From the above and results in [35] and [76] follows the algebraic characterisation: a module is totally transcendental if and only if it is \( \Sigma \)-pure-injective. I will include a direct proof of this below (2.1 (v) \( \Leftrightarrow \) (ix))—note that the direction "\( \Rightarrow \)" is easy from [31, Lemma 6].

In general a complete theory \( T \) is totally transcendental if and only if all types (in any \( S^T_n(A) \)) have Morley rank. The definition of this rank involves a natural topology on \( S^T_n(A) \) (see below, before 3.2) and also requires consideration of \( S^T_n(B) \) for various \( B \supseteq A \) [49]. What I will do here is to define a rank which, in the case I need, may be shown (see [53]) to coincide with Morley rank.

Suppose that \( T = T^{\aleph_0} \); that is to say by previous remarks, suppose the class of models of \( T \) is closed under products. It is equivalent to require that all the invariants \( \text{Inv}(T, \phi, \psi) \) are 1 or \( \infty \), as \( \phi \) and \( \psi \) range over the set of pp formulas in one free variable. These invariants are defined as follows [11, 31, 48].
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\[ \text{Inv}(T, \phi, \psi) = \left| \frac{M^\phi}{M^\phi \cap M^\psi} \right| \quad \text{if this is finite,} \]
\[ = \infty \quad \text{otherwise,} \]

where \( M \) is any model of \( T \), and the quotient is one of abelian groups. These are indeed invariants of \( T \) since:

\[ M \equiv N \text{ if and only if, for all pp formulas } \phi \text{ and } \psi \text{ in one free variable,} \]
\[ \left| \frac{M^\phi}{M^\phi \cap M^\psi} \right| = \left| \frac{N^\phi}{N^\phi \cap N^\psi} \right|, \]

provided either side is finite.

Also, if \( N <_1^1 M \), then \( \text{Inv}(T_N, \phi, \psi) \leq \text{Inv}(T_M, \phi, \psi) \) for all pp formulas \( \phi \) and \( \psi \).

Define, if \( T = T^{\aleph_0} \), a rank \( MR \) on \( S^T(0) \) as follows.

(i) \( MR(tp(0)) = 0 \) (clearly \( tp(0) \) is the maximum element of \( S^T(0) \)).

(ii) \( MR(p) = \alpha \) if, for all \( q > p \) in \( S^T(0) \), \( MR(q) \) is defined and \( MR(q) < \alpha \), and \( \alpha \) is least with this property.

(iii) \( MR(p) = \alpha \) if, for every ordinal \( \alpha \), \( MR(p) \neq \alpha \).

This rank, in this context, generalises that defined in [9].

Clearly \( MR(p) < \infty \) if and only if \( S^T(0) \) (equivalently, the corresponding part of \( P^\alpha_p \)) has the ascending chain condition on types above \( p \).

I will be interested here in the case where \( T \) is the “largest” complete theory \( T^* \) of \( R \)-modules. This is defined as follows.

\[ T^* = \text{Th}(\{ M_T : T \text{ is a complete theory of } R \text{-modules, and } M_T \text{ is chosen to be any model of } T \}). \]

Then it is easily checked that every module purely embeds in some model of \( T^* \). In particular, within some sufficiently saturated model \( M \) of \( T^* \) may be found many realisations of each pp-type in each \( P^\alpha_n \). In particular, the map \( S^T(0) \rightarrow P_1 \) given by \( p \mapsto p^+ \) is a bijection.

Using 1.2 it is easily checked that the following holds.

**Lemma 1.4.** \( T^* \) is totally transcendental if and only if every module is totally transcendental.

If \( \Phi \) is any set of pp formulas, in the free variables \( \bar{v} \), and if \( M \) is any saturated model of \( T^* \), then define

\[ M^\Phi = \bigcap \{ M^\phi : \phi \in \Phi \}. \]

Then clearly \( p^+ \mapsto M^p^+ \) defines an order-reversing embedding of \( P_1 \) into the
lattice of possibly infinitely pp-definable subgroups of $M$. Therefore the next lemma follows from 1.4 and the remarks before.

**Lemma 1.5.** Every $R$-module is totally transcendental if and only if $P_i$ has the ascending chain condition.

I will refer to the rank $MR$, defined above, as Morley rank, since, in the cases I consider, it does coincide with the rank usually given that name. In fact it is more clearly (by [53]) the $U$-rank of Lascar [40] but (scc [53]) these ranks coincide in modules whenever both are defined.

Then 1.5 is the well-known result that a theory $T$ is totally transcendental if and only if every 1-type has Morley rank.

Finally, define the Morley rank of $T$, $MR(T)$, to be $\sup \{MR(p): p \in S^*_T(0)\}$.

2.

A ring is right pure semisimple, rt. pure s.s., if there is a set of indecomposable modules, such that every right $R$-module is isomorphic to a direct sum of copies of these indecomposables.

For example, a semisimple artinian ring satisfies this condition. The commutative right pure semisimple rings are precisely the commutative artinian principal ideal rings. Other examples are: if $k$ is a field of characteristic $p$ and if $G$ is a finite group of order $n$, with $n$ a multiple of $p$, then the group ring $k[G]$ is right pure s.s. provided $G$ has a cyclic Sylow $p$-subgroup; serial rings are rt. pure s.s.; if $k$ is a field then the ring of 2-by-2 upper triangular matrices over $k$ is rt. pure s.s.

This definition as given is stronger than it needs to be: by 2.1 below, the requirement that the collection of indecomposables be a set may be dropped. Alternatively one may dispense with the requirement of indecomposability. One may not, however, drop the decomposition requirement: for any commutative (von Neumann) regular ring satisfies the condition that there is just a set of indecomposable modules, yet (by 2.1(a) below), if non-artinian, such a ring cannot be right pure semisimple.

The model theory may be seen entering in the equivalent to right pure semisimplicity: that every module be totally transcendental (equivalently, that every module be pure-injective). Thus, by 1.5, right pure semisimplicity may be seen as an extremely strong form of the right noetherian condition.

The following Theorem 2.1 draws together a number of equivalents to, and consequences of, right pure semisimplicity. All these, except perhaps (iv), may be found in the literature, but the overall approach in, and a number of parts of, the proof are new. Those arguments which may be found elsewhere
I have, to some extent, summarized rather than giving all details. After the proof, I have further listed some of the (numerous) equivalents and where they may be found.

**Theorem 2.1.** The following conditions on a ring $R$ are equivalent.

(i) $R$ is right pure semisimple.

(ii) Every module is a direct sum of indecomposable modules.

(iii) There is a cardinal $\kappa$, such that every module is a direct sum of modules of cardinality less than, or equal to $\kappa$.

(iv) There is a cardinal $\kappa$, such that every module purely embeds in a direct sum of modules of cardinality less than, or equal to $\kappa$.

(v) Every module is totally transcendental.

(vi) $P^\mathfrak{a}$ has the ascending chain condition.

(vii) Every module is pure-injective.

(viii) Every direct sum of pure-injective modules is pure-injective.

(ix) Every pure-injective module is $\Sigma$-pure-injective.

(x) Every module is superstable.

If $R$ satisfies these equivalent conditions then it further satisfies the following.

(a) $R$ is right artinian.

(b) There are at most $|R| + \aleph_0$ indecomposable modules.

(c) Every indecomposable module is finitely generated.

**Proof:**

(i) $\Rightarrow$ (iii) This is immediate from the definitions.

(iii) $\Rightarrow$ (iv) This is trivial.

(iv) $\Rightarrow$ (x) (This is based on [32, Lemma 4]; also compare with Chase's [16, 3.1]; see also [22, 20.20]).

From 1.3 and comments on the invariants, clearly it will be enough to prove (x) for models of the largest theory $T^*$ of $R$-modules.

Let $A$ be any submodule of $\bar{M}$—a very saturated model of $T^*$. Let $N_\lambda$ be any copy of the hull of $A$ (containing $A$, in $\bar{M}$). By hypothesis, there is a pure embedding $N_\lambda \subseteq \bigoplus \lambda N_\lambda = N''$ (say), where $|N_\lambda| \leq \kappa$, for $\lambda \in A$. Moreover, we may suppose that this latter module is purely embedded in $\bar{M}$.

Now, each element $a \in A$ is contained in a finite sub-sum of $N''$. Hence $A$ is contained in a direct summand $N' = \bigoplus \lambda N_\lambda$ of $N''$, with $|A| \leq |A| + \aleph_0$. Thus, $|N'| \leq \kappa(|A| + \aleph_0)$.

Consider the composite morphism $A \subseteq N_\lambda \subseteq \bigoplus \lambda N_\lambda \subseteq N'$. Since the projection is split, it is clear that $tp^{\mathfrak{a}}(N')^+ = tp^{\mathfrak{a}}(A)^+ = tp^{\mathfrak{a}}(A)^+$. Hence, by
1.1 the composite morphism $N_A \subseteq \oplus \lambda N_\lambda \rightarrow N'$ is an embedding. (For if $b \in N_A$, $b \neq 0$, then there is some pp $\phi$ linking $b$ and $A$—say, $\phi(b, \bar{a}) \wedge \neg \phi(0, \bar{a})$ holds, where $a \in A$. Then $b$ cannot be sent to 0 in $N'$, otherwise we would have $\phi(0, \bar{a}) \in tp^{N'}(A) = tp^\bar{A}(A)$, a contradiction.) Hence $|N_A| \leq |N'| \leq \kappa |A|$ (assuming, as we may do, that $\kappa$ is infinite).

Now count the types over $A$. Write $M$ as $N_A \oplus M$, for some copy $N_A$ of the hull of $A$ in $\bar{M}$, and some $M$. If $p \in S^T_1(A)$, then realise $p$, in $\bar{M}$, by $c = (a_0, b) \in N_A \oplus M$. That $p$ may be realised in $\bar{M}$ follows by saturation of $\bar{M}$.

Suppose that $q \in S^T_1(A)$ is realised by $c' = (a_0, b') \in N_A \oplus M$, where $tp(b) = tp(b')$. Then for $\phi(v, \bar{a})$, a pp formula with $\bar{a} \in A$. The following assertions are equivalent: $\phi(v, \bar{a}) \in p^+(v)$; $\phi(c, \bar{a})$ holds; $\phi(a_0 + b, \bar{a})$ holds; (on projecting) $\phi(a_0, \bar{a})$ and $\phi(b', \bar{0})$ hold; (by assumption) $\phi(a_0, \bar{a})$ and $\phi(b', \bar{0})$ hold; (by linearity) $\phi(a_0 + b', \bar{a})$ holds; $\phi(c', \bar{a})$ holds; $\phi(v, \bar{a}) \in q^+(v)$.

Therefore, by Baur's result [11], $p = q$.

There are, however, at most $\kappa |A|$ choices for $a_0$ (as was shown above), and at most $|S^T_1(A)| \leq 2^{1| + \aleph_0} = \mu$ (say) choices for $tp(b)$. The latter follows since $tp(b)$ is a collection of formulas in the language—which has cardinality $|R| + \aleph_0$. Hence $|S^T_1(A)| \leq \kappa |A| \mu$.

Therefore, provided $|A| \geq \kappa \mu$ (a constant), $|S^T_1(A)| \leq |A|$.

That is, $T$ is $\lambda$-stable for every cardinal $\lambda \geq \kappa \mu$. Hence (see Section 1), $T$ is superstable, as required.

(x) \Rightarrow (vi) (This is due to Macintyre [43, Lemma 3]. The proof is that of [31, Lemma 6]; I will summarise it here.)

Suppose that $P^*_\lambda$ does not have acc. Then (by 1.5 and 1.2) choose pp formulas $\phi_n$ ($n \in \omega$), such that $M_0^n > M_0^{n+1} > \cdots > M_0^\omega > \cdots$, where $M_0$ is an arbitrarily chosen model of $T^*$. It is clear, from the definition of $T^*$, that $(T^*)^\aleph_0 = T^*$. Hence each index $|M_0^n/M_0^{n+1}|$ is infinite.

Choose a cardinal $\lambda > 2^{1R + \aleph_0}$, such that $\lambda^{\aleph_0} > \lambda$—any $\lambda > 2^{1R + \aleph_0}$ of cofinality $\aleph_0$ will do. Choose a $\lambda$-saturated model $M$ of $T^*$ with $M > M_0$. It is easy to check, from the definition of $\lambda$-saturated, that $|M_0^n/M_0^{n+1}| \geq \lambda$, for each $n \in \omega$.

Then it is possible to define $\lambda^{\aleph_0}$ ($\geq \lambda$) distinct 1-types, using only $\lambda$ parameters: thereby contradicting $\lambda$-stability, hence superstability.

These types correspond to nested decreasing sequences of cosets, defined as follows.

For each $n \in \omega$, choose elements $a_{\alpha a} \in M_0^\alpha \setminus M_0^{\alpha + 1}$ ($\alpha < \lambda$), such that the union of cosets $\bigcup \{a_{\alpha a} + M_0^{\alpha + 1} : \alpha < \lambda\}$ is disjoint. If $\eta \in \lambda^{n + 1}$, set $b_\eta = \sum_{i=0}^n a_{\eta(i),i}$, and let $\phi_n(v)$ be the pp formula $\phi_{n+1}(v - b_\eta)$, which defines the coset $b_\eta + M_0^{\alpha + 1}$. Then for $\eta \in \lambda^{\aleph_0}$, set $p_\eta(v) = \{\phi_{\eta \eta(n+1)}(v) : n \in \omega\}$. The $p_\eta$, as $\eta$ ranges over $\lambda^{\aleph_0}$, are mutually contradictory, so extend to $\lambda^{\aleph_0}$ many distinct types in $S^T_1(\bigcup \{a_{\alpha a} : n \in \omega, \alpha < \lambda\})$. 
A tree diagram, where the "partial type" $p'_\eta$ corresponds to the branch defined by $\eta \in \lambda^{<\omega}$, may make this construction clearer.

\[ \begin{array}{c}
M^{\omega_0} \\
\vdots \\
a_{00} + a_{01} + M^{\omega_1} \\
\vdots \\
a_{00} + a_{10} + a_{01} + M^{\omega_2} \\
\vdots \\
a_{00} + a_{10} + a_{11} + M^{\omega_2} \\
\vdots \\
a_{01} + a_{10} + a_{11} + M^{\omega_2} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \]

(vi) $\Rightarrow$ (v) (This is [31, Lemma 5]. As stated in [32, Theorem 1], countability of $R$ is not required.)

Let $p \in S_T^*(A)$. From acc on $P_1^R$ it is easy to see that there exists a single pp formula $\phi \in p^+$, with parameters from $A$, logically equivalent under $T^*$ to all of $p^+$. Now, there are only $|A| + |R| + \aleph_0$ such formulas. Hence $|S_T^*(A)| \leq |A| + |R| + \aleph_0$. Therefore $T^*$ is totally transcendental. Hence, by 1.4, every module is totally transcendental.

(v) $\Rightarrow$ (x) This is always so (see Section 1).

(vi) $\Rightarrow$ (vii) As noted already, (vi) implies that every pp-type is equivalent to a single pp formula. The result is then immediate from the fact [73, Theorem 2] that the pure-injective modules are just the algebraically compact ones.

Alternatively, note that since we have (vi) $\Rightarrow$ (v), we may quote [31, Lemma 6].

(vii) $\Rightarrow$ (viii) This is trivial.

(viii) $\Rightarrow$ (ix) This is trivial.

(ix) $\Rightarrow$ (vi) (This is based on an argument of Bass, originally for injective modules, given in [16, 4.1]. Also compare [76, 3.4] and [78].)

Suppose that $P_1^R$ fails to have the ascending chain condition. Let $M$ be any pure-injective model of $T^*$. Choose pp formulas $\phi_n(v) \ (n \in \omega)$, such that the chain $M^{\omega_0} > M^{\omega_1} > \cdots > M^{\omega_\eta} > \cdots$ is strictly decreasing. Pick $a_i \in M^{\omega_i}\setminus M^{\omega_i+1}$, and set $b_n = (a_0, a_1, \ldots, a_{n-1}, 0, 0, \ldots) \in M^{(\aleph_0)}$.

Consider the set $\Phi(v) = \{\phi_n(v - b_n) : n \geq 1\}$, of pp formulas defined over $M^{(\aleph_0)}$. This set is finitely realised in $M^{(\aleph_0)}$, since $M^{(\aleph_0)} \models \phi_k(b_n - b_k)$ for $k < n$—note that $b_n - b_k = (0, \ldots, a_k, \ldots, a_{n-1}, 0, \ldots)$. By hypothesis, $M^{(\aleph_0)}$ is pure-injective, so algebraically compact. Therefore, since the $\phi_n$ are pp, there is $c = (c_n)_{n \in \omega} \in M^{(\aleph_0)}$ which realises simultaneously all the formulas in $\Phi(v)$.

Since $c$ lies in the direct sum $M^{(\aleph_0)}$, there is $n \in \omega$ with $c_m = 0$, whenever $m > n$. But then $\phi_{n+1}(c - b_{n+1})$ is $\phi_{n+1}(0, \ldots, 0, -a_n, 0, \ldots)$, which on projecting gives $\phi_{n+1}(-a_n)$—contrary to choice of $a_n$. 
First of all: every module $N$ has an indecomposable direct summand. For choose $a \in N$ such that $tp(a)^+$ is maximal in $P^K_a$, among the set of such $pp$ types, for nonzero elements $a$ of $N$. This exists by acc on $P^K_a$. Since $N$ is pure-injective ($(vii) \Rightarrow (vii)$), there is a decomposition $N = N_a \oplus N'$, for some copy $N_a$ of the hull of $a$.

Then $N_a$ is indecomposable. For otherwise, set $N_a = N_1 \oplus N_2$, with $N_1, N_2 \neq 0$. Write $a = a_1 + a_2$, with $a_1 \in N_1, a_2 \in N_2$. By [55, 2.5] both $a_1, a_2$ are nonzero. Suppose $\phi \in tp(a)^+$. Then, on projecting, from $\phi(a)$ we obtain $\phi(a_1)$ and $\phi(a_2)$. Thus $tp(a_i)^+ \supseteq tp(a)^+$, so by maximality of $tp(a)^+$, $tp(a_i)^+ = tp(a)^+ (i = 1, 2)$.

On the other hand, since $a_1 \in N_a$ there is ([55, 2.5], say) some $pp$ formula linking $a$ and $a_1$; say $\phi(a, a_1)$ holds but $\phi(0, a_1)$ does not hold. Projecting this first formula yields $\phi(a_1, a_1)$ and $\phi(a_2, 0)$. Then, since $tp(a_i)^+ = tp(a_2)^+$, it must be that $\phi(a_1, 0)$ holds. Subtracting this from $\phi(a_1, a_1)$ yields $\phi(0, a_1)$, a contradiction.

The first claim is therefore established.

Now let $N_1$ be a direct sum of indecomposable modules, with $N_1 <^+_1 N$. Suppose that $N_1$ is maximal such, by Zorn's Lemma. Since $N_1$ is, by assumption, pure-injective, $N_1$ is a direct summand of $N$. Say $N = N_1 \oplus N'$. By the first claim, and maximality of $N_1$, it must be that $N' = 0$, as required.

(ii) $\Rightarrow$ (i) Note first that (ii) $\Rightarrow$ (ix). For let $N = \oplus \lambda N_\lambda$ be a pure-injective module with (using (ii)) each $N_\lambda$ indecomposable, and pure-injective. Then $N^{(N_\lambda)} = \oplus \lambda N_\lambda^{(N_\lambda)}$. Further, by hypothesis, $\oplus \lambda N_\lambda^{(N_\lambda)} \cong \oplus \mu N_\mu$, for suitable indecomposable pure-injective modules $N_\mu$.

Now, indecomposable pure-injective modules have local endomorphism rings ([79, Theorem 9] or [36, 1.2 and 1.3], or, for a model-theoretic proof, [55, 4.8] or [75, 4.3]). Therefore by E. Fisher's extension of Azumaya's Theorem (see [33, Lemma 3]), alternatively by [36, 1.2] plus the usual Azumaya's Theorem ([22, 2.16], for example), the two decompositions of $N^{(N_\lambda)}$ are just reorderings of each other, in the sense that there is a one-to-one correspondence between the copies of the $N_\lambda$'s and the $N_\mu$'s. Therefore $N^{(N_\lambda)} = (\oplus \lambda N_\lambda)^{(N_\lambda)} \cong (\oplus \lambda N_\lambda^{(N_\lambda)}) \cong (\oplus \mu N_\mu) \cong N^{(N_\lambda)}$. Thus $N^{(N_\lambda)}$ already is pure-injective, as required.

Therefore, every module is a direct sum of indecomposable pure-injectives since (ix) $\Rightarrow$ (vii). But there are at most $|P^K|$ indecomposable pure-injectives, since each has the form $N_p$ for some $pp$-type $p$. Therefore (i) follows.

The equivalence of (i)-(x) has been established. I proceed to derive (a), (b), (c) as consequences.

(a) Here we may simply quote [22, 20.23], which states that (ii) $\Rightarrow$ (a); alternatively [1, Corollary 9], or, for a more model-theoretic proof, [8a, 9.11]. I will, however, include a proof (compare [22, pp. 118, 119]) which is fairly transparent.
Notice first that (vi) immediately implies that $R$ is right noetherian, since the lattice of right ideals of $R$ embeds in $P^R_1$ (for example, by $I \mapsto t^{p_k(r)}(1 + I)$).

Let $N$ be the nilradical of $R$. Then the module $E(R/N)$ is finitely generated. For by (ii), $E(R/N) = \bigoplus E_i$, for suitable indecomposable injectives $E_i$. Since $R/N$ is cyclic, it is contained in some finite sub-sum which, since $R/N$ is essential in $E(R/N)$, must be all of $E(R/N)$. By (c) below, each $E_i$ is finitely generated. Hence $E(R/N)$ is finitely generated. Now [22, 20.12] finishes the proof: I include the argument.

Let $E'$ be the injective hull of $R/N$ as an $R/N$-module. Then it is easy to see that $R/N \leq E' \leq E(R/N)$. Therefore, since $E'$ is a submodule of a finitely generated $R$-module, and since $R$ is right noetherian, $E'_R$ is finitely generated.

But by Goldie's Theorem, $E'$ has the structure of a ring of fractions of $R/N$. Therefore, if $R/N \neq E'$, then there is a non-zero-divisor $c \subseteq R/N$ without an inverse in $R/N$. Therefore, we have inside $E'$, $R/N \leq c^{-1}(R/N) \leq c^{-2}(R/N) \leq \cdots$. So, since $E'$ has acc on submodules, there is some $n \in \omega$ and some $b \in R/N$, with $c^{-n+1} = c^{-n}b$. Hence $c^{-1} = b \in R/N$, a contradiction. Thus, every non-zero-divisor of $R/N$ already is invertible in $R/N$. Hence $R/N$ is its own ring of fractions. Therefore, by Goldie's Theorem, $R/N$ is a semisimple artinian ring.

Furthermore, since $R$ is right noetherian, $N^k = 0$, for some $k \in \omega$. Moreover, each $N^i/N^{i+1}$ is a finitely generated right $R$, hence $R/N$-, module. Therefore, $R$ is a finitely generated right $R/N$ module. So, since the latter is artinian, $R$ is right artinian.

(b) Either use (c), or more directly, note that every indecomposable module is, by (vii), pure-injective, hence is the hull of some pp-type in $P^R_1$. By (vi), each pp-type is determined by a single formula. There are $|R| + \aleph_0$ formulas. Hence there are at most $|R| + \aleph_0$ indecomposables.

(c) Let $N$ be indecomposable, and choose any non-zero element $a \subseteq N$. Note that $N = N_a$. Let $p = t^p(a)$ and, by (vi), choose a pp formula $\phi$ which is equivalent to $p^*$. Then $\phi(v)$ has the form $\exists w \psi(v, w)$, with $\psi$ a conjunction of linear equations.

Since $N \models \phi(a)$, there is $\bar{b} = (b_1, \ldots, b_n)$, say, in $N$, such that $N \models \psi(a, \bar{b})$. Set $A = aR + \sum_{i=1}^n b_iR$. Clearly $A \models \psi(a, \bar{b})$, so $A \models \phi(a)$. Since $A \subseteq N$, it is clear that $t^p(a) \leq t^p(a) = p$. On the other hand, $\phi \in t^p(a)$. So $t^p(a) = p$.

Therefore, since $A$ is, by (vii), pure-injective, it follows, from the basic properties of hulls [55, 2.10], that $A$ must be a direct summand of $N$. Hence $A = N$, and $N$ is finitely generated, as required.

Note the following corollary of the proof of 2.1(iv) ⇒ (x).

**Corollary 2.2.** $R$ is right pure semisimple if and only if
(xi) there exists a cardinal $\kappa$ such that, for every module $A(\leq M)$, $|N_A| \leq \kappa|A|$ (where $N_A$ refers to the hull of $A$ in $M$).

This is not entirely an algebraic characterisation, since $N_A$ depends not just on the isomorphism type of $A$ but also on the pp-type of $A$ (that is, on how $A$ is embedded in the over-module $M$). In the case where $R$ is (von Neumann) regular, however, $T^*$ has complete elimination of quantifiers (by [64b], say). Therefore we have the following corollary to 2.2.

**Corollary 2.3.** Let $R$ be a regular ring which is not artinian. Then for every cardinal $\kappa$, there is some module $A$, such that $|E(A)| > \kappa|A|$.

Any right noetherian, non-artinian, ring shows that some condition such as regularity is necessary in 2.3.

Note that by 2.1(i) $\rightarrow$ (iii) any factor ring of a rt. pure s.s. ring also is rt. pure s.s.

It is an open question whether or not 2.1(a) and (c) may be strengthened, respectively, to:

(a)* $R$ is right and left artinian,

(c)* there are only finitely many indecomposable modules.

Indeed, it is open whether right pure semisimplicity implies left pure semisimplicity.

There are numerous further equivalents to right pure semisimplicity. I list some of these, and indicate where they may be found. Since the equivalence of the conditions (i)–(x) of 2.1 was established only gradually, sometimes the original statements of these further equivalents contained a certain amount of redundancy.

A decomposition $M = \bigoplus \alpha N_A$, with the $N_A$ all nonzero, is said to complement direct summands if, whenever $N$ is a direct summand of $M$, there is $A' \subseteq A$ with $M = N \oplus \bigoplus A N_A$. Note that this implies that the $N_A$ are indecomposable. Fuller [29] gives the following equivalent, among others.

(xii) Every module has a decomposition which complements direct summands.

Zimmermann-Huisgen [78] has an alternative proof of this, and, in fact, obtains a local version.

For any module $M$, the following are equivalent.

(i) $M^{\mathbb{N}_0}$ is a direct sum of submodules with local endomorphism rings.

(ii) $M^I$ has a decomposition which complements direct summands, for any $I$.

(iii) $M$ is $\Sigma$-pure-injective.

Fuller [29] also has the following equivalent.
(xiii) $1 \in R$ is a sum of orthogonal primitive idempotents, and every family $N_0 \rightarrow f_0 N_1 \rightarrow f_1 N_2 \rightarrow \cdots$ of homomorphisms between finitely generated indecomposable modules is noetherian (that is, eventually $f_n f_{n-1} \cdots f_0 = 0$, or eventually the $f_n$'s are isomorphisms).

The first part of (xiii) is immediate from (ii). The second part may be derived as follows. If, for each $n$, $f_n f_{n-1} \cdots f_0 \neq 0$, then choose $a_i \in N_i$ such that for each $n$, $a_{n+1} = f_n a_n \neq 0$. This is possible since $N_0$ is finitely generated. Let $p_k = tp^{N_k}(a_k)$. Then $p_0 \leq p_1 \leq \cdots$. Since $P^R_i$ has acc, eventually $p_n = p_{n+1} = \cdots$. Hence ([55, 2.11], or compare with the proof of 2.1(c)), $f_n f_{n-1} \cdots$, all are isomorphisms, as required.

It is also remarked in [29] that, by results of Auslander, equivalent is:

(xiv) $R$ is right artinian, and every family of monomorphisms, between finitely generated indecomposable modules, is noetherian.

Noting that $R$ right artinian implies that every finitely generated module is a direct sum of indecomposables, it is clear that (xiv) $\Rightarrow$ (vi). The other direction follows as above.

Yet another equivalent is the following.

(xv) Every module is pure-projective.

This may be deduced from 2.1 and [73, Corollary 3], or from [25, 10.1] which contains the equivalent: every module is a direct sum of countably generated pure-projective modules.

An equivalent, due to Bican [13], in terms of abstract notions of purity, is given as 2.2 in the survey article [70].

A number of equivalents, some in terms of categories of functors from finitely presented modules to abelian groups, are given in [69, 2.3].

A number of references contain relevant information. Of those not mentioned elsewhere in the paper there are, for example: [17, 23, 34, 68]; see also the volumes [18, 19], and particular the bibliography and survey article [59] therein. Let me also take the opportunity to point out the paper [38], which contains applications of model theory, though of a rather different sort.

It has been seen, in 2.1(b), that if $R$ is right pure semisimple, then there are at most $|R| + \aleph_0$ indecomposable modules, up to isomorphism.

The ring $R$ is said to be of finite representation type, FRT, if $R$ is right pure semisimple and there are only finitely many indecomposable modules.

The following result characterises this property in various ways. There are some connections between the model-theoretic methods used here and the category-theoretic methods (see [2, 3, 5]) used to establish 2.4. I will examine these connections in the sequel. Note, in particular, that 2.4 shows that being of finite representation type is equivalent to being both right and left pure semisimple—thus it is a right/left symmetric property.
Theorem 2.4. The following conditions on a right artinian ring $R$ are equivalent.

(i) $R$ is of finite representation type.

(i)\textsuperscript{op} $R$ is of finite representation type on the left.

(ii) There are only finitely many indecomposable modules.

(iii) There are only finitely many finitely generated indecomposable modules.

(iv) $R$ is both right and left pure semisimple.

(v) Every right, and every left, module is totally transcendental.

(vi)(a) For every family $N_0 \rightarrowtail f_0 N_1 \rightarrowtail f_1 N_2 \rightarrowtail f_2 \ldots$ of monomorphisms, between finitely generated indecomposables, the $f_n$'s are eventually isomorphisms.

(b) For every family $\cdots \rightarrow f_2 N_2 \rightarrow f_1 N_1 \rightarrow f_0 N_0$ of epimorphisms between finitely generated indecomposables, the $f_n$'s are eventually isomorphisms.

(vii)(a) Every object of the functor category $\mathcal{D}$ has a simple subobject.

(b) Every simple object of the functor category $\mathcal{D}$ is finitely presented.

Here $\mathcal{D}$ is the (Grothendieck abelian) category of functors from the full subcategory of $\mathcal{M}_R$, whose objects are the finitely generated modules, to the category of abelian groups.

The equivalence of (i) and (i)\textsuperscript{op} is [20, 1.2], also see [17]. That (iv) implies (iii) (and also (ii) of 2.1) is [71, 9.5], also see [72]. The equivalence of (iv) and (v) is immediate from 2.1. For (iv) $\leftrightarrow$ (i), see [30, 60]. Then [3] contains the rest. I have included a direct proof of (iii) $\Rightarrow$ (i) below for the special case of Artin algebras.

The answer to the following question is unknown.

Q1. If $R$ is right pure semisimple, does it follow that $R$ is of finite representation type?

Even the next question is open, as was noted above.

Q2. If $R$ is right pure semisimple, is $R$ necessarily left artinian?

The answer to Q1, and so to Q2, is known [4] to be in the affirmative if $R$ is an Artin algebra.

It is also known [14b] that the answer to Q1 is negative if, in place of $R$, one allows a ringoid with infinitely many objects (see [46, 47]).

More precisely, let $I$ be a totally ordered set. Let $K$ be a field. Define the ringoid $K[I]$ to have as objects the elements of $I$, and to have the morphism
groups $K[I]_{(i,j)} = K$ if $i \leq j$, and 0 otherwise. Let $(I, \mathbb{M}_K)$ denote the
category of functors (additive, of course) from $I$ to the category of $K$-vector
spaces.

Now Brune [14b, Theorem 1] shows that, for $I$ totally ordered, $(I^{\text{op}},
\mathbb{M}_K) \simeq \mathbb{M}_{K[I]}$ is pure semisimple if and only if $I^{\text{op}}$ is well-ordered. Therefore
taking $I^{\text{op}}$ to be any infinite ordinal, we obtain a right but not left pure
semisimple ringoid.

Brune's Theorem 1 seems to have some relation to 3.9 below.
Note that, using 2.4(i) $\Leftrightarrow$ (iv), Q1 has a model-theoretic formulation.

Q1'. If every right $R$-module is totally transcendental, does it follow that
every left $R$-module is totally transcendental?
The following proposition turns out to be useful.

**Proposition 2.5.** Suppose that there is a totally transcendental module
$C$, such that every finitely generated module purely embeds in $C$. Then $R$ is
right pure semisimple.

**Proof.** Let $M$ be any module. Let $a$ be any nonzero element of $M$. Set
$p = tp^M(a)$. Set $A_0 = aR$, and put $p_0 = tp^{A_0}(a)$.

If possible, choose a sequence $A_0 < A_1 < \cdots < A_n < \cdots$ of finitely
generated submodules of $M$, such that if $p_i = tp^{A_i}(a)$, then $p_0 < p_1 < \cdots <
p_n < \cdots (\leq p)$.

By hypothesis there is, for each $i$, a pure embedding $f_i: A_i \rightarrow C$. Note that
$tp^C(f_i a_i) = p_i$. Since $C$ is totally transcendental, so has "act on types"
(1.2), eventually $p_i = p_{i+1}$, a contradiction.

Thus there is $A \leq M$ with $A$ finitely generated, containing $a$, and with
$tp^A(a) = p = tp^M(a)$. Since $A$ purely embeds in $C$, it is totally transcendental.
Hence $A$ is pure-injective. Therefore (see Section 1), $A$ purely contains a
copy of the hull $N_a$ of $a$ in $M$. But by [55, 2.5] $N_a$ must itself be pure in $M$,
so must be a direct summand of $M$.

Then, as in the last part of the proof of 2.1(v) + (vi) $\Rightarrow$ (ii), $M$ is a direct
sum of indecomposable totally transcendental modules, noting, from the
argument above, that in this context all such direct sums are totally transcen-
dental, hence are pure-injective.

The result is therefore established.

**Corollary 2.6.** $R$ is right pure semisimple if and only if the module
$\bigoplus \{M : M \text{ is finitely generated}\}$ is totally transcendental.

**Corollary 2.7.** Suppose that every finitely generated module is totally
transcendental, and that there are only finitely many indecomposable finitely
generated modules. Then $R$ is of finite representation type.
The ring $R$ is an artin algebra if it has artinian centre, and is finitely generated, as a module, over its centre. So such a ring is right and left artinian.

**Lemma 2.8.** Let $R$ be an artin algebra. Then every finitely generated module is totally transcendental (so, in particular, is pure-injective).

**Proof.** Suppose $M_R$ is finitely generated. Let $Z$ be the centre of $R$. So $Z$ is artinian, and $R_Z$ is finitely generated. Therefore $M_Z$ is finitely generated, hence is artinian. Since $Z$ is commutative, every pp-definable subgroup of $M_R$ is (easily seen to be) a $Z$-submodule of $M$. Therefore, $M_R$ has the descending chain condition on pp-definable subgroups. Hence, by 1.2, $M_R$ is totally transcendental.

**Corollary 2.9.** Suppose that $R$ is an artin algebra. If there are only finitely many indecomposable finitely generated modules, then $R$ is of finite representation type.

**Proof:** This is immediate from 2.8 and 2.7.

Of course, for the conclusion of 2.9 to hold, it is enough that $R$ be right artinian, rather than actually an artin algebra; but this gives an alternative proof in this special case.

It turns out that, for countable rings $R$, being of finite representation type is equivalent to there being only countably many countable modules (up to isomorphism). Let $n(\kappa, T_U)$ be the number of $R$-modules of cardinality $\kappa$.

**Theorem 2.10 (Baldwin and McKenzie [8a, 8.7]).** Suppose $R$ is countable. Then the following conditions are equivalent.

(i) $n(\mathbb{N}, T_U) < \aleph_0$.

(ii) $n(\mathbb{N}, T_U) < 2^{\aleph_0}$.

(iii) $R$ is of finite representation type.

(i)^{op}, (ii)^{op}: as (i), (ii), but with $R^{op}$ in place of $R$.

**Proof.** Suppose that $n(\mathbb{N}, T_U) < 2^{\aleph_0}$. Then I show that every finitely generated module $A$ is totally transcendental. For were $A$ to contain an infinite properly descending chain of pp-definable subgroups then (compare proof of 2.1(x) \Rightarrow (vi)) $S^T_A(A)$ would have cardinality $2^{\aleph_0}$. Now, if $A = \sum_{i=1}^n a_iR$ (say), then clearly $S^T_A(a_1, \ldots, a_n)$: for every element of $A$ is a term in $a_1, \ldots, a_n$. Therefore $|S^T_A(0)| = 2^{\aleph_0}$: for if $b, c$ realise different types in $S^T_A(a_1, \ldots, a_n)$, then $(a_1, \ldots, a_n, b)$ and $(a_1, \ldots, a_n, c)$ realise different types in $S^T_A(0)$.

Now, any countable model of $T_A$, "contains" only countably many
(n + 1)-tuples, hence realises only countably many types in $S^T_{n+1}(0)$. Therefore, there must be $2^{\aleph_0}$ countable models of $T_A$. In particular $n(\aleph_0, T sure that

$$P_f \text{ is equivalent under } T \text{ to } \tau; \text{ that is, } T \text{ proves } \tau \rightarrow p \text{ for every } \tau \in P_f.$$ In such a case I will use the notation $T \vdash \tau \rightarrow p$.

The reason for this terminology lies in the fact that each set of types $S^T_n(A)$ carries a natural topology: a typical basic set has the form $O_\tau = \{ p : \tau \in P_f \}$. Thus $p$ is isolated if and only if it is isolated in the topological sense. It is a standard fact that, with this topology, $S^T_n(A)$ is a totally disconnected, compact, Hausdorff space.

**Lemma 3.2.** Let $T$ be totally transcendental. Suppose $p \in S^T_1(0)$. Then
either $p$ is isolated, or $p = \bigwedge \{ q \in S^T_1(0) : q > p \}$, that is, either $p$ is isolated, or $p^+ = \bigcap \{ q^+ \in S^T_1(0) \text{ and } q^+ \not\subset p^+ \}$.

Proof. Let $\Psi = \bigcap \{ q^+ : q > p \}$. Choose a pp formula $\phi \in p^+$ so that $\phi$ is $T$-equivalent to $p^+$: This is possible, since $T$ is totally transcendental.

If $\Psi = p^+$, then $p = \bigwedge \{ q : q > p \}$, as required.

Suppose then that there is some pp formula $\psi \in \Psi \setminus p^+$. I will show that $\Psi$ is $T$-equivalent to $\{ \phi, \psi \}$, that is, to $\phi \land \psi$. Since $\phi, \psi \in \Psi$, it is clear that $T \vdash \Psi \rightarrow \phi \land \psi$. On the other hand, let $a$ be any element, in a model of $T$, satisfying $\phi \land \psi$. Since, therefore, $\phi(a)$ holds, it must be that $tp(a) \geq p$. Since $\psi(a)$ holds, and $\psi \not\in p^+$, it follows that $tp(a) > p$. Hence $\Psi(a)$ holds. Thus $T \vdash \Psi \leftrightarrow \phi \land \psi$.

It is clear then that the formula $\phi \land \neg \psi$ isolates $p$. In more detail: if $a$ satisfies $\phi \land \neg \psi$, then, since $\phi(a)$ holds, certainly we have $q = tp(a) \geq p$. Were $q > p$, then $\Psi(a)$, so in particular $\psi(a)$ would hold, a contradiction. Thus $q = p$, as required.

It is known that if $T$ is totally transcendental, then $T$ has a (unique to isomorphism) prime model $M_0$. That is, $M_0$ elementarily embeds in every model of $T$. I will require the following description of those types which are realised in this prime model.

**Theorem 3.3 ([66, 21.2], say).** Let $T$ be totally transcendental, with prime model $M_0$. Then the type $p \in S^T_1(0)$ is realised in $M_0$ if and only if $p$ is isolated.

Elsewhere [54], I have termed a module $M$ an elementary cogenerator, if, for all $M' \equiv M^{N_0}$, there is some index set $I$, and some pure embedding $M' <^+ I M^I$.

**Corollary 3.4.** Let $T$ be totally transcendental. Then the prime model $M_0$ of $T$, and hence every model of $T$, is an elementary cogenerator.

Proof. Note first that, if $M < M'$ and if $M$ is an elementary cogenerator, then so is $M'$, since products of elementary embeddings are elementary [24]. Therefore, it is indeed sufficient to consider just the prime model $M_0$ of $T$. Moreover, it may be supposed that $T = T^{N_0}$. For: clearly the prime model of $T^{N_0}$ (which also is totally transcendental) is an elementary substructure of $M^{N_0}_0$; and further, $(M^{N_0}_0)^I \simeq M^{N_0 \times I}_0$, for any set $I$.

Suppose, for a contradiction, that the result fails. Since $T$ is totally transcendental, we have acc on types. So choose $p \in S^T_1(0)$ maximal with respect to the property that its hull $N_p$ is purely embeddable in no power of $M_0$: note that if all such hulls were so embeddable, then every model, being a direct sum of such hulls, would also be so embeddable. Since $p$ is, in
particular, not realised in $M_0$, $p$ must, by 3.3, be non-isolated. So by 3.2, $p = \bigwedge \{q \in S^T_1(0) : q > p\} = \bigwedge \{p_\lambda \in A\}$, say.

Let $M = M_0 \oplus \alpha N_{p_\lambda}$. Since $T = T^{\aleph_0}$, $M \models T$. So by the maximality assumption on $p$, each $N_{p_\lambda}$ purely embeds in some power of $M_0$. So, clearly, $M <_{i+} M_0^{\lambda}$, for some index set $I$.

But then $N_p <_{i+} (M_0^{\lambda})^\lambda$. To see this, pick, for each $\lambda \in A$, some $a_\lambda \in M$, with $tp^M(a_\lambda) = p_\lambda$. Then set $a = (a_\lambda)_{\lambda \in A} \in M^\lambda <_{i+} (M_0^{\lambda})^\lambda$. Then it is easy to check that $tp(a)^+ = \bigcap \lambda \in A tp(a_\lambda)^+ = \bigcap \lambda \in A p_\lambda = p^+$. That is, $tp(a) = p$.

Therefore $N_p \cong N_a <_{i+} (M_0^{\lambda})^\lambda \cong M_0^{\lambda ^+ \lambda}$: which is a contradiction, as required.

Before going on, I need the next definition and result. A type $p \in S^T_1(0)$ is irreducible (called indecomposable by Ziegler [75]) if its hull $N_p$ is indecomposable: this generalises the notion of (meet-) irreducible right ideal.

**Theorem 3.5** [75]. Let $T$ be a complete theory. Let $p \in S^T_1(0)$.

(i) The type $p$ is irreducible if and only if

(+) whenever $\psi_1, \psi_2$ are $pp$ formulas not in $p$, there is some $pp$ formula $\phi \in p$ such that $\psi_1 \land \phi + \psi_2 \land \phi \notin p$.

(ii) Suppose $T$ is totally transcendental, with $p^+$ equivalent to the $pp$ formula $\phi$, say. Then $p$ is irreducible if and only if

(+++) whenever $\psi_1, \psi_2$ are $pp$ formulas both of which imply $\phi$, but which are not in $p$, then $\psi_1 + \psi_2 \notin p$.

(iii) If $p$ is irreducible and isolated, then it is isolated by a formula of the form $\phi \land \neg \psi$, with $\phi$ equivalent to $p^+$, and $\psi$ $pp$.

**Proof.** For (i) see [75, 4.4]. Part (ii) is essentially a reformulation of (i) in the particular case. Part (iii) is easily deduced from (ii) in the t.t. case. For the general case it follows from (i) and [57].

**Theorem 3.6.** Suppose $T = T^{\aleph_0}$ is totally transcendental. Then the following conditions are equivalent.

(i) $(*)^*_T$ Whenever $N, N_\lambda (\lambda \in \Lambda)$ are indecomposable direct summands of models of $T$, and $N <_{i+}^* \Pi_{\lambda \in \Lambda} N_\lambda$, then $N \cong N_\lambda$, for some $\lambda \in \Lambda$.

(ii) Every irreducible type $p \in S^T_1(0)$ is isolated.

(iii) For any $N <_{i+}^* M = T$, if $T' = Th(N)$, then every irreducible type in $S^T_1(0)$ is isolated.

**Proof.** (i) $\Rightarrow$ (ii) Let $p \in S^T_1(0)$ be irreducible. Now $N_p$ is a direct summand of some $M \models T$. So by 3.4, there is a pure embedding $N_p <_{i+}^* M_0'$, for some index set $I$, where $M_0$ is the prime model of $T$.

By [32, Theorem 4], $M_0$ has a direct sum decomposition; say, $M_0 = \oplus \lambda N_\lambda$, with each $N_\lambda$ indecomposable.
Now, clearly, $\bigoplus \lambda N_{\lambda} \preceq_i^+ \Pi_{\lambda} N_{\lambda}$. Thus we have $M_0' = (\bigoplus \lambda N_{\lambda})' \preceq_i^+$ $(\Pi_{\lambda} N_{\lambda})'$. Therefore, we have $N_p \preceq_i^+ (\Pi_{\lambda} N_{\lambda})'$. So, by ($\ast$)$_T$, there is an isomorphism $N_p \cong N_{\lambda}$, for some $\lambda$.

Therefore $N_p$ is isomorphic to a direct summand of the prime model $M_0$.

In particular, $p$ is realised in $M_0$, so by 3.3, $p$ is isolated—as required.

(ii) $\Rightarrow$ (iii) Suppose $p \in S'_T(0)$ is irreducible. It is easy to see (for example, [55, 3.11]) that there is some $p_1 \in S'_T(0)$, with $p_1^+ = p_1^+$. Since $N_p \preceq N_{p_1}$, $p_1$ is of course irreducible. So by assumption, and 3.5(iii), there are pp formulas $\phi, \psi$ with $\phi \in p_1^+ = p^+$, and $p_1$ $T$-equivalent to $\phi \wedge \neg \psi$.

Then $p$ is $T'$-equivalent to $\phi \wedge \neg \psi$. To see this, note that in general, if $\theta$ is a pp formula, and if $N \preceq M$, then $N^\theta \preceq N \cap M^\theta$. Therefore, it easily follows that, in $T'$, the formula $\phi \wedge \neg \psi$ defines either the empty set, or a complete type (necessarily $p$). But, since $p^+ = p_1^+$, the formula $\phi \wedge \neg \psi$ is in $p$—so cannot define the empty set.

(iii) $\Rightarrow$ (i) Given $N, N_{\lambda}$ ($\lambda \in \Lambda$) as in ($\ast$)$_T$, set $T' = Th(\Pi_{\lambda} N_{\lambda})^{\aleph_0}$. By assumption, $N$ purely embeds in a model of $T'$. So we may choose $p \in S'_T(0)$ with $N \preceq N_p$. By the hypothesis applied to $T'$, $p$ is isolated (note that $p$ is irreducible, since $N$ is indecomposable).

Therefore $p$ is realised in the prime model $M_0$ of $T'$. Hence $N$ is a direct summand of this model. Now $\bigoplus \lambda N_{\lambda}^{\aleph_0} \models T$. So $M_0 \preceq_i^+ \bigoplus \lambda N_{\lambda}^{\aleph_0}$. Therefore $N \preceq_i^+ \bigoplus \lambda N_{\lambda}^{\aleph_0}$. But, by uniqueness of decomposition, this implies that $N \preceq N_{\lambda}$, for some $\lambda \in \Lambda$, as required.

The global version of this is as follows.

**Theorem 3.7.** Let $R$ be right pure semisimple. Then the following conditions are equivalent.

(i) $R$ is of finite representation type.

(ii) Every irreducible type in $S'_T(0)$ is isolated.

(iii) For any complete theory $T$ of $R$-modules, every irreducible type in $S'_T(0)$ is isolated.

**Proof:** This is immediate from 3.1 and 3.6 applied to $T^*$.

The next result appears in [58], but I repeat the proof here for completeness, minus some verifications.

**Theorem 3.8.** Suppose $T = T^{\aleph_0}$ is a complete totally transcendental theory of modules. Suppose $MR(T) \geq \omega$. Then there is $p \in S'_T(0)$ which is irreducible and non-isolated.

**Proof:** Before sketching the proof, I should mention that the assumptions $T = T^{\aleph_0}$ and $T$ t.t. are here simply for convenience.

Define a “height” function on pp formulas $\phi$, as follows. Let $M$ be any
model of $T$, and set $h_0 = \alpha$ if, for all pp $\psi$, with $M^\psi < M^\alpha$, we have $h\psi < \alpha$, and if $\alpha$ is minimal such (we begin by setting $h(v = 0) = 0$).

Clearly, $\phi$ is assigned a height if and only if $M$ has the d.c.c. on pp-definable subgroups which lie inside $M^\phi$. It is also easy to see that $MR(T) < \omega$ if and only if every pp formula has finite height, that is, if and only if the formula "$v = 0"$ has finite height.

So by hypothesis, $h(v = 0) \leq \omega$. Choose $p \in S^T_1(0)$ maximal with respect to not containing any pp formula of finite height (by t.t., or Zorn's Lemma). Let $\phi$ be a pp formula $T$-equivalent to $p^+$. Suppose $\psi$ is pp and not in $p$. Were $h(\phi \land \psi) \leq \omega$, then $\{\phi \land \psi\} \cup \{-\theta: h\theta < \omega\}$ would be consistent (this is easily checked). Then this set would extend to a type strictly above $p$, but containing no formula of finite height—contradicting maximality of $p$.

Therefore, if $\psi_1, \psi_2$ are pp formulas not in $p$, then $h(\phi \land \psi_1), h(\phi \land \psi_2)$ both are finite. So, by modularity of the lattice of pp formulas, the formula $\phi \land \psi_1 + \phi \land \psi_2$ has finite height. In particular, $\phi \land \psi_1 + \phi \land \psi_2$ is not in $p$. So by 3.5(i), $p$ is irreducible.

Moreover, $p$ is non-isolated. For suppose $p$ were isolated, say, by the formula $d \land \neg \psi$, where (3.4(iii)) $\psi$ is pp. Note that $\phi \land \neg \psi$ is equivalent to $\phi \land \neg (\phi \land \psi)$. By the above, since $\psi \not\in p$, we have $h(\phi \land \psi) = n$, for some $n < \omega$. But, since this formula isolates $p$, it is clear that there can be no pp formula strictly between $\phi$ and $\phi \land \psi$. Therefore, again by modularity, $h\phi = n + 1$, a contradiction, as required.

Now, I can prove the characterisation of rings of finite representation type in terms of the Morley rank of their modules.

**Theorem 3.9.** The following conditions on a ring $R$ are equivalent.

(i) $R$ is of finite representation type.

(ii) $MR(T^*) < \omega$.

(iii) Every $R$-module has finite Morley rank.

(iv) $P_1^R$ has finite length.

**Proof.** (i) $\Rightarrow$ (ii) If $MR(T^*) \leq \omega$, then by 3.8, there is some non-isolated irreducible type in $S^T_1(0)$. So by 3.7, $R$ is not of finite representation type.

(ii) $\Rightarrow$ (iii) Every module is a direct summand of a model of $T^*$. Since pure embeddings do not decrease Morley rank (directly, or [53]), the conclusion is clear.

(iii) $\Rightarrow$ (iv) In Section 1 we observed that $P_1^R$ is lattice-isomorphic to $S^T_1(0)$. Since, by assumption, $T^*$ is t.t., this lattice has a.c.c. On the other hand, the d.c.c. follows from finite Morley rank (direct computation, or see [53]).

(iv) $\Rightarrow$ (i) Since the lattice of right ideals of $R$ embeds in $P_1^R$, certainly
$R$ is right artinian. I check the conditions of 2.4(vi): an equally straightforward verification of condition (ii) of 3.7 is an alternative.

Part (a) of 2.4(vi) is immediate from a.c.c. on $P_R$. Part (b) is similar: with notation as there, choose a nonzero element $a_i \in N_0$. Inductively, choose $a_{i+1} \in N_{i+1}$, such that $f_i a_{i+1} = a_i$. Set $p_i = tp^{\lambda}(a_i)$. Then $p_{i+1} \leq p_i$, for each i. But the $N_i$ are indecomposable, so if $p_{i+1} = p_i$, then $f_i$ is an isomorphism. Since $P_\lambda$ has d.c.c., eventually this is the case.

May condition (iv) above be weakened to: (iv)' $P_R$ has the descending chain condition?

I will now give a proof of one direction of the result 3.1 of Auslander. In fact I will prove a relativised version. It seems likely that there should be a relatively short model-theoretic proof of the other direction: the direction which I actually used above in the proof of 3.7, and hence of 3.9. In fact, a relativised version, applying to universal Horn theories of modules, should hold.

The proof of 3.1 [4] relies heavily on the results and techniques developed in [2], [3], and [5]. The proof below is a compactness argument. Say that types $p$ and $q$ are related, $p \sim q$, if $N_p \cong N_q$.

**Theorem 3.10.** Suppose $T$ is totally transcendental. Let $M_0$ be the prime model of $T$. Suppose $M_0$ realises infinitely many non-related irreducible types. Then there is a non-isolated irreducible type in $S_T'\Omega$.

**Proof.** Suppose $M_0 \cong \bigoplus \lambda N_\lambda$, for suitable cardinals $\lambda$, where each $N_\lambda$ is indecomposable, and where $\mu \neq \lambda$ implies $N_\mu \cong N_\lambda$.

For each $\lambda \in \Lambda$, choose a (irreducible) type $p_\lambda \in S_T'\Omega$ with $N_\lambda \cong N_{p_\lambda}$. Set $\mathcal{P} = \{ p_\lambda : \lambda \in \Lambda \}$. Let $\mathcal{P}^*$ denote the set of accumulation points of the set $\mathcal{P}$.

Since $S_T'\Omega$ is compact and Hausdorff, and by hypothesis $\mathcal{P}$ is infinite, it follows that $\mathcal{P}^* \neq \emptyset$. But each $p_\lambda$ is realised in the prime model $M_0$, so, by 3.3, is isolated. Therefore $\mathcal{P}^* \cap \mathcal{P} \neq \emptyset$. Choose $q \in \mathcal{P}^* \setminus \mathcal{P}$. Choose $q \in \mathcal{P}^* \setminus \mathcal{P}$.

Now, since $T$ is totally transcendental, $N_q$ is ([32, Theorem 4] and [55, 1.6], say) a finite direct sum of indecomposable modules. If $S_T'\Omega$ contained no non-isolated irreducible types, then each of these indecomposables would be the hull of an isolated irreducible type: so each of these indecomposables would be isomorphic to some $N_\lambda$. So if $q$ realises $p$, in some model, then we may find some $\lambda \in \Lambda$ and $b \in N_q$, with $tp(b) = p_\lambda$.

Since $b \in N_q$, there is some pp formula $\phi$ such that $\phi(a, b)$ holds, but $\phi(a, 0)$ does not hold. Let $\chi$ be a formula which isolates $p_\lambda$.

Then we have $\phi(a, b) \land \neg \phi(a, 0) \land \chi(b)$. Hence $\theta(v) \in q(v)$, where $\theta(v)$ is the formula $\exists w (\phi(v,w) \land \chi(w))$. Therefore, since $q \in \mathcal{P}^*$, there exist infinitely many $\mu \in \Lambda$ such that $\theta \in p_{\mu}$ (see the earlier definition of the topology).
If, however, \( \theta \in \rho_\mu \), and if \( a_\mu \) realises \( \rho_\mu \), then there is some element \( b_\mu \) such that \( \phi(a_\mu, b_\mu) \land \neg \phi(a_\mu, 0) \land \chi(b_\mu) \) holds. Since \( \chi \) isolates \( \rho_\lambda \), it follows that \( tp(b_\mu) = \rho_\lambda \). Since \( \rho_\mu, \rho_\lambda \) both are irreducible, it follows ([56, 18], say; note that the sum \( N_{a_\mu} + N_{b_\mu} \) cannot be direct) that \( \rho_\mu \) is related to \( \rho_\lambda \) (that is, \( N_{\rho_\mu} \simeq N_{\rho_\lambda} \)). This cannot be the case for more than one value of \( \mu \), a contradiction, as required.

**COROLLARY 3.11.** Suppose that \( T = T^{\pi_0} \) is totally transcendental and suppose

\[
(*)_T \quad \text{whenever } N, B_\lambda (\lambda \in \Lambda) \text{ are indecomposable direct summands of models of } T, \text{ and } N <^+_1 \prod_\lambda N_\lambda, \text{ then } N \simeq N_\lambda, \text{ for some } \lambda \in \Lambda.
\]

Then there are only finitely many indecomposable direct summands of models of \( T \), up to isomorphism.

**Proof.** By 3.6, \((*)_T\) is equivalent to the condition that every irreducible type in \( S^*_T(0) \) be isolated. So the result follows by 3.10.

**COROLLARY 3.12.** Suppose that \( R \) is right pure semisimple, and suppose

\[
(*) \quad \text{whenever } N, N_\lambda (\lambda \in \Lambda) \text{ are indecomposable modules, and } N <^+_1 \prod_\lambda N_\lambda, \text{ then } N \simeq N_\lambda, \text{ for some } \lambda \in \Lambda.
\]

Then \( R \) is of finite representation type.

**Proof.** This is the global case of 3.11: take \( T = T^* \).

I finish with some results related to those above, but mainly concerned with the projective, rather than all, modules.

Consider the ring \( R \) as a (right) module over itself. It is clear that any finitely generated left ideal of \( R \) is a (right) pp-definable subgroup: if \( L = \sum_{i=1}^n R a_i \), then \( L = R^\circ \), where \( \phi(v) \) is the formula \( \exists v_1, \ldots, v_n (v = \sum_{i=1}^n v_i a_i) \). The converse (that is, every right pp-definable subgroup of \( R \) is a finitely generated left ideal) holds if and only if \( R \) is left coherent [62, Proposition 9]. Therefore, as a corollary to 1.2 we have the following (compare [65, 32, 62])

**PROPOSITION 3.13.** (a) If \( R_R \) is totally transcendental, then \( R \) is right perfect (i.e., has d.c.c. on finitely generated left ideals, see [22, 22.29]).

(b) If \( R \) is left coherent, then \( R_R \) is totally transcendental if and only if \( R \) is right perfect.

If \( R \) is not left coherent, then \( R \) being right perfect need not entail \( R_R \) being totally transcendental: Zimmerman [77, Section 2] provides an example of a ring \( R \) which is right artinian (so is semiprimary—in particular
is right and left perfect), but which is not right t.t. Note that, since the example is therefore right coherent and left perfect, it is left t.t. Note that, by [64c, p. 867, Corollary 1], any left coherent, right perfect (so right t.t.) ring is automatically left perfect.

Note the following example. Let $R$ be the upper triangular matrix ring

$$
\begin{pmatrix}
Q & R \\
0 & R
\end{pmatrix}.
$$

Then $R$ is right artinian, right and left hereditary, left and right coherent, right and left t.t., but not left artinian.

Projective and flat modules over left coherent, and over left coherent, right perfect rings, enjoy a number of model-theoretically desirable properties (see [62, 65]). Below, I note some relevant points.

Since any projective module is a direct summand of a free module, the following is immediate from 3.13 and Section 1.

**Proposition 3.14.** If $R_R$ is totally transcendental, then every projective module is totally transcendental: in fact any pure submodule of any power of $R$ is totally transcendental (if $R$ were also left coherent, then such modules would be projective).

From [53], or directly (compare [14, Sections 1–4]), it may be seen that chains of types, in the theory of $R_{R^p}$, of increasing Morley rank, correspond to increasing chains of finitely generated left ideals of $R$, at least if $R$ is left coherent.

**Proposition 3.15.** If $R_R$ is totally transcendental, and Th($R_{R^p}^N$) has finite Morley rank, then $R$ is left artinian.

**Proof.** By comments just above, $R$ has a.c.c. on finitely generated left ideals, and this with a fixed bound (given by the Morley rank of $R_{R^p}^N$). But then $R$ is left noetherian, and the fixed bound gives $R$ actually left artinian.

The following then follows easily.

**Proposition 3.16.** If $R$ is left coherent, then the following conditions are equivalent.

(i) $R_R$ is totally transcendental, and Th($R_{R^p}^N$) has finite Morley rank.

(ii) $R$ is left artinian.

It is not enough, in 3.16(i), to suppose that $R_R$ has finite Morley rank, as illustrated by the following example.

Let $R = K[x_i (i \in \omega): x_i x_j = 0, \text{ for all } i, j \in \omega]$, where $K$ is a finite field. Then $R$ is commutative and totally transcendental [79, Theorem 5], and may
be seen to have Morley rank 1. It does not, of course, have a.c.c. on finitely generated ideals: let \( I_n = \oplus_{i=0}^{n} R x_i \leq R \).

The point here is that "finite gaps" \( R^o/R^o \) do not contribute to Morley rank.

Let \( A = A_r R \) denote the left Artin radical of the ring \( R \): that is, \( A \) is the sum of all left ideals of \( R \) of finite length. It is easy to see that \( A \) is a two-sided ideal of \( R \), and that \( R \) is a left Artinian ring if and only if \( A = R \).

**Theorem 3.17.** Suppose that \( S^*_T(0) \) contains no non-isolated irreducible types, where \( T = T_r \). Then the left Artin radical \( A \) of \( R \) is left finitely generated.

*Proof.* For each finitely generated left submodule \( L \) of \( A \), let \( \psi_L(v) \) be a pp formula such that \( R^o L \leq L \).

Suppose \( A \) were not finitely generated. In particular then, \( A \neq R \), and so the set of formulas

\[
\{ -\psi_L(v) : L \leq R A, L \text{ finitely generated} \}
\]

is consistent. Therefore it extends to at least one type in \( S^*_T(0) \).

Among all such types choose one, \( p \), say, which is maximal, that is, with \( p^* \) maximal. The existence of such a type follows from Zorn's Lemma, using the compactness theorem (that if a set of formulas is inconsistent, then some finite subset is inconsistent). Note that this just extends the technique of choosing a right ideal maximal with respect to not containing certain elements of the ring.

I will show that \( p \) is irreducible and non-isolated.

Suppose that \( \bar{\psi}_1, \bar{\psi}_2 \) are pp formulas not in \( p \). Then there are \( \phi_1, \phi_2 \in p^+ \) such that, for \( i = 1, 2 \), \( \{ \phi_i \land \bar{\psi}_i \} \cup \{ -\psi_L(v) \}_L \) is inconsistent. That is, for suitable finitely generated \( L_1, \ldots, L_n \) contained in \( A \), we have \( \phi_i \land \bar{\psi}_i \rightarrow \lor_{j=1}^{n} \psi_j \), where \( \psi_j \) is \( \psi_{L_j} \). That is, \( R^o \land \phi_i \leq \lor_{j=1}^{n} R^o \land \psi_j \leq L_j \leq \sum_{j=1}^{n} L_j = L_0 \) (say) \( < A \) (by hypothesis \( A \) is not finitely generated).

Let \( \phi = \phi_1 \land \phi_2 \). So \( R^o \land \phi_0 \leq R^o \land \phi \leq L_0 \) \( < A \). Therefore, if \( \phi' \) is the pp formula \( \phi \land \bar{\psi}_1 + \phi \land \bar{\psi}_2 \), then \( R^o = R^o \land \phi_0 \leq R^o \land \phi \leq L_0 = R^o \) (note that \( L_0 \) is finitely generated). Hence \( \phi' \in p \), since \( \psi_0 \notin p \); and \( \phi = \phi_1 \land \phi_2 \in p^+ \).

So by Ziegler's criterion 3.5(i), \( p \) is irreducible.

Now suppose \( p \) were isolated. By the above paragraphs, and 3.5(iii), the isolating formula may be supposed to be of the form \( \phi \land \neg \psi \), with \( \phi, \psi \) pp. Replacing \( \psi \) by \( \psi \land \phi \), it may be supposed that \( R^o < R^o \).

As above, since \( \psi \notin p \), a finitely generated left ideal \( L < A \) may be found, with \( R^o \leq L \).

Since \( L \) has finite length, so has \( R^o \). Since \( R^o \) does not have finite length, \( R^o \land \phi \) cannot be a simple (left) module. Therefore pick \( L' \) finitely generated, with \( R^o < L' < R^o \). Then \( R^o \land \phi \) is split by the formula \( \phi_L \), — contradicting that \( \phi \land \neg \psi \) isolates a type.
COROLLARY 3.18. Suppose that $R$ is right perfect. Then there are no non-isolated irreducible types in $S^T_1(0)$ if and only if $R$ is left artinian.

Proof. ($\Rightarrow$) Since $R$ is right perfect, $R$ is semiartinian. Therefore $r(R/A)$, if nonzero, has a simple submodule: say $L > A$ is a left ideal with $r(L/A)$ simple. By 3.17, $A$ is finitely generated, hence is of finite length. Therefore $rL$ has finite length, so $L = A$, a contradiction.

Thus $R = A$, and $R$ is indeed left artinian.

($\Leftarrow$) If $R$ is left artinian, then, since it is also right perfect, $R_R$ is totally transcendental (by 3.13(b)).

Let $p \in S^T_1(0)$ be an irreducible type. Say $p^+$ is equivalent to the pp formula $\phi$. Let $\Psi$ be the set of pp formulas $\psi$, with $R^* < R^*$.

By 3.5(ii), clearly $\Psi$ is closed under addition. But the lattice of pp formulas is, by assumption and earlier observations, of finite length. So $\Psi$ has a maximum element $\psi$, say. Clearly $\phi \land \neg\psi$ isolates $p$.

Note that any left coherent, right pure semisimple ring, in particular, any hereditary right pure semisimple ring, satisfies the conditions of the result below. Note that 3.18, 3.19 relate to Q2 of Section 2 in more or less the same way that 3.7 relates to Q1 of that section.

THEOREM 3.19. Let $R$ be left coherent and right perfect. (So that every indecomposable projective is finitely generated—hence, is a direct summand of $R$ and so, in particular, there are only finitely many of these). Then the following are equivalent.

(i) $R$ is left artinian.

(ii) There are no non-isolated irreducible types in $S^T_1(0)$.

(iii) $N_A = R$: that is, if $R = I_R \oplus J_R$, with $A \leq I$, then $J = 0$.

Proof: (i) $\Rightarrow$ (ii) This is 3.18.

(ii) $\Rightarrow$ (iii) This is trivial, since (i) means that $A = R$.

(iii) $\Rightarrow$ (ii) Suppose $p$ were a non-isolated irreducible type in $S^T_1(0)$. Note that, by the hypothesis, $R_R$ is totally transcendental (3.13(b)). So we may set $R = I_R \oplus J_R$, where $I$ is the prime model of $T_R$. But $p$ cannot be realised in $I$, by 3.3, since $p$ is non-isolated. Yet the hypothesis implies that $N_p$ must be a direct summand of $R$ (since $N_p$ is projective—see [65, Theorem 5] or use 3.4 and [16, 3.3]). Therefore $J \neq 0$.

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