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ANALYTIC TORSION AND THE ARITHMETIC TODD GENUS

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with an Appendix by D. ZAGIER

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INTRODUCTION

THE AIM of this article is to state a *conjectural* Grothendieck–Riemann–Roch theorem for metrized bundles on arithmetic varieties, which would extend the known results of Arakelov [3], Faltings [19] and Deligne [17] in the case of arithmetic surfaces. The project of looking for such a theorem was first advocated by Manin in [29].

Let X be an arithmetic variety (i.e. a regular scheme, quasi-projective and flat over $\mathbb{Z}$). In a previous paper [20] we defined *arithmetic Chow groups* $\widehat{CH}^p(X)$ for every integer $p \geq 0$, generated by pairs of cycles and "Green currents" (*loc. cit.*). We showed that these groups have basically the same formal properties as the classical Chow groups. They are covariant for proper maps (with a degree shift). In [21] we attached to any algebraic vector bundle E on X, endowed with an hermitian metric h on the associated holomorphic vector bundle, characteristic classes

$$\hat{\phi}(E, h) \in \bigoplus_{p \geq 0} \widehat{CH}^p(X) \otimes \mathbb{Q} = \widehat{CH}(X)_{\mathbb{Q}},$$

for every symmetric power series $\phi(T_1, \ldots, T_{rk(E)})$ with coefficients in $\mathbb{Q}$. For instance we have Chern character $\widehat{ch}(E, h) \in \widehat{CH}(X)_{\mathbb{Q}}$. We also introduced in [21] a group $\hat{K}_0(X)$ of *virtual hermitian vector bundles* on X and extended $\widehat{ch}$ to $\hat{K}_0(X)$.

To state a Grothendieck–Riemann–Roch theorem one still needs two notions. First, given a smooth projective morphism $f\colon X \to Y$ between arithmetic varieties, one needs a *direct image morphism*

$$f_!\colon \hat{K}_0(X) \to \hat{K}_0(Y).$$

Given (E, h) on X, to get the determinant of $f_!(E, h)$ amounts to defining a metric on the determinant of the cohomology of E (on the fibers of f). This question was solved by Quillen [35] using the Ray–Singer analytic torsion [36]. In §3 below we shall define higher analogs of Ray–Singer analytic torsion and get a reasonable definition of $f_!$ (this is a variant of ideas from our work with Bismut [8, 9, 10]).

The second question we have to ask is what will play the role of the Todd genus. For this we proceed in a way familiar to algebraic geometry (see for instance [25]), namely we compute both sides of the putative Riemann–Roch formula for the trivial line bundle on the projective spaces $\mathbb{P}^n$ over $\mathbb{Z}$, $n \geq 1$. This normalizes the *arithmetic Todd genus* uniquely. To

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the obvious candidate $\widehat{Td}(E, h)$, with

$$Td(x) = \frac{x}{1-e^{-x}} = 1 - \sum_{n \geq 1} \zeta(1-n) \frac{x^n}{(n-1)!},$$

where $\zeta(s)$ is the Riemann zeta function, it turns out that a secondary characteristic class has to be added. It is constructed using the following characteristic power series (see 1.2.3)

$$R(x) = \sum_{\substack{m \text{ odd} \\ m \geq 1}} (2\zeta'(-m) + \zeta(-m)(1 + \tfrac{1}{2} + \ldots + \tfrac{1}{m}))\frac{x^m}{m!}.$$

The computations which yield this power series are pretty involved. We got the first coefficients of $R(x)$ using a computer. To check the general expression we reduced the problem to a difficult combinatorial identity, that D. Zagier was able to prove in general (Appendix). The conclusion is that the Grothendieck–Riemann–Roch theorem we conjecture is true for the trivial line bundle on $\mathbb{P}^n$ (Theorem 2.1.1.).

The paper is organized as follows. In §1 we define Quillen's metric on the determinant of cohomology, recall the definitions from [20] and [21], and define the arithmetic Todd genus. We then give a conjecture computing the Quillen metric (1.3). The holomorphic variation of this equality is known to be true [35] [8, 9, 10] (sec. 1.4). When specialized to the moduli space of curves of a given genus, the conjecture 1.3 gives the value of some unknown constants in string theory (1.5). In fact, our computation on $\mathbb{P}^n$ extends the work of the string theorist Weisberger [38] on this question when $n = 1$.

In §2 we prove conjecture 1.3 for the trivial line bundle on $\mathbb{P}^n$ (Theorem 2.1.1) by reduction to an identity of Zagier. In §3 we define higher analytic torsion using results of [9], compute its holomorphic variation (3.1) and define the map $f_!$ (3.2). We then conjecture a general arithmetic Grothendieck–Riemann–Roch identity (3.3) the holomorphic variation of which holds.

§1. ON THE DETERMINANT OF COHOMOLOGY

1.1. Quillen's metric

Let X be a compact complex manifold of complex dimension n, g a Kähler metric on X, E an holomorphic vector bundle on X, and h a smooth hermitian metric on E. We orient X using the convention that $\mathbb{C}^n$ is oriented by $dx_1 dy_1 dx_2 dy_2 \ldots dx_n dy_n$, with $z_\alpha = x_\alpha + iy_\alpha$, $\alpha = 1, \ldots, n$, the complex coordinates. Define the normalized Kähler form ω on X to be

$$\omega = \frac{i}{2\pi} \sum_{\alpha, \beta} g\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}\right) dz_\alpha d\bar{z}_\beta,$$

for any choice of local coordinates z_α, $\alpha = 1, \ldots, n$. Let $\mu = \omega^n/n!$.

Consider the Dolbeault complex

$$\ldots \to A^{0q}(X, E) \xrightarrow{\bar\partial} A^{0,q+1}(X, E) \to \ldots,$$

where $A^{pq}(X, E)$ is the vector space of smooth forms of type (p, q) with coefficients in E, and $\bar\partial$ is the Cauchy–Riemann operator. For each $q \geq 0$ we define the hermitian scalar product on $A^{0q}(X, E)$ by the formula

$$\langle \eta, \eta' \rangle_{L^2} \int_X \langle \eta(x), \eta'(x) \rangle \mu, \tag{1}$$

where $\langle \eta(x), \eta'(x) \rangle$ is the pointwise scalar product coming from the metric on E and the metric on differential forms induced by the metric on X.

The operator $\bar{\partial}$ admits an adjoint $\bar{\partial}^*$ for this scalar product:

$$\langle \bar{\partial}\eta, \eta' \rangle_{L^2} = \langle \eta, \bar{\partial}^*\eta' \rangle_{L^2}, \quad \eta \in A^{0q}(X, E), \ \eta' \in A^{0,q+1}(X, E).$$

Let $\Delta_q = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ be the Laplace operator on $A^{0q}(X, E)$ and $\mathscr{H}^{0q}(X, E) = \operatorname{Ker} \Delta_q$ the set of harmonic forms. Let $\lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots$ be the eigenvalues of Δ_q on the orthogonal complement to $\mathscr{H}^{0q}(X, E)$, indexed in increasing order and taking into account multiplicities (these are finite and $\lambda_n > 0$ for all $n \ge 1$). For every complex number s such that $Re(s) > \dim_{\mathbb{C}} X$, the series

$$\zeta_q(s) = \sum_{n \ge 1} \lambda_n^{-s} \quad (= \zeta_q(X, E, s))$$

converges absolutely. This function of s admits a meromorphic continuation to the whole complex plane, the zeta function of the operator Δ_q. This function is holomorphic at $s = 0$, so it makes sense to consider its derivative $\zeta_q'(0)$. Following Ray and Singer [36] one considers the analytic torsion

$$\tau(E)^0 = \sum_{q \ge 0} (-1)^q q \zeta_q'(0).$$

Remark. Notice that $\tau(E)^0$ depends on the metrics chosen on E and X. The number $\exp(-\zeta_q'(0))$ may be taken as a definition for $\det' \Delta_q$, the determinant of Δ_q restricted to the orthogonal complement of $\mathscr{H}^{0q}(X, E)$, since, for every finite sequence $0 < \mu_1 \le \mu_2 \le \ldots \le \mu_M$ of positive real numbers, the following holds:

$$\mu_1 \mu_2 \ldots \mu_M = \exp\left(-\frac{d}{ds}\left(\sum_{n=1}^{M} \mu_n^{-s} \right)\bigg|_{s=0} \right).$$

1.1.2. Consider the cohomology groups $H^q(X, E)$ of X with coefficients in E, and the one-dimensional complex vector space

$$\lambda(E) = \bigoplus_{q \ge 0} \Lambda^{\max} H^q(X, E)^{(-1)^q}$$

(when L is a line bundle we denote by L^{-1} its dual). Since $H^q(X, E)$ is canonically isomorphic to the cohomology of the Dolbeault complex, hence to $\mathscr{H}^{0q}(X, E)$, the scalar product $\langle \ , \ \rangle_{L^2}$ gives rise to a metric h_{L^2} on $\lambda(E)$. Quillen [35] defined a new metric h_Q on $\lambda(E)$ by the formula

$$h_Q = h_{L^2} \exp\left(\sum_{q \ge 0} (-1)^{q+1} q \zeta_q'(0) \right) = h_{L^2} \exp(-\tau(E)^0).$$

1.1.3. Now let $f: X \to Y$ be a smooth proper map of complex analytic manifolds. Assume that every point $y \in Y$ has an open neighborhood U such that $f^{-1}(U)$ can be endowed with a Kähler structure.

On the relative tangent space $T_{X/Y}$ (a bundle on X) choose an hermitian metric $h_{X/Y}$ whose restriction to each fiber $X_y = f^{-1}(y)$, $y \in Y$, gives a Kähler metric. Let E be a holomorphic vector bundle on X and h an hermitian metric on E.

Let $\lambda(E) = \det Rf_*(E)$ be the determinant of the direct image of E, as defined in [28] and [10]. This is a holomorphic line bundle $\lambda(E)$ on Y such that, for every $y \in Y$,

$$\lambda(E)_y = \bigotimes_{q \ge 0} \Lambda^{\max} H^q(X_y, E)^{(-1)^q}$$

It is shown in [8, 9, 10], Theorem 0.1, that the Quillen metric h_Q on $\lambda(E)$ (defined fiberwise as in 1.1.2) is smooth. Furthermore its curvature was computed in *loc. cit.*

1.2. Characteristic classes

1.2.1. Let (A, Σ, F_∞) be an arithmetic ring in the sense of [20], i.e. A is an excellent noetherian integral domain, Σ is a non-empty finite set of imbeddings $\sigma: A \to \mathbb{C}$, and F_∞: $\mathbb{C}^\Sigma \to \mathbb{C}^\Sigma$ is a conjugate linear involution fixing A (imbedded diagonally into $\mathbb{C}^\Sigma$). Let F be the fraction field of A.

Let X be an arithmetic variety (*loc. cit*) i.e. a regular quasi-projective flat scheme over A. Assume the generic fiber X_F is projective. Let X_σ be the set of complex points of X defined using the imbedding $\sigma \in \Sigma$ and $X_\infty = \coprod_{\sigma\in\Sigma} X_\sigma$. In [20] we defined arithmetic Chow groups $\widehat{CH}^p(X)$ for every integer $p \geq 0$, which generalize those introduced by Arakelov [2] for arithmetic surfaces. The group $\widehat{CH}^p(X)$ is generated by pairs (Z, g), where Z is a cycle of codimension p on X and g is a "Green current" for the corresponding cycle on X_∞ (i.e. $dd^c g$ plus the current given by integration on Z_∞ is a smooth form; see *loc. cit.* for the relations). There is a canonical morphism z: $\widehat{CH}^p(X) \to CH^p(X)$ to the usual Chow group of codimension p sending (Z, g) to Z. On the other hand, let $A^{pp}(X_{\mathbb{R}})$ be the set of real forms ω of type (p, p) on X_∞ such that $F_\infty^*(\omega) = (-1)^p\omega$. Denote by $\tilde{A}^{pp}(X_{\mathbb{R}})$ the quotient of $A^{pp}(X_{\mathbb{R}})$ by $\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial$, $Z^{pp}(X_{\mathbb{R}})$ the kernel of $\partial\bar\partial$ in $A^{pp}(X_{\mathbb{R}})$ and $H^{pp}(X_{\mathbb{R}})$ the quotient of $Z^{pp}(X_{\mathbb{R}})$ by $\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial$. According to [20] there is a morphism ω: $\widehat{CH}^p(X) \to Z^{pp}(X_{\mathbb{R}})$ and canonical exact sequences:

$$\tilde{A}^{p-1,p-1}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{z} CH^p(X) \to 0 \qquad (1)$$

$$H^{p-1,p-1}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{z\oplus\omega} CH^p(X) \oplus Z^{pp}(X_{\mathbb{R}}) \qquad (3)$$

Any projective map f: $X \to Y$ of arithmetic varieties which is smooth on X_F induces a direct image morphism f_*: $\widehat{CH}^p(X) \to \widehat{CH}^{p+\delta}(Y)$, where $-\delta$ is the relative dimension. Furthermore

$$\widehat{CH}(X)_{\mathbb{Q}} = \bigoplus_{p\geq 0} \widehat{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

has a graded ring structure, contravariant for all morphisms of arithmetic varieties. The product on $\widehat{CH}(X)_{\mathbb{Q}}$ satisfies the formula

$$a(x)y = a(x\omega(y)),\ x \in \tilde{A}(X) = \bigoplus_{p\geq 1} A^{p-1,p-1}(X_{\mathbb{R}}),\ y \in \widehat{CH}(X)_{\mathbb{Q}}. \qquad (4)$$

In particular $\mathrm{Im}\, a$ is a square zero ideal in $\widehat{CH}(X)_{\mathbb{Q}}$.

1.2.2. Let E be a vector bundle of rank n on the arithmetic variety X and h an hermitian metric on the associated holomorphic vector bundle E_∞ on X_∞. We assume h is invariant under F_∞. (We say that (E, h) is an hermitian vector bundle on X.) When $E = L$ is a line bundle one can define its first Chern class $\hat{c}_1(L, h) \in \widehat{CH}^1(X)$ ([17], [21] 2.5). More generally, let $\phi \in \mathbb{Q}[[T_1, \ldots, T_n]]$ be a symmetric power series in n variables. In [21] §4 we defined a class $\hat\phi(E, h) \in \widehat{CH}(X)_{\mathbb{Q}}$, characterized by the following properties:

(i) $\hat\phi(f^*E, f^*h) = f^*\hat\phi(E, h)$

(ii) Let ϕ_i, $i \geq 0$, be defined by the identity

$$\phi(T_1 + T, \ldots, T_n + T) = \sum_{i\geq 0} \phi_i(T_1, \ldots, T_n)T^i.$$

Then

$$\hat{\phi}(E \otimes L, h \otimes h') = \sum_{i \geq 0} \hat{\phi}_i(E, h)\hat{c}_1(L, h')^i,$$

for every line bundle L.

(iii) Given two metrics h and h' on E,

$$\hat{\phi}(E, h) - \hat{\phi}(E, h') = a(\tilde{\phi}(E, h, h')),$$

where $\tilde{\Phi}(E, h, h') \in \tilde{A}(X_{\mathbb{R}})$ is a secondary characteristic class introduced by Bott and Chern ([15, 18, 8, 21]).

(iv) When $(E, h) = (L_1 \oplus \ldots \oplus L_n, h_1 \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} h_n)$ is an orthogonal direct sum of hermitian line bundles,

$$\hat{\phi}(E, h) = \phi(\hat{c}_1(L_1, h_1), \ldots, \hat{c}_1(L_n, h_n)).$$

In particular the Chern character $\widehat{ch}(E, h) \in \widehat{CH}(X)_{\mathbb{Q}}$ is defined using

$$ch(T_1, \ldots, T_n) = \sum_{i=1}^{n} \exp(T_i)$$

and the Todd class $\widehat{Td}(E, h) \in \widehat{CH}(X)_{\mathbb{Q}}$ by means of

$$Td(T_1, \ldots, T_n) = \prod_{i=1}^{n} (T_i/(1 - \exp(-T_i))).$$

1.2.3. Let E be a holomorphic bundle on a complex manifold X. Let us define a characteristic class $R(E) \in H^{ev}(X)$ in the even complex cohomology of X by the following properties:

(i) $R(f^*E) = f^*R(E)$

(ii) Given any exact sequence $0 \to S \to E \to Q \to 0$ of vector bundles over X, we have

$$R(E) = R(S) + R(Q)$$

(iii) When L is a line bundle on X with $x = c_1(L) \in H^2(X)$ its first Chern class,

$$R(L) = \sum_{\substack{m \text{ odd} \\ m \geq 1}} (2\zeta'(-m) + \zeta(-m)(1 + \tfrac{1}{2} + \ldots + \tfrac{1}{m}))\frac{x^m}{m!} \tag{5}$$

Here $\zeta(s)$ is the Riemann zeta function and $\zeta'(s)$ its derivative.

Assume now (E, h) is a hermitian vector bundle on an arithmetic variety X as in 1.2.2. Then $R(E_\infty)$ lies in $H(X) = \bigoplus_{p \geq 1} H^{p-1,p-1}(X_{\mathbb{R}})$. We define the *arithmetic Todd genus* of (E, h) to be

$$Td^A(E, h) = \widehat{Td}(E, h)(1 - a(R(E_\infty))) \text{ in } \widehat{CH}(X)_{\mathbb{Q}}. \tag{6}$$

1.3. A conjecture.

Let $f\colon X \to Y$ be a smooth projective morphism of arithmetic varieties. Choose a hermitian metric $h_{X/Y}$ on the relative tangent space $T_{X/Y}$ which induces a Kähler metric on each fiber $f^{-1}(y)$, $y \in Y_\infty$. Let (E, h) be an hermitian vector bundle on X. The determinant line bundle

$$\lambda(E) = \det Rf_* E$$

on Y [28] is endowed with the Quillen metric h_Q as in 1.1.3. Given $x \in \widehat{CH}(Y)_{\mathbb{Q}}$, denote $x^{(p)}$ its component in $\widehat{CH}^p(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Conjecture 1.3. $\hat{c}_1(\lambda(E), h_Q) = f_*(\widehat{ch}(E, h)Td^A(T_{X/Y}, h_{X/Y}))^{(1)}$

1.4. Some evidence for the conjecture

Let

$$\delta(E) = \hat{c}_1(\lambda(E), h_Q) - f_*(\widehat{ch}(E, h)Td^A(T_{X/Y}, h_{X/Y}))^{(1)}.$$

THEOREM 1.4

(i) [8, 9, 10] *The element* $\delta(E)$ *lies in* $a(H(Y)) \subset \widehat{CH}(Y)_{\mathbb{Q}}$. *It is independent of* h *and* $h_{X/Y}$. *Given any short exact sequence* $0 \to S \to E \to Q \to 0$ *on* X, *then* $\delta(E) = \delta(S) + \delta(Q)$.

(ii) *Let* E^* *be the dual of* E, d *the rank of* $T_{X/Y}$, *and* $K = \Lambda^d T^*_{X/Y}$ *the relative dualizing bundle. Then*

$$\delta(E) = (-1)^{d+1}\delta(K \otimes E^*).$$

(iii) *Let* E' *be any bundle on* Y. *Then*

$$\delta(E \otimes f^* E') = rk(E')\delta(E).$$

(iv) [17] *When* f *has relative dimension one and* Σ *contains a real imbedding, one has*

$$\delta(E) = c \,.\, rk(E)$$

where $c \in \mathbb{R}$ *depends only on the genus of the fibers of* f.

Proof

(i) By the Grothendieck–Riemann–Roch theorem for higher Chow groups [24] we get $z(\delta(E)) = 0$. On the other hand, we know from [21] 4.1. that

$$\omega(\hat{\phi}(E, h)) = \phi(E, h) \tag{7}$$

is the closed form in $A(X) = \bigoplus_{p \geq 0} A^{pp}(X_{\mathbb{R}})$ representing the ϕ-characteristic class of E_{∞} which is attached to the hermitian holomorphic connection on E_{∞}. Since ω is multiplicative and commutes with f_* [20] we get

$$\omega(\delta(E)) = c_1(\lambda(E), h_Q) - f_*(ch(E, h)Td(T_{X/Y}, h_{X/Y}))^{(2)}.$$

This is zero by [8, 9, 10], Theorem 0.1. We conclude from (7) that $\delta(E)$ lies in the image of a.

When $h_{X/Y}$ is replaced by $h'_{X/Y}$ we have

$$\hat{c}_1(\lambda(E), h_Q) - \hat{c}_1(\lambda(E), h'_Q) = a(\tilde{c}_1(\lambda(E), h_Q, h'_Q))$$

by 1.2.2 (iii). Similarly

$$Td^A(T_{X/Y}, h_{X/Y}) - Td^A(T_{X/Y}, h'_{X/Y}) = a(\widetilde{Td}(T_{X/Y}, h_{X/Y}, h'_{X/Y})).$$

Using (6) we get

$$\delta(E) - \delta'(E) = a(\tilde{c}_1(\lambda(E), h_Q, h'_Q) - f_*(ch(E, h)\widetilde{Td}(T_{X/Y}, h_{X/Y}, h'_{X/Y}))).$$

Theorem 0.3 in [8, 9, 10] gives $\delta(E) - \delta'(E) = 0$. By a similar argument, Theorem 0.2 in [8, 9, 10] implies that $\delta(E)$ does not depend on the metric h on E and $\delta(E) = \delta(S) + \delta(Q)$ for every exact sequence

$$0 \to S \to E \to Q \to 0.$$

(ii) For every point $y \in Y$, Serre's duality identifies $H^q(X_y, E)$ with the dual of $H^{d-q}(X_y, K \otimes E^*)$, $q \geq 0$. Hence one gets an isomorphism of line bundles on Y

$$\lambda(E) \simeq \lambda(K \otimes E^*)^{(-1)^{d+1}} \tag{8}$$

Up to sign, on X_∞, Serre's duality is induced by the pairing of Dolbeault complexes

$$A^{0q}(X_y, E) \otimes A^{0,d-q}(X_y, K \otimes E^*) \to \mathbb{C} \tag{9}$$

sending $\eta \otimes \eta'$ to

$$\left(\frac{i}{2\pi}\right)^d \int_{X_y} \eta \wedge \eta'.$$

(we forget the subscript ∞). Let us endow E with a hermitian metric h. From the definition of the (normalized) Kähler form ω and the L^2-metric (1.1), we see that the pairing (9) gives an isometry

$$A^{0q}(X_y, E) \xrightarrow{\sim} A^{0,d-q}(X_y, K \otimes E^*)$$

for the L^2 metrics. Furthermore

$$\zeta_q(X_y, E, s) = \zeta_{d-q}(X_y, K \otimes E^*, s).$$

Therefore (8) is an isometry for Quillen's metrics. Let $x \to x^\vee$ be the involution on $\widehat{CH}$ equal to $(-1)^p$ on $\widehat{CH}^p$. Then $\widehat{ch}((E,h)^*) = \widehat{ch}(E,h)^\vee$ ([21] 4.9) and, by a standard computation

$$\begin{aligned} f_*(\widehat{ch}(E,h)\widehat{Td}(T_{X/Y}, h_{X/Y}))^\vee &= (-1)^d f_*(\widehat{ch}(E,h)^\vee\, \widehat{Td}(T_{X/Y}, h_{X/Y})^\vee) \\ &= (-1)^d f_*(\widehat{ch}((E,h)^*)\,\widehat{ch}(K, \Lambda^d h_{X/Y})\widehat{Td}(T_{X/Y}, h_{X/Y})^\vee) \end{aligned}$$

Therefore

$$f_*(\widehat{ch}(E,h)\widehat{Td}(T_{X/Y}, h_{X/Y}))^\vee = (-1)^d f_*(\widehat{ch}((K \otimes E^*, \Lambda^d h_{X/Y} \otimes h^*)\widehat{Td}(T_{X/Y}, h_{X/Y})). \tag{10}$$

Furthermore

$$a(f_*(ch(E)Td(T_{X/Y})R(T_{X/Y})))^\vee = (-1)^d a f_*(ch(E)^\vee\, Td(T_{X/Y})^\vee\, R(T_{X/Y})^\vee).$$

Since $R(x) = -R(-x)$ we get $R(T_{X/Y})^\vee = R(T_{X/Y})$, hence

$$f_*(ch(E)Td(T_{X/Y})R(T_{X/Y}))^\vee = (-1)^d f_*(ch(K \otimes E^*)Td(T_{X/Y})R(T_{X/Y})). \tag{11}$$

Applying (10) and (11) in degree one we get

$$\begin{aligned} &f_*(\widehat{ch}(E,h)Td^A(T_{X/Y}, h_{X/Y}))^{(1)} \\ &\qquad = (-1)^{d+1} f_*(\widehat{ch}(K \otimes E^*, \Lambda^d h_{X/Y} \otimes h^*)\widehat{Td}(T_{X/Y}, h_{X/Y}))^{(1)} \end{aligned}$$

and (i) follows.

(iii) From the algebraic isomorphisms

$$H^q(X_y, E \otimes f^*E') \simeq H^q(X_y, E) \otimes E'_y, \quad y \in Y, \ q \geq 0$$

we get

$$\lambda(E \otimes f^*E') \simeq \lambda(E)^{rk(E')}(\det E')^{\chi(E)}. \tag{12}$$

Let us endow E and E' with hermitian metrics h and h'. On X_∞ the isomorphism (12) is induced by L^2 isometries

$$A^{0q}(X_y, E \otimes f^*E') \simeq A^{0q}(X_y, E) \otimes E'_y$$

from which we conclude that (12) is an isometry for Quillen's metric.

On the other hand

$$f_*(\widehat{ch}((E,h)\otimes f^*(E',h'))Td^A(T_{X/Y},h_{X/Y}))^{(1)}$$
$$=[f_*(\widehat{ch}(E,h)Td^A(T_{X/Y},h_{X/Y}))\widehat{ch}(E',h')]^{(1)}$$
$$=\chi(E)\hat{c}_1(E',h')+rk(E')f_*(\widehat{ch}(E,h)Td^A(T_{X/Y},h_{X/Y}))^{(1)}.$$

This proves (iii).

The statement (iv) is Deligne's result [17], since, by [21] Theorem 4.10.1., the right hand side of Conjecture 1.3 is the class of the corresponding metrized line bundle introduced in [17]. q.e.d.

1.5. Consequences of the Conjecture

1.5.1. Under the hypotheses of 1.3 assume that the rank of $T_{X/Y}$ is one, i.e. f is a family of curves. Then Conjecture 1.3 is equivalent to the following:

Conjecture 1.5

$$\hat{c}_1(\lambda(E),h_Q)=f_*(ch(E,h)\widehat{Td}(T_{X/Y},h_{X/Y}))^{(1)}-a(rk(E)(1-g)(4\zeta'(-1)-\tfrac{1}{6})),$$

where $g(y)$ is the genus of $f^{-1}(y)$ for every $y\in Y_\infty$.

To see that the conjectures are equivalent notice that

$$f_*(\widehat{ch}(E,h)\widehat{Td}(T_{X/Y},h_{X/Y})a(R(T_{X/Y})))^{(1)}=a(f_*(ch(E)Td(T_{X/Y})R(T_{X/Y})))^{(1)}$$

by (4). Since $R(T_{X/Y})$ has degree at least 2 we get

$$a(f_*(rk(E)r_1c_1(T_{X/Y})))^{(1)}$$

where

$$r_1=2\zeta'(-1)+\zeta(-1)=2\zeta'(-1)-\tfrac{1}{12}$$

By the classical Riemann-Roch theorem in cohomology:

$$1-g=f_*(Td(T_{X/Y}))^{(0)}=\tfrac{1}{2}f_*(c_1(T_{X/Y})).$$

Hence Conjecture 1.5 is equivalent to Conjecture 1.3 when $rk(T_{X/Y})=1$.

1.5.2. We keep the hypotheses of 1.5.1 and let ω be the dual of $T_{X/Y}$, with the dual metric.

PROPOSITION 1.5.2. *Assume Conjecture 1.5 holds. Then, for every $j\geq 1$, there is an isomorphism,*

$$M\colon \lambda(\omega^j)\to\lambda(\omega)^{6j^2-6j+1}$$

such that

$$h_Q(M(s),M(s'))=h_Q(s,s')\exp((1-g)(j^2-j)(24\zeta'(-1)-1)). \tag{13}$$

Proof. The algebraic isomorphism $\lambda(\omega^j)\simeq\lambda(\omega)^{6j^2-6j+1}$ is due to Mumford [32]. By a standard computation ([32, 12]) we get

$$f_*(\widehat{ch}(\omega^j)\widehat{Td}(T_{X/Y},h_{X/Y}))^{(1)}=(6j^2-6j+1)f_*(\widehat{ch}(\omega)\widehat{Td}(T_{X/Y},h_{X/Y}))^{(1)}.$$

Therefore, by applying the Conjecture 1.5 to ω and ω^j,

$$\hat{c}_1(\lambda(\omega^j),h_Q)=(6j^2-6j+1)\hat{c}_1(\lambda(\omega),h_Q)+(6j-6j^2)(1-g)(4\zeta'(-1)-\tfrac{1}{6}).$$

Let $\widehat{\mathrm{Pic}}(Y)$ be the group of hermitian line bundles on Y, modulo the algebraic isomorphisms which preserve the metrics. From [17] and [21] 2.5, we know that

$$\hat{c}_1 : \widehat{\mathrm{Pic}}(Y) \to \widehat{CH}^1(Y)$$

is an isomorphism. Hence the Proposition follows.

1.5.3. The Mumford isomorphism $M: \lambda(\omega^j) \simeq \lambda(\omega)^{6j^2-6j+1}$ is fixed up to sign when the base is $A = \mathbb{Z}$. In particular there is a unique metric on $\lambda(\omega^j)$ such that M is an isometry when $\lambda(\omega)$ has its L^2 metric. As shown in [5], when $j=2$, this metric on $\lambda(\omega^2)$ gives rise to the Polyakov measure on the moduli space of curves of genus g (cf. also [12]). If Conjecture 1.5 would hold, it would then normalize the constant which appears in several expressions for the Polyakov measure ([5, 30, 14, 4, 31, 11, 1]). The meaning of such a normalization over $\mathbb{Z}$ for string theory is *a priori* unclear. However Weisberger in [38] argues that "unitarity" can be used to normalize the expression of the Polyakov measure. His method is based on the computation of the determinant of the Laplace operator on $\mathbb{P}^1$ (as in paragraph 2 below) and the constants he gets are similar to those in (13) (Proposition 1.5.2).

§2. PROJECTIVE SPACES

2.1. Statement of the results

2.1.1. THEOREM 2.1.1. *For every $n \geq 0$ let $f: \mathbb{P}^n \to \mathrm{Spec}(\mathbb{Z})$ be the projective space of dimension n over $\mathbb{Z}$. Then the conjecture 1.3 holds when E is the trivial line bundle $\mathcal{O}_{\mathbb{P}^n}$ on $\mathbb{P}^n$.*

2.1.2. *Remarks.* As shown in Theorem 1.4, it is enough to prove 2.1.1 with one choice of metrics. On $\mathbb{P}^n(\mathbb{C})$ we shall take the Fubini Study metric $h_{\mathbb{P}^n}$; and on $\mathcal{O}_{\mathbb{P}^n}$ we take the trivial metric.

Let $R'(x) = r_0 + r_1 x + r_2 x^2 + \ldots \in \mathbb{R}[[x]]$ be an arbitrary power series with real coefficients. Define a characteristic class $R'(E) \in H(X)$ as in 1.2.3, with $R(L) = R'(c_1(L))$ instead of (5). Let

$$Td^{A'}(E, h) = \widehat{Td}(E, h)(1 - a(R'(E_{\mathbb{C}}))).$$

In $\widehat{CH}^1(\mathrm{Spec}\ \mathbb{Z}) = \mathbb{R}$ consider the equation

$$\hat{c}_1(\lambda(\mathcal{O}_{\mathbb{P}^n}), h_Q) = f_*(Td^{A'}(T_{\mathbb{P}^n}, h_{\mathbb{P}^n}))^{(1)}. \tag{14}$$

For every $n \geq 0$ this is a linear equation in the variables $r_0, r_1, \ldots, r_n$ and the coefficient of r_n is not zero. Therefore there is a unique sequence $r_0, r_1, r_2, \ldots$ such that (14) holds for all $n \geq 0$. Theorem 2.1.1 computes these numbers, proving that $R' = R$ must be given by formula (14). This is quite similar to the way the Todd genus is defined in [25] for instance: the Todd genus is the unique multiplicative characteristic class such that the Riemann-Roch theorem holds for the trivial line bundle on $\mathbb{P}^n$ (cf. Lemma 1.7.1 in *loc. cit.*).

2.1.3. COROLLARY. *The conjecture 1.3 holds when f is the projection $\mathbb{P}^1_{\mathbb{Z}} \to \mathrm{Spec}\ \mathbb{Z}$.*

Proof. Let $\mathcal{O}(1)$ be the standard line bundle on $\mathbb{P}^1$. From Theorem 1.4(i) we only need to check $\delta(E) = 0$ for any E in $K_0(\mathbb{P}^1) = \mathbb{Z}^2$. Since $\delta(\mathcal{O}_{\mathbb{P}^1}) = 0$ (Theorem 2.1.1) we are left with showing $\delta(\mathcal{O}(1)) = 0$. From Theorem 1.4(ii) and Theorem 2.1.1 we get $\delta(K) = 0$.

From the relation

$$[\mathcal{O}_{\mathbb{P}^1}]-[K]+2[\mathcal{O}(1)]=0$$

in $K_0(\mathbb{P}^1)$ (see (15) below) the Corollary follows. q.e.d.

2.2. The right hand side

In this paragraph we shall compute

$$f_*(Td^A(T_{\mathbb{P}^n}, h_{\mathbb{P}^n}))^{(1)} \quad \text{in } \widehat{CH}^1(\operatorname{Spec} \mathbb{Z})=\mathbb{R}$$

(the identification being given by the map a of 1.2.1.).

2.2.1. First we compute

$$R_n=f_*(\widehat{Td}(T_{\mathbb{P}^n}, h_{\mathbb{P}^n})a(R(T_{\mathbb{P}^n})))=\int_{\mathbb{P}^n(\mathbb{C})} Td(T_{\mathbb{P}^n})R(T_{\mathbb{P}^n})$$

by (4) and (7). Consider the canonical exact sequence on $\mathbb{P}^n$:

$$\mathcal{E}_n: 0\to\mathbb{C}\to\mathcal{O}(1)^{n+1}\to T_{\mathbb{P}^n}\to 0. \tag{15}$$

Let $x=c_1(\mathcal{O}(1))\in H^2(\mathbb{P}^n)$. We get, from (15), $R(T_{\mathbb{P}^n})=(n+1)R(x)$. On the other hand

$$Td(T_{\mathbb{P}^n})=Td(x)^{n+1}, \quad \text{where } Td(x)=x/(1-e^{-x}),$$

and

$$\int_{\mathbb{P}^n(\mathbb{C})} x^k=\begin{cases} 1 & \text{if } k=n \\ 0 & \text{otherwise}\end{cases}.$$

So we have

LEMMA 2.2.1. $R_n=$ *coefficient of* x^n *in* $(n+1)\left(\frac{x}{1-e^{-x}}\right)^{n+1} R(x)$.

2.2.2. Let us equip all bundles in $\mathcal{E}_n$ with the standard metric invariant under $SU(n+1)$. From [21] Theorem 4.8. and (15) we get

$$\widehat{Td}(T_{\mathbb{P}^n}, h_{\mathbb{P}^n})\widehat{Td}(\mathbb{C},|\cdot|)=\widehat{Td}(\mathcal{O}(1)^{n+1})+a(\widetilde{Td}(\mathcal{E}_n)). \tag{16}$$

Here $\widetilde{Td}(\mathcal{E}_n)$ is the secondary characteristic class considered in 1.2.2 (iii). Let

$$\hat{x}=\hat{c}_1(\mathcal{O}(1)).$$

We get from (16), since $\widehat{Td}(\mathbb{C},|.|)=1$,

$$\widehat{Td}(T_{\mathbb{P}^n}, h_{\mathbb{P}^n})=Td(\hat{x})^{n+1}+a(\widetilde{Td}(\mathcal{E}_n)).$$

In [21] 5.4.6. we computed

$$f_*(\hat{x}^k)^{(1)}=\begin{cases} \displaystyle\sum_{p=1}^{n}\sum_{j=1}^{p}\frac{1}{j} & \text{if } k=n+1 \\ 0 & \text{otherwise.}\end{cases}$$

So we have proved

LEMMA 2.2.2. *Let*

$$t_n = f_*(\widehat{Td}(\mathcal{O}(1))^{n+1})^{(1)}.$$

Then

$$t_n = \left(\sum_{p=1}^{n} \sum_{j=1}^{p} \frac{1}{j}\right) \cdot \left(\text{coefficient of } x^{n+1} \text{ in } \left(\frac{x}{1-e^{-x}}\right)^{n+1}\right).$$

2.2.3. We still need to compute

$$\tilde{Td}_n = f_* a(\tilde{Td}(\mathcal{E}_n))^{(1)}.$$

PROPOSITION 2.2.3. $\tilde{Td}_n$ = *coefficient of* x^n *in* $\displaystyle\int_0^1 \frac{\phi(t)-\phi(0)}{t}\,dt$,

where

$$\phi(t) = \left[\frac{1}{tx} - \frac{e^{-tx}}{1-e^{-tx}}\right]\left(\frac{x}{1-e^{-x}}\right)^{n+1}$$

Proof. To compute $\tilde{Td}(\mathcal{E}_n)$ we apply the method of [15], §4. Let

$$\mathcal{E}_n:\ 0 \to S \to E \to Q \to 0$$

be the exact sequence (15). The metrics on $S = \mathbb{C}$ and $Q = T_{\mathbb{P}^n}$ are induced by the metric on $E = \mathcal{O}(1)^{n+1}$, as in *loc. cit.* Let us write E as the orthogonal direct sum of S and $S^\perp \simeq Q$. The curvature of E $\left(\text{multiplied by } \frac{i}{2\pi}\right)$ decomposes as a 2 by 2 matrix $K = (K_{ij})$. Let K_S (resp. K_Q) the curvature of S (resp. Q) multiplied by $\frac{i}{2\pi}$. Let $Td(A) = \det(A/(1-e^{-A}))$ for any square matrix A. For every $t \in [0, 1]$ consider

$$\phi(t) = \text{coefficient of } \lambda \text{ in } Td\left[\begin{array}{c|c} tK_{11}+(1-t)K_S+\lambda & tK_{12} \\ \hline K_{21} & tK_{22}+(1-t)K_Q \end{array}\right].$$

and

$$I = \int_0^1 \frac{\phi(t)-\phi(0)}{t}\,dt.$$

As in [15] *loc. cit.* one checks that

$$\frac{i}{2\pi}\,\bar{\partial}\partial(I) = Td(K) - Td(K_S \oplus K_Q).$$

Moreover the characteristic properties of $\tilde{Td}$ given in [8] are easily seen to be satisfied by the class of I in $\tilde{A}(\mathbb{P}^n)$. Therefore $\tilde{Td}(\mathcal{E}_n) \equiv I$, modulo $Im\,\partial + Im\,\bar{\partial}$.

In our case K is equal (in any frame) to the product of the first Chern form ω of $\mathcal{O}(1)$ by the identity matrix. Furthermore S has rank one and $K_S = 0$. Therefore we get

$$\phi(t) = \text{coefficient of } \lambda \text{ in } Td\left[\begin{array}{c|c} t\omega+\lambda & 0 \\ \hline 0 & t\omega+(1-t)K_Q \end{array}\right].$$

Since $Td(A \oplus B) = Td(A)Td(B)$ we get

$$\phi(t) = \frac{d}{d\lambda}\left[\frac{t\omega+\lambda}{1-e^{-t\omega-\lambda}}\right]_{|\lambda=0} Td(t\omega+(1-t)K_Q). \tag{17}$$

We define a characteristic class $Td_{u,v}(E)$ with coefficients in the ring $\mathbb{Q}[[u,v]]$ of power series in two variables by the formula

$$Td_{u,v}(E) = \det \frac{uK+v}{1-e^{-uK-v}}.$$

Then $Td_{u,v}$ is multiplicative on exact sequences. From $\mathscr{E}_n$ we get

$$Td_{u,v}(K_Q)\frac{v}{1-e^{-v}} = \left[\frac{u\omega+v}{1-e^{-u\omega-v}}\right]^{n+1}.$$

Specializing to $u=1-t$ and $v=t\omega$ we get

$$Td(t\omega+(1-t)K_Q) = \frac{1-e^{-t\omega}}{t\omega}\left[\frac{\omega}{1-e^{-\omega}}\right]^{n+1}. \tag{18}$$

From (17) and (18) we get

$$\phi(t) = \left[\frac{1}{t\omega} - \frac{e^{-t\omega}}{1-e^{-t\omega}}\right]\left[\frac{\omega}{1-e^{-\omega}}\right]^{n+1}$$

Since

$$\int_{\mathbb{P}^n(\mathbb{C})} \omega^k = \begin{cases} 1 & \text{when } k=n \\ 0 & \text{otherwise,} \end{cases}$$

the Proposition 2.2.3 follows.

2.3. *The left hand side.*

2.3.1. Let ω be the (1, 1) form of the Fubini Study metric on $\mathbb{P}^n(\mathbb{C})$. By definition, ω is the first Chern form of $\mathcal{O}(1)$ (with its standard metric), with cohomology class $x = c_1(\mathcal{O}(1))$. The associated density is $\mu = \omega^n/n!$, hence

$$\int_{\mathbb{P}^n(\mathbb{C})} \mu = 1/n!. \tag{19}$$

Since

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \begin{cases} \mathbb{Z} & \text{if } q=0 \\ 0 & \text{otherwise,} \end{cases}$$

the line bundle $\lambda(\mathcal{O}_{\mathbb{P}^n})$ is trivial, with section $1 \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ of L^2-norm

$$h_{L^2}(1,1) = \int_{\mathbb{P}^n(\mathbb{C})} \mu = 1/n!.$$

Therefore

$$-\hat{c}_1(\lambda(\mathcal{O}_{\mathbb{P}^n}), h_Q) = \log h_Q(1,1) = -\log(n!) + \sum_{q\geq 0} (-1)^{q+1} q\zeta_q'(0), \tag{20}$$

where $\zeta_q(s)$ is the zeta function of the Laplace operator $\Delta_q = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ acting upon $A^{0q}(\mathbb{P}^n)$, i.e. forms of type $(0, q)$ on $\mathbb{P}^n(\mathbb{C})$.

2.3.2. The spectrum of Δ_q was computed by Ikeda and Taniguchi in [27]. Let $\Lambda_i = x_1 + \ldots + x_i$, $1 \leq i \leq n$, be the standard fundamental weights of the group $SU(n+1)$

and $\Lambda_0=0$ (hence $x_1, \ldots, x_{n+1}$ are the usual characters of the diagonal subgroup of $SU(n+1)$, and $x_1+x_2+\ldots+x_{n+1}=0$). When $k\geq q\geq 0$ denote by $\Lambda(k,0,q)$ the irreducible representation of $SU(n+1)$ of highest weight

$$(k-q)\Lambda_1+\Lambda_q+k\Lambda_n.$$

According to [27], Theorem 5.2, $A^{0q}(\mathbb{P}^n)$ contains as a dense subspace (stable under $SU(n+1)$) the following infinite direct sum:

$$\bigoplus_{k\geq 0} \Lambda(k,0,0) \qquad \text{when } q=0$$

$$\left(\bigoplus_{k\geq q} \Lambda(k,0,q)\right)\oplus\left(\bigoplus_{k\geq q+1} \Lambda(k,0,q+1)\right) \quad \text{when } 1\leq q<n,$$

and

$$\bigoplus_{k\geq n} \Lambda(k,0,n) \quad \text{when } q=n.$$

Furthermore the Laplace operator Δ_q acting on $\Lambda(k,0,q)$, $q>0$, is the multiplication by $k(k+n+1-q)$ (the subspace $\Lambda(k,0,q+1)$ of $A^{0q}(\mathbb{P}^n)$ is mapped isomorphically by $\bar\partial$ to $\Lambda(k,0,q+1)\subset A^{0,q+1}(\mathbb{P}^n)$, $q<n$). We define

$$d_{n,q}(k)=\dim_{\mathbb{C}}\Lambda(k,0,q).$$

Therefore

$$\sum_{q\geq 0}(-1)^{q+1}q\zeta_q(s)=\sum_{\substack{q\geq 1\\ k\geq q}}(-1)^{q+1}\frac{d_{n,q}(k)}{(k(k+n+1-q))^s}. \tag{21}$$

2.3.3. LEMMA 2.3.3. *When $k\geq q$ and $n\geq q$,*

$$d_{n,q}(k)=\left(\frac{1}{k}+\frac{1}{k+n+1-q}\right)\frac{(k+n)!\,(k+n-q)!}{k!\,(k-q)!\,n!\,(n-q)!\,(q-1)!}. \tag{22}$$

Proof. We apply the Hermann-Weyl formula. Let $\lambda=(k-q)\Lambda_1+\Lambda_q+k\Lambda_n$, δ the half sum of positive roots, and $(\,,)$ the invariant scalar product on the root system of $SU(n+1)$. Then

$$d_{n,q}(k)=\frac{\prod_{\alpha>0}(\lambda+\delta,\alpha)}{\prod_{\alpha>0}(\delta,\alpha)}.$$

The standard positive roots of $SU(n+1)$ are x_i-x_j, $1\leq i<j\leq n+1$. The basis Λ_i is dual for $(\,,)$ to the basis x_i-x_{i+1}, $i=1,\ldots,n$. We have [26],

$$\prod_{\alpha>0}(\delta,\alpha)=\prod_{1\leq i<j\leq n+1}(j-i)$$

and, if

$$\lambda=\sum_{i=1}^{n} m_i\Lambda_i,$$

$$\prod_{\alpha>0}(\lambda+\delta,\alpha)=\prod_{1\leq i<j\leq n+1}\left(\sum_{l=i}^{j-1}(m_l+1)\right).$$

The factor $\sum_{l=i}^{j-1}(m_l+1)$ is equal to $j-i$ unless $i=1, j=n+1$ or $1<i\leq q<j\leq n$. Hence we get

$$\begin{aligned} d_{n,q}(k) &= \left(\prod_{1<j\leq q}\frac{k-q+j-1}{j-1}\right)\cdot\left(\prod_{q<j<n+1}\frac{k-q+j}{j-1}\right) \\ &\cdot\left(\frac{2k-q+n+1}{n}\right)\cdot\left(\prod_{1<i\leq q}\frac{k+n+2-i}{n+1-i}\right) \\ &\cdot\left(\sum_{q<i\leq n}\frac{k+n+1-i}{n+1-i}\right)\cdot\left(\prod_{1<i\leq q<j\leq n}\frac{1+j-i}{j-i}\right) \\ &= \frac{2k-q+n+1}{k(k+n-q+1)}\cdot\frac{(k+n)!\,(k+n-q)!}{k!\,(k-q)!\,n!(n-q)!(q-1)!} \end{aligned}$$

q.e.d.

2.3.4. To compute $\hat{c}_1(\lambda(\mathcal{O}_{\mathbb{P}^n}), h_Q)$ we need to know $\zeta_q'(0)$. For this we use a result of Vardi [37] (see also [38] when $n=1$, and an unpublished work of Bost [13]. Let $P(X)\in\mathbb{C}[X]$ be a polynomial and $a\geq 0$ an integer. Let $P(X)=\sum_{n\geq 0}c_nX^n$. Consider the real numbers

$$\zeta P=\sum_{n\geq 0}c_n\zeta'(-n),$$

where $\zeta(s)$ is the Riemann zeta function and $\zeta'(s)$ its derivative, and

$$P^*(a)=\sum_{n\geq 1}c_n\frac{a^{n+1}}{n+1}\left(1+\frac{1}{2}+\frac{1}{3}+\ \dots\ +\frac{1}{n}\right).$$

The series

$$Z(s)=\sum_{k\geq 1}P(k)\,(k(k+a))^{-s}$$

converges absolutely when $Re(s)$ is big enough, and extends meromorphically to the whole complex plane.

Proposition 2.3.4 ([37], Prop. 3.1.)†

$$Z'(0)=\sum_{m=1}^{a}P(m-a)\log m+\zeta P+\zeta P(.-a)-\tfrac{1}{2}P^*(-a). \tag{23}$$

Remark. Consider the formal sum

$$\begin{aligned} &-\sum_{k\geq 1}P(k)\log k-\sum_{k\geq 1}P(k)\log(k+a) \\ &\qquad=-\sum_{k\geq 1}P(k)\log k-\sum_{m\geq 1}P(m-a)\log m+\sum_{m=1}^{a}P(m-a)\log m. \end{aligned}$$

If we replace in this formula $-\sum_{k\geq 1}k^n\log k$, $n\geq 0$, by $\zeta'(-n)$ we get $\zeta P+\zeta P(.-a)+\sum_{m=1}^{a}P(m-a)\log m$. The extra term $-\frac{1}{2}P^*(-a)$ in (23) is the effect of regularizing this sum by means of a zeta function.

†M. Wodzicki tells us he had proved this result in 1982.

2.4. *Proof of Theorem 2.1.1.*

When $1 \leq q \leq n$ we denote by $d_{n,q}(X)$ the polynomial such that $d_{n,q}(k)$ is given by (22) when k is an integer and $k \geq q$.

2.4.1. LEMMA 2.4.1.

(i) $d_{n,q}(k)=0$ *when* $k=q-n,\ q-n+1, \ldots,$ *or* $q-1$, *and* $k \neq 0$; $d_{n,q}(0)=(-1)^{q+1}$.

(ii) $d_{n,q}(X-n-1+q)=-d_{n,q}(-X)$

(iii) $1+\sum_{q \geq 1}(-1)^{q+1} d_{n,q}(k)=(n+1)\dfrac{(k+n)!}{n!\,k!}$

(k integer $\geq n$).

Proof.

(i) This follows from

$$d_{n,q}(k)=\left(\frac{1}{k}+\frac{1}{k+n-q+1}\right)\binom{n+k}{n}\frac{1}{(n-q)!\,(q-1)!}$$
$$\cdot(k+n-q)(k+n-q-1)\ldots(k-q+1).$$

(ii) One checks that

$$d_{n,q}(X)=\left(\frac{1}{X}+\frac{1}{X+n-q+1}\right)\phi(X)$$

with

$$\phi(X-n-1+q)=\phi(-X).$$

(iii) (The following proof, simpler than our original one, is due to D. Zagier). First notice that

$$\begin{aligned}
&d_{n,q}(k)+d_{n-1,q-1}(k)\\
&=\frac{(n-1)!}{(n-q)!\,(q-1)!}\left(\frac{1}{k}+\frac{1}{k+n-q}\right)\frac{(k+n-1)!\,(k+n-q)!}{k!\,(k-1)!\,(n-1)!\,(n-q)!}\left[\frac{k+n}{n}+\frac{q-1}{k-q+1}\right]\\
&=\binom{n-1}{q-1}\left[\binom{k+n-1}{n}\binom{k+n-q}{n-1}+\binom{k+n-1}{n-1}\binom{k+n-q+1}{n}\right].
\end{aligned}$$

Call L_n the left hand side of (iii). We get

$$\begin{aligned}
L_n-L_{n-1}&=\binom{k+n-1}{n}\left[\sum_{q=1}^{n}(-1)^{q+1}\binom{n-1}{q-1}\binom{k+n-q}{n-1}\right]\\
&\quad+\binom{k+n-1}{n-1}\left[\sum_{q=1}^{n}(-1)^{q+1}\binom{n-1}{q-1}\binom{k+n-q+1}{n}\right]\\
&=\binom{k+n-1}{n}+\binom{k+n-1}{n-1}(k+1)\\
&=(n+1)\binom{k+n}{n}-n\binom{k+n-1}{n-1}.
\end{aligned}$$

When $n=1$, (iii) is easily checked, therefore it follows by induction on n.

2.4.2. To compute $\sum_{q \geq 0}(-1)^{q+1} q\zeta_q'(0)$ using (21), (22) and Proposition 2.3.4, we have to study two different terms.

The first term involves logarithms of integers. This is

$$\sum_{q=1}^{n}(-1)^{q+1}\sum_{m=1}^{n-q+1}d_{n,q}(m-n-1+q)\log m$$
$$=\sum_{q=1}^{n}(-1)^{q}(-1)^{q}\log(n+1-q)\quad\text{(by Lemma 2.4.1 (i))}$$
$$=\log(n!).$$

This term cancels with $\log h_{L^2}(1,1)=-\log(n!)$ in (20).

The second term involves values of $\zeta'(s)$. First, since $d_{n,q}(X-n-1+q)=-d_{n,q}(-X)$ (Lemma 2.4.1 (ii)), the Proposition 2.3.4, when applied to $P=d_{n,q}$ and $a=n+1-q$, gives

$$\hat{c}_1(\lambda(\mathcal{O}_{\mathbb{P}^n}), h_Q)=2\sum_{q=1}^{n}(-1)^{q+1}\zeta(d_{n,q}^{\text{odd}})-\frac{1}{2}\sum_{q=1}^{n}(-1)^{q+1}d_{n,q}^{*}(q-n-1),$$

where

$$2d_{n,q}^{\text{odd}}(X)=d_{n,q}(X)-d_{n,q}(-X).$$

From Lemma 2.4.1 (iii) we get

$$2\sum_{q=1}^{n}(-1)^{q+1}\zeta(d_{n,q}^{\text{odd}})=2(n+1)\zeta\left(\left(\frac{(k+n)!}{k!\,n!}\right)^{\text{odd}}\right). \tag{24}$$

On the right hand side, from Lemma 2.2.1 and the definition (5) we get

$$R_n=\zeta P-s_n \tag{25}$$

with $P(k)=$ coefficient of x^n in

$$2(n+1)\left(\frac{x}{1-e^{-x}}\right)^{n+1}\sum_{\substack{m\text{ odd}\\ m\geq 1}}k^m\frac{x^m}{m!}$$

and

$$s_m=\text{coefficient of } x^n \text{ in} \tag{26}$$

$$(n+1)\left(\frac{x}{1-e^{-x}}\right)^{n+1}\left[\sum_{\substack{m\text{ odd}\\ m\geq 1}}-\zeta(-m)\left(1+\frac{1}{2}+\ \dots\ +\frac{1}{m}\right)\frac{x^m}{m!}\right]$$

Clearly

$$P(k)=\left[\text{coefficient of } x^n \text{ in } 2(n+1)\left(\frac{x}{1-e^{-x}}\right)^{n+1}e^{kx}\right]^{\text{odd}}$$
$$=2(n+1)\left[\int\frac{e^{kx}}{(1-e^{-x})^{n+1}}\,dx\right]^{\text{odd}}$$

where the integral is taken on a small circle around the origin in the complex plane. We perform the change of variable $u=1-e^{-x}$ and we get

$$\int\frac{e^{kx}}{(1-e^{-x})^{n+1}}\,dx=\int\frac{(1-u)^{-k-1}}{u^{n+1}}\,du=\frac{(n+k)!}{k!\,n!}.$$

Hence

$$P(k)=2(n+1)\left[\frac{(k+n)!}{k!\,n!}\right]^{\text{odd}} \tag{27}$$

From (21), (23), 2.4.1(iii), (25) and (27) we conclude that the terms involving $\zeta'(s)$ are the same on the left and right hand sides, and Theorem 2.1.1 is equivalent to the identity

$$\frac{1}{2}\sum_{q=1}^{n}(-1)^{q+1}d^{*}_{n,q}(q-n-1)=s_n+t_n+\tilde{Td}_n, \tag{28}$$

where s_n, t_n and $\tilde{Td}_n$ are defined in (26), 2.2.2, and 2.2.3 respectively.

2.4.3. LEMMA 2.4.3. *Let T, y be two variables related by $T=1-e^{-y}$. Define coefficients β_l, σ_n and λ_n by the generating functions*

$$\sum_{l\geq 0}\beta_l y^l=y(1-T)/T$$

$$\sum_{n\geq 0}\sigma_n T^n=y/(1-T),$$

and

$$\sum_{n\geq 0}\lambda_n T^n=y^{-1}T/(1-T).$$

Then the following holds:

(i) $\displaystyle\sum_{n\geq 1}\frac{s_n}{n+1}T^n=\frac{1}{1-T}\sum_{k\geq 2}\sigma_{k-1}\beta_k y_{k-1}$

(ii) $\displaystyle\sum_{n\geq 1}\frac{\tilde{Td}_n}{n+1}T^{n+1}=-\sum_{k\geq 2}\frac{\beta_k}{k(k-1)}y^k$

(iii) $t_n=(n+1)\lambda_{n+1}(\sigma_{n+1}-1)$

Proof.

(i) Let

$$\psi(x)=\sum_{\substack{m\ \mathrm{odd}\\ m\geq 1}}-\zeta(-m)\left(1+\frac{1}{2}+\ldots+\frac{1}{m}\right)\frac{x^m}{m!}.$$

From (26) we get

$$s_n=(n+1)\int_C\psi(x)\frac{dx}{(1-e^{-x})^{n+1}},$$

the integral being taken on a small oriented loop C around $0\in\mathbb{C}$. Define a new variable $u=1-e^{-x}$. Then

$$\sum_{n\geq 1}\frac{s_n}{n+1}T^n=\sum_{n\geq 1}\left[\int_C\frac{\psi(x)}{1-u}\frac{du}{u^{n+1}}\right]T^n$$
$$=\frac{1}{1-T}\psi(y).$$

Since

$$\sigma_k=1+\frac{1}{2}+\ldots+\frac{1}{k}$$

and

$$\beta_k=\begin{cases}-\zeta(-m)/(m!) & \text{when } m=k-1 \text{ is odd}\\ 0 & \text{otherwise,}\end{cases}$$

we get (i).

(ii) We have

$$\frac{e^{-tx}}{1-e^{-tx}}-\frac{1}{tx}=\sum_{l\geq 1}\beta_l t^{l-1}x^{l-1}.$$

Define

$$\psi(x)=-\int_0^1 \sum_{l\geq 2}\beta_l t^{l-2}x^{l-1}\,dt.$$

From Proposition 2.2.3 we get

$$\tilde{Td}_n=\int_C \psi(x)\frac{dx}{(1-e^{-x})^{n+1}},$$

hence, as in (i) above,

$$\sum_{n\geq 1}\tilde{Td}_n T^n=\frac{1}{1-T}\psi(y).$$

Therefore

$$\sum_{n\geq 1}\frac{\tilde{Td}_n}{n+1}T^{n+1}=\int_0^y \psi(z)dz$$
$$=-\sum_{k\geq 2}\frac{\beta_k}{k(k-1)}y^k.$$

(iii) We have

$$\sum_{p=1}^{n}\sum_{j=1}^{p}\frac{1}{j}=(n+1)\left(1+\frac{1}{2}+\ \dots\ +\frac{1}{n}\right)-n$$
$$=(n+1)(\sigma_{n+1}-1).$$

Furthermore, the coefficient of x^{n+1} in $(x/(1-e^{-x}))^{n+1}$ is

$$\int\frac{dx}{x(1-e^{-x})^{n+1}}=\int\frac{du}{x(u-1)u^{n+1}}=\lambda_{n+1}.$$

Using Lemma 2.2.2, we get (iii). q.e.d.

2.4.4. The equality (28), $n\geq 1$, is proved by D. Zagier in the Appendix (using the notation $\delta_{n,r}=d_{n,n+1-r}$ and the definition of s_n, $\tilde{Td}_n$ and t_n coming from Lemma 2.4.3). This concludes the proof of Theorem 2.1.1.

3. HIGHER DIRECT IMAGES OF HERMITIAN HOLOMORPHIC VECTOR BUNDLES

3.1. Higher analytic torsion.

Let $f\colon X\to Y$ be a smooth proper map of complex analytic manifolds, TX the tangent bundle to X, and $T_{X/Y}$ the relative tangent bundle. Let $h_{X/Y}$ be a metric on $T_{X/Y}$ whose restriction to each fiber $X_y=f^{-1}(y)$, $y\in Y$, is Kähler. Call $\omega_{X/Y}$ the associated (1, 1) form.

Let $T^H X$ be a smooth sub-bundle of TX such that $TX=T_{X/Y}\oplus T^H X$. We shall assume that $(f, h_{X/Y}, T^H X)$ is a *Kähler fibration* in the sense of [9], Def. 2.4. p. 50, i.e. there exists a closed form ω on X such that $T_{X/Y}$ and $T^H X$ are orthogonal with respect to ω, and ω restricts to $\omega_{X/Y}$ on $T_{X/Y}$.

Let now E be a holomorphic vector bundle on X which is *f-acyclic* i.e. the coherent sheaf $R^q f_* E$ vanishes for every $q\geq 1$, and h a metric on E. By the semi-continuity of the Euler characteristic the sheaf $f_* E = R^0 f_* E$ is locally free on Y. We shall define a form $\tau(E)$ in

$$A(Y, \mathbb{C}) = \bigoplus_{p \geq 0} A^{pp}(Y, \mathbb{C})$$

whose component of degree zero is the Ray–Singer analytic torsion $\tau(E)^0$ considered in §1. Let $T^*_{X/Y}$ be the complexification of the dual of $T_{X/Y}$, $T^{*(0,1)}_{X/Y}$ its antiholomorphic component, $\Lambda^q T^{*(0,1)}_{X/Y}$ its q-th exterior power, $q \geq 0$, and $\mathscr{D}^q$ the infinite dimensional C^∞ bundle on Y whose sections on any open $U \subset Y$ are

$$\mathscr{D}^q(U) = C^\infty(f^{-1}(U), \Lambda^q T^{*(0,1)}_{X/Y} \otimes E). \tag{29}$$

Let $\bar{\partial}: \mathscr{D}^q \to \mathscr{D}^{q+1}$ be the Cauchy–Riemann operator of E along the fibers of f. We shall be interested in the *relative Dolbeault complex*:

$$\mathscr{D}^0 \xrightarrow{\bar{\partial}} \mathscr{D}^1 \xrightarrow{\bar{\partial}} \mathscr{D}^2 \to \dots .$$

Let $\mathscr{D}$ be the graded bundle $\bigoplus_{q \geq 0} \mathscr{D}^q$. Each fiber X_y having a Kähler metric, hence a density μ_y as in 1.1.1, we may define an L^2-metric on $\mathscr{D}^q_y = A^{oq}(X_y, E)$ by the formula (1) in 1.1.1. We let $f_*(h)$ denote the metric on the smooth bundle $f_*(E)_\infty \subset \mathscr{D}^0$ attached to $f_*(E)$ which is induced by the L^2-metric on $\mathscr{D}$, and we denote by $\bar{\partial}^*$ the adjoint of $\bar{\partial}$.

We now turn $\Lambda T^{*(0,1)}_{X/Y} \otimes E$ into a Clifford module under the action of the smooth sections of $T_{X/Y}$ as follows. ([9] (2.42) and (2.43)). If v is a relative tangent vector of type (1, 0), let $v^* \in T^{*(0,1)}_{X/Y}$ be the one form sending $w \in T_{X/Y}$ to its scalar product with v, and $c(v)$ the endomorphism of $\Lambda T^{*(0,1)}_{X/Y} \otimes E$ sending η to $2\, v^* \wedge \eta$. On the other hand, when v is a relative tangent vector of type (0, 1), we let $c(v)$ be the interior product by $-2v$. The map c extends by linearity to the whole tangent space.

Now let v and w be two vector fields on y. Call v^H and w^H the vector fields on X obtained by lifting v and w to $T^H X$. Let $[v, w]$ be their commutator and $T(v, w) \in T_{X/Y}$ be the projection along $T^H X$ of $-[v, w]$. The map T defines a tensor in $C^\infty(X, T_{X/Y} \otimes \Lambda^2(T^H X)^*)$. The action of T by Clifford multiplication on $\mathscr{D}$ and the exterior product of forms on Y define an operator $c(T)$ in the algebra

$$\operatorname{End}_{\mathbb{C}}(\mathscr{D}) \underset{\mathbb{C}}{\otimes} A^*(Y, \mathbb{C}),$$

where

$$A^*(Y, \mathbb{C}) = \bigoplus_{n \geq 0} A^n(Y, \mathbb{C})$$

(see [7], 3.Def.1.8., and [9] (3.7.) p. 69).

Let $T = T^{(1,0)} + T^{(0,1)}$ be the decomposition of T according to its type in $T_{X/Y}$ and

$$c(T) = c(T^{(1,0)}) + c(T^{(0,1)})$$

the corresponding decomposition of $c(T)$.

We now define a connection $\tilde{\nabla}$ on the bundles $\mathscr{D}^q$, $q \geq 0$ ([7] Def. 1.10., [9] Def. 2.1.3.). The metric on $T_{X/Y}$ gives an isomorphism between $T^{*(0,1)}_{X/Y}$ and the holomorphic relative tangent bundle $T^{(1,0)}_{X/Y}$, hence a holomorphic structure on every bundle $\Lambda^q T^{*(0,1)}_{X/Y}$, $q \geq 0$. We let ∇ be the unique unitary connection on $\Lambda^q T^{*(0,1)}_{X/Y} \otimes E$ which is compatible to its holomorphic structure. Let σ be a smooth section of $\mathscr{D}^q$ on some open subset $U \subset Y$, i.e. a section of $\Lambda^q T^{*(0,1)}_{X/Y} \otimes E$ over $f^{-1}(U)$ (cf. (29)). For every $x \in X$, $y = f(x)$ and $v \in T_y Y$ denote by $v^H \in T_x^H X$ the horizontal lifting of v. We define

$$\tilde{\nabla}_v(\sigma) = \nabla_{v^H}(\sigma).$$

From [9], Theorem 1.14., we know that $\tilde{\nabla}$ is unitary.

Let now p: $X \times \mathbb{C}^* \to X$ be the first projection, δ the differential on $\mathbb{P}^1$, and $\tilde{\nabla} + \delta$ the connexion on $p^*\mathcal{D}$ induced by $\tilde{\nabla}$. By the Leibnitz' rule we extend $\tilde{\nabla} + \delta$ to get an operator in

$$\mathcal{A} = \mathrm{End}_{\mathbb{C}}(\mathcal{D} \underset{\mathbb{C}}{\otimes} A^*(Y \times \mathbb{C}^*, \mathbb{C})). \tag{30}$$

For every non zero complex number $z \in \mathbb{C}^*$ we consider the following element of $\mathcal{A}$ (a superconnection in the sense of Quillen [34]):

$$A_z = \tilde{\nabla} + \delta + z\bar{\partial} + \bar{z}\bar{\partial}^* - \frac{1}{4\bar{z}} c(T^{(1,0)}) - \frac{1}{4z} c(T^{(0,1)}).$$

The curvature $-A_z^2$ defines an element in $\mathrm{End}(\mathcal{D}) \otimes A(Y \times \mathbb{C}^*, \mathbb{C})$ whose exponential $\exp(-A_z^2)$ happens to be trace class (see below). Let

$$a(z) = Tr_s \exp(-A_z^2) \in A^*(Y \times \mathbb{C}^*, \mathbb{C}). \tag{31}$$

be its supertrace for the $\mathbb{Z}/2$ grading on $\mathcal{D} \otimes A(Y, \mathbb{C})$. For every positive real number $\varepsilon > 0$, we let

$$I(\varepsilon) = \int_{|z| > \varepsilon} a(z) \log |z|^2$$

in $A(Y, \mathbb{C})$. As we shall see below this integral happens to converge and to have a finite asymptotic development of the type

$$I(\varepsilon) = \sum_{j \le 0} a_j \varepsilon^j + \sum_{j \le 0} b_j \varepsilon^j \log \varepsilon + O(\varepsilon) \tag{32}$$

which is uniform on every compact subset of Y. We let $I(0) = a_0$ be the finite part of $I(\varepsilon)$.

We now define two new characteristic classes ch' and Td' as follows. The first is

$$ch' = \sum_{q \ge 0} (-1)^q q\, ch_q,$$

where ch_q is the component of degree q of the Chern character. The second one is the one coming from the invariant polynomial function on square matrices A:

$$Td'(A) = \frac{d}{dt} Td(A + t\, Id).$$

Given any form

$$\eta = \sum_{p \ge 0} \eta^p \quad \text{in} \bigotimes_{p \ge 0} A^{2p}(Y, \mathbb{C}) \qquad \text{and } \lambda \in \mathbb{C}^*$$

we let

$$\delta_\lambda(\eta) = \sum_{p \ge 0} \lambda^{-p} \eta^p.$$

If γ is the Euler constant, we define

$$\tau(E) = \delta_{2i\pi} I(0) + \gamma f_*(ch(E, h)\, Td'(T_{X/Y}, h_{X/Y})) - \gamma\, ch'(f_* E, f_* h).$$

The following is a variant of [10] Theorem 1.27.

THEOREM 3.1.

(i) *The form* $\tau(E)$ *lies in* $A(Y,\mathbb{C})=\bigoplus_{p\geq 0} A^{pp}(Y,\mathbb{C})$ *and satisfies the equation*

$$dd^c\tau(E)=f_*(ch(E,h)\,Td(T_{X/Y},h_{X/Y}))-ch(f_*E,f_*h). \tag{33}$$

(ii) *The degree zero component of* $\tau(E)$ *is the Ray–Singer analytic torsion* $\tau(E)^0$.

Proof

(i) Since f_* commutes with dd^c and since $(ch(E,h),Td'(T_{X/Y},h_{X/Y}))$ and $ch'(f_*E,f_*h)$ are killed by dd^c, we just need to prove (33) with $\tau(E)$ replaced by $\delta_{2i\pi}I(0)$.

The first thing we show is that $\exp(-A_z^2)$ is trace class for $z\in\mathbb{C}^*$. For this we notice that

$$A_z^2=|z|^2\Delta+\Phi,$$

where $\Delta=\bar\partial\bar\partial^*+\bar\partial^*\bar\partial$ and Φ is nilpotent (since it has positive degree as form over Y). From Duhamel's formula we can write $\exp(-A_z^2)$ as a finite sum

$$\exp(-A_z^2)=\sum_{n\geq 0}\int_{0\leq t_1\leq\ldots\leq t_n\leq 1} e^{-t_1|z|^2\Delta}\Phi e^{(t_1-t_2)|z|^2\Delta}\Phi\ldots\Phi e^{(t_n-1)|z|^2\Delta}dt_1\ldots dt_n.$$

Using the fact that the heat kernel $e^{-|z|^2\Delta}$ is a smooth family of smoothing operators, we conclude that $\exp(-A_z^2)$ is trace class. Furthermore a similar argument for derivatives with respect to $Y\times\mathbb{C}^*$ shows that $a(z)$ is a smooth form on $Y\times\mathbb{C}^*$.

The form $a(z)$ is closed since

$$d\,tr_s\exp(-A_z^2)=tr_s[A_z,\exp(-A_z^2)]=0$$

where we use the fact that the supertrace vanishes on supercommutators (denoted $[\,,]_s$). For more details on this argument see [34] and [7] Prop. 2.9.

From the identities of [9] Theorem 2.6, we get

$$A_z^2=\left[\tilde\nabla^{(1,0)}+\frac{\partial}{\partial z}dz+z\bar\partial-\frac{1}{4\bar z}c(T^{(1,0)}),\tilde\nabla^{(0,1)}+\frac{\partial}{\partial\bar z}d\bar z+\bar z\bar\partial^*-\frac{1}{4z}c(T^{(0,1)})\right]. \tag{34}$$

For any $\theta\in\mathbb{R}$ let $r_\theta\colon Y\times\mathbb{C}^*\to Y\times\mathbb{C}^*$ be the automorphism sending (y,z) to $(y,e^{i\theta}z)$.

The vector space $\mathscr{S}\otimes A^*(Y\times\mathbb{C}^*,\mathbb{C})$ is graded by $\mathbb{N}^3$ with

$$\mathscr{S}\otimes A^*(Y\times\mathbb{C}^*,\mathbb{C}))^{(n,p,q)}=\mathscr{S}^n\otimes_{\mathbb{C}}A^{pq}(Y\times\mathbb{C}^*,\mathbb{C}).$$

Therefore the algebra $\mathscr{A}$ is also graded by $\mathbb{N}^3$. We denote by $\mathscr{B}\subset\mathscr{A}$ the subalgebra generated (as complex vector space) by elements α of degree (n,p,q) such that $q=p+n$ and $r_\theta^*(\alpha)=e^{in\theta}\alpha$ (see [8]). From (34) we conclude that A_z^2 lies in $\mathscr{B}$. Since tr_s vanishes on the subspace of $\mathscr{B}$ spanned by elements of degree (n,p,q), $n>0$, we conclude that $a(Z)$ lies in

$$A(Y\times\mathbb{C}^*,\mathbb{C})=\bigotimes_{p\geq 0}A^{pp}(Y\times\mathbb{C}^*,\mathbb{C})$$

and

$$r_\theta^*(a(z))=a(z).$$

Let us study the behaviour of $a(z)$ as $r=|z|$ goes to infinity. For this we use a result of Berline and Vergne ([5], Theorem 2.4). For any real number $t>0$ let δ_t be the automorphism of $A^n(Y\times\mathbb{C}^*,\mathbb{C})$ acting by multiplication by $t^{-n/2}$. Extend δ_t to an automorphism of $\mathscr{A}$. Then (*loc. cit.*) as t goes to infinity the operator $\delta_t\exp(-t\,A_z^2)$ converges (for the operator norm in $\mathscr{A}$) to the orthogonal projection of $\exp(-(\tilde{\nabla}+\delta)^2)$ on the kernel of $\bar{\partial}$. In particular the component $[\delta_t \exp(-t\, A_z^2)]^{(1,1)}$ of degree (1, 1) with respect to $\mathbb{C}^*$ converges to zero. In fact, looking at the proof of *loc. cit.* (see Lemma 1.1.1) we get

$$[\delta_t \exp(-t\, A_z^2)]^{(1,1)} = O(t^{-1/2})$$

as $t \to \infty$.

Now, since $r_\theta^*(a(z)) = a(z)$ for every θ, we can write, with $z = re^{i\theta}$,

$$a(z) = tr_s \exp - (R(r) + S(r) dr),$$

where

$$R(r) = \left(\tilde{\nabla} + \frac{\partial}{\partial \theta} d\theta + r(\bar{\partial} + \bar{\partial}^*) - \frac{1}{4r} c(T)\right)^2$$

and

$$S(r) = \frac{\partial}{\partial r}(A_r) = \bar{\partial} + \bar{\partial}^* + \frac{1}{4r^2} c(T)$$

Therefore the component involving dr in

$$tr_s(\delta_t \exp(-t A_z^2)) dr$$

is

$$\begin{aligned} & tr_s \delta_t(-tS(r) \exp(-tR(r))) \\ & = tr_s(((\sqrt{t}(\bar{\partial} + \bar{\partial}^*) + \frac{1}{4r^2\sqrt{t}} c(T)) \exp(-R(r\sqrt{t}))) dr \\ & = O(t^{-1/2}) dr. \end{aligned}$$

Dividing by $\sqrt{t}$ and putting $r = 1$ and $s = t^{-1/2}$ we get

$$tr_s(((\bar{\partial} + \bar{\partial}^*) + \frac{s^2}{4} c(T)) \exp(-R(1/s))) = O(s^{-2}),$$

i.e. the form $tr_s(S(r) \exp(-R(r))) dr$ is bounded as r goes to infinity (take $s = 1/r$). Similarly $tr_s(\exp(-R(r)))$ is bounded as r goes to infinity, i.e. $a(z)$ remains bounded as a form on $X \times \mathbb{C}^*$ as $|z|$ goes to infinity. Similar arguments apply to any derivative of $a(z)$ with respect to the parameter space $Y \times S^1$ and the bounds we get are uniform on any compact subset of $Y \times S^1$.

We now consider the behaviour of $a(z)$ as $r = |z|$ goes to zero. We have

$$\delta_r(A_z^2) = r^2 A_1^2 = r^2(A - B dr),$$

where A and B are smooth families of differential operators on $Y \times S^1$, with A elliptic and positive definite. From [22] and [23] we conclude that $tr_s \exp(-r^2 A)$ and $tr_s(B \exp(-r^2 A))$ have finite asymptotic expansions in powers of r^2, which converge uniformly on any compact subset of $Y \times S^1$ as r goes to zero. Therefore the same holds for

$$a(z) = \delta_r^{-1}(tr_s \exp(-r^2 A) + tr_s(B \exp(-r^2 A)) dr).$$

Now let $\varepsilon > 0$ be any positive real number. The integral

$$I(\varepsilon) = \int_{|z| > \varepsilon} a(z) \log |z|^2$$

converges and is C^∞ on Y since $a(z)$ and its derivatives with respect to Y are bounded as $|z|$ goes to infinity. From the asymptotic development of $a(z)$ we may write $I(\varepsilon)$ as in (32), the convergence being uniform on any compact subset of Y. Since $a(z)$ lies in $A(Y\times\mathbb{C}^*,\mathbb{C})$, we conclude that $I(\varepsilon)$ and $I(0)$ lie in $A(Y,\mathbb{C})$.

Now we compute $dd^cI(0)$. Since $a(z)$ is closed on $Y\times\mathbb{C}^*$ we have

$$(d+\delta)(a(z))=0 \tag{35}$$

If δ^c is the "complex conjugate" of δ, it follows that

$$(d^c+\delta^c)(a(z))=0, \tag{36}$$

since $a(z)$ lies in $A(Y\times\mathbb{C}^*,\mathbb{C})$. Let $R>\varepsilon$ be any real number and

$$I(\varepsilon,R)=\int_{\varepsilon<|z|<R} a(z)\log|z|^2.$$

We get from (35) and (36)

$$\begin{aligned} dd^cI(\varepsilon,R)&=\int_{|z|=\varepsilon}\delta^c(a(z))\log|z|^2-\int_{|z|=R}\delta^c(a(z))\log|z|^2\\ &-\frac{1}{2\pi i}\int_{|z|=\varepsilon}a(z)\delta\log|z|^2+\frac{1}{2\pi i}\int_{|z|=R}a(z)\delta\log|z|^2\\ &+\int_{\varepsilon<|z|<R}a(z)\delta^c\delta\log|z|^2. \end{aligned}\tag{37}$$

Now $\delta^c\delta\log|z|^2=0$, $\delta^c a(z)=-d^c a(z)$ is bounded as $|z|$ goes to infinity, and the component of $a(z)$ of degree zero with respect to $\mathbb{P}^1$ has a limit $a(\infty)$ when z goes to infinity ([6] Thm. 2.4.). Therefore, letting R go to infinity in (37), we get

$$dd^cI(\varepsilon)=-2\log(\varepsilon)\int_{|z|=\varepsilon}d^ca(z)-\frac{1}{2\pi i}\int_{|z|=\varepsilon}a(z)\delta\log|z|^2+a(\infty). \tag{38}$$

The component of $d^ca(z)$ which does not involve dr has a finite asymptotic development in powers of r, therefore the first summand in (38) is a sum of type

$$\sum_{k\gg-\infty}\beta_k\log(\varepsilon)\varepsilon^k+0(\varepsilon).$$

The component of $a(z)$ of degree zero with respect to $\mathbb{P}^1$ is

$$tr_s\exp-\left(\tilde{\nabla}+r(\bar{\partial}+\bar{\partial}^*)-\frac{1}{4r}c(T)\right)^2.$$

By the local index theorem for families [7] we know that this form has a limit $a(0)$ as r goes to zero. By the unicity of the asymptotic development of $I(\varepsilon)$ we get from (38);

$$dd^cI(0)=a(0)-a(\infty).$$

Now from [7] we have

$$\delta_{2\pi i}a(0)=f_*(ch(E,h)Td(T_{X/Y},h_{X/Y}))$$

and from [6]

$$\delta_{2\pi i}a(\infty)=ch(f_*E,f_*h).$$ q.e.d.

(ii) Let $a(z)^0$ be the component of $a(z)$ of degree zero with respect to Y. We have

$$a(z)^0 = tr_s \exp -(\delta + z\bar{c} + \bar{z}\bar{c}^*)^2.$$

Since $a(z)^0$ is invariant under the rotations r_θ we get

$$a(z)^0 = tr_s \exp -(r^2\Delta + (\bar{c} + \bar{c}^*)dr + ir(\bar{c} - \bar{c}^*)d\theta).$$

Since $\bar{c} + \bar{c}^*$ commutes with Δ, the component of $a(z)^0$ of degree 2 with respect to $\mathbb{C}^*$ is

$$a(z)^{02} = tr_s \exp[(\bar{c} - \bar{c}^*)(\bar{c} + \bar{c}^*)\exp(-r^2\Delta)]r\,dr\,d\theta. \tag{39}$$

Let N be the operator acting on $\mathscr{D}^q$ by multiplication by q. We have (see [8]).

$$[N, \bar{c}] = \bar{c},$$

and

$$[N, \bar{c}^*] = -\bar{c}^*,$$

hence

$$\bar{c} - \bar{c}^* = [N, \bar{c} + \bar{c}^*]. \tag{40}$$

Since tr_s vanishes on supercommutators and $\bar{c} + \bar{c}^*$ commutes with Δ we get from (39) and (40):

$$a(z)^{02} = 2tr_s(N\Delta \exp(-r^2\Delta))r\,dr\,d\theta.$$

Let $u = r^2$. We get

$$\int_{\mathbb{C}^*} a(z)^0 \log|z|^2 = \int_0^\infty \int_0^{2\pi} tr_s(N\Delta \exp(-r^2\Delta)) \log(u)\,du\,dr$$
$$= 2\pi \int_0^\infty tr_s(N\Delta \exp(-u\Delta)) \log(u)\,du.$$

Now let Q be the orthogonal projection of $\mathscr{D}$ onto the orthogonal complement of $f_*(E)_{\mathbb{C}}$. Define

$$\zeta(s) = \frac{-1}{\Gamma(s)} \int_0^\infty tr_s(QN\exp(-u\Delta))u^{s-1}\,du. \tag{42}$$

Clearly

$$\zeta(s) = \sum_{q \geq 0} (-1)^q q\zeta_q(s),$$

where $\zeta_q(s)$ is the zeta function of the Laplace operator Δ_q as in §1.1. From the fact that $Tr_S(QN\exp(-u\Delta))$ has a finite aymptotic development in powers of u as u goes to zero, it follows that

$$J(\varepsilon) = \int_\varepsilon^\infty u^{-1} tr_s(QN\exp(-u\Delta))du$$

has a finite asymptotic development in terms of $\varepsilon^k \log\varepsilon$ and ε^k, $k \in \mathbb{Z}$. Furthermore its finite part $J(0)$ satisfies

$$J(0) = \zeta'(0) + \gamma\alpha_0, \tag{43}$$

where γ is the Euler constant and α_0 is the finite part of $tr_s(QN\exp(-u\Delta))$ as u goes to zero (for a similar argument, see [16] 3.5).

Integrating $J(\varepsilon)$ by parts we get

$$J(\varepsilon) = [\log(u)tr_s(QM\exp(-u\Delta))]_\varepsilon^\infty - \int_\varepsilon^\infty \log(u)Tr_s(QN\Delta\exp(-u\Delta))du$$

The first term in this expression has a finite asymptotic development in terms of $\log(\varepsilon)\,\varepsilon^k$ and $Q\Delta=\Delta$, therefore

$$J(0)=-\int_0^\infty tr_s(N\Delta\exp(-u\Delta))\log(u)du$$

$$=-2\pi\int_{\mathbb{C}^*} a(z)^0\log|z|^2.$$

Since $\Delta=0$ on $f_*(E)_\infty=(1-Q)(\mathscr{S})$, we get

$$tr_s(QN\exp(-u\Delta))=tr_s(N\exp(-u\Delta))-tr_s(N \text{ on } f_*(E)_\infty).$$

It was shown in [9], Thm. 3.1.6, p. 87, that the finite part of $tr_s(N\exp(-u\Delta))$ as u goes to zero is the component of degree zero in

$$f_*(ch(E,h)Td'(T_{X/Y},h_{X/Y})).$$

Since $Tr_s(N\,|\,f_*(E)_\infty)$ is the component of degree zero of $ch'(f_*E,f_*h)$ we conclude, using (43) and (44), that

$$\tau(E)^0=\delta_{2i\pi}\zeta'(0)$$

is the analytic torsion considered in §1.1.

Remarks. Assume $R^0f_*E=0$. Then an argument similar to the proof of (ii) above shows that $\tau(E)=\delta_{2i\pi}\tilde{\zeta}'(0)$, where $\tilde{\zeta}(s)$ is the form-valued zeta function considered in [9] Thm. 3.20. Therefore (i) follows from *loc. cit.*

One may wonder whether the class of $\tau(E)$ in $\tilde{A}(Y,\mathbb{C})$ depends on the choice of the horizontal tangent space T^HX (see Conjecture 3.3 below).

3.2. *Arithmetic K-theory*

Let (A,Σ,F_∞) be an arithmetic ring (1.2.1). Given any arithmetic variety X over A we defined in [21], §6, a group $\hat{K}_0(X)$ of *virtual hermitian vector bundles* over X as follows. A generator of $\hat{K}_0(X)$ is a triple (E,h,η), where (E,h) is a hermitian vector bundle on X and $\eta\in\tilde{A}(X)$. The relations are the following. Let

$$\mathscr{E}: 0\to S\to E\to Q\to 0$$

be an exact sequence of vector bundles on X, h',h,h'' arbitrary metrics on S,E,Q respectively and $\bar{\mathscr{E}}=(\mathscr{E},h',h,h'')$. Then, given any $\eta',\eta''\in\tilde{A}(X)$ one has, in $\hat{K}_0(X)$,

$$(S,h',\eta')+(Q,h'',\eta'')=(E,h,\eta'+\eta''+\tilde{ch}(\bar{\mathscr{E}})).$$

Here $\tilde{ch}(\bar{\mathscr{E}})\in\tilde{A}(X)$ denotes (as in 1.2.2 above) the solution of the equation

$$-dd^c\tilde{ch}(\bar{\mathscr{E}})=ch(E,h)-ch(S,h')-ch(Q,h'') \qquad (45)$$

defined by Bott and Chern in [15], and studied in [18], [8] and [20] §1. One can then define a morphism

$$ch\colon \hat{K}_0(X)\to A(X)$$

by the formula

$$ch(E,h,\eta)=ch(E,h)+dd^c(\eta)$$

(by (45) this is compatible with the relations defining $\hat{K}_0(X)$).

Let now

$$f\colon X\to Y$$

be a smooth projective map between arithmetic varieties over A. Let $T_{X/Y}$ be the relative tangent bundle, and $h_{X/Y}$ a metric on the associated holomorphic bundle on X_∞. Let $f_\infty: X_\infty \to Y_\infty$ be the map of complex varieties induced by f and $T^H X_\infty$ a smooth sub-bundle of TX_∞ such that the triple $(f_\infty, h_{X/Y}, T^H X_\infty)$ is a Kähler fibration in the sense of 3.1.

We shall now define a direct image morphism from $\hat{K}_0(X)$ to $\hat{K}_0(Y)$. Given any triple (E, h, η) on X with $R^q f_* E = 0$ when $q > 0$, we define $f_!(E, h, \eta)$ in $\hat{K}_0(Y)$ to be the class of

$$(f_* E, f_* h, \tau(E) + f_!(\eta)),$$

where $f_* h$ is defined as in 3.1. (the L^2-metric on $f_* E$), $\tau(E)$ is the class in $\tilde{A}(Y)$ of the higher analytic torsion introduced in Theorem 3.1 and

$$f_!(\eta) = f_*(\eta\, Td(T_{X/Y}, h_{X/Y})) \in \tilde{A}(Y).$$

THEOREM 3.2. *The map $f_!$ induces a group morphism*

$$f_!: \hat{K}_0(X) \to \hat{K}_0(Y)$$

such that the following formula holds in $A(Y)$:

$$ch(f_!(\alpha)) = f_*(ch(\alpha)\, Td(T_{X/Y}, h_{X/Y})) \tag{46}$$

for any $\alpha \in \hat{K}_0(X)$.

Proof. We know already from Theorem 3.1 and the definition of ch that formula (46) holds when α is replaced by (E, h, η), with $R^q f_* E = 0$ when $q > 0$.

Consider an exact sequence

$$\mathscr{E}: 0 \to S \to E \to Q \to 0$$

of bundles on X, with $R^q f_* S = R^q f_* E = R^q f_* Q = 0$ for every $q > 0$. Choose arbitrary metrics h', h, h'' on S, E, Q respectively. Taking the direct images by f we get an exact sequence of vector bundles on Y:

$$f_* \mathscr{E}: 0 \to f_* S \to f_* E \to f_* Q \to 0$$

with metrics $f_* h', f_* h, f_* h''$. Let

$$\bar{\mathscr{E}} = (\mathscr{E}, h', h, h'') \quad \text{and} \quad f_* \bar{\mathscr{E}} = (f_* \mathscr{E}, f_* h', f_* h, f_* h'').$$

We shall prove below that the following equation holds in $\tilde{A}(Y)$:

$$\tau(E) - \tau(S) - \tau(Q) - \tilde{ch}(f_* \bar{\mathscr{E}}) = -f_!(\tilde{ch}(\bar{\mathscr{E}})). \tag{47}$$

Since f is projective, any vector bundle on X has a finite resolution by vector bundles E which are acyclic for f, i.e. $R^q f_* E = 0$ when $q > 0$ ([33], 7.27). Therefore $\hat{K}_0(X)$ is generated by triples (E, h, η) with E acyclic for f, and the relation (47) means that, in $\hat{K}_0(Y)$,

$$\begin{aligned} f_!(S, h', 0) + f_!(Q, h'', 0) &= (f_* S, f_* h', \tau(S)) + (f_* Q, f_* h'', \tau(Q)) \\ &= (f_* E, f_* h, \tau(S) + \tau(Q) + \tilde{ch}(f_* \bar{\mathscr{E}})) \\ &= (f_* E, f_* h, \tau(E) + f_!(\tilde{ch}(\bar{\mathscr{E}}))) \\ &= f_!(E, h, \tilde{ch}(\bar{\mathscr{E}})). \end{aligned}$$

In other words, $f_!$ preserves the defining relations in $\hat{K}_0(X)$, and by (33) (Theorem 3.1), Theorem 3.2 follows.

So let us prove (47). For this we may assume that the ground ring is $\mathbb{C}$. We use a definition of $\tilde{ch}(\bar{\mathscr{E}})$ introduced in [8] and [21]. Let $\mathbb{P}^1$ be the complex projective line, $\mathcal{O}(1)$

the standard line bundle of degree one on $\mathbb{P}^1$ and σ a section of $\mathcal{O}(1)$ vanishing only at infinity. Let z be the standard complex parameter on $\mathbb{P}^1$, and $i_z\colon X \to X \times \mathbb{P}^1$ the map sending x to (x, z). On $X \times \mathbb{P}^1$ consider the bundle

$$\tilde{E} = (E \oplus S(1))/S,$$

where S is embedded in E as in $\mathscr{E}$ and in $S(1) = S \otimes \mathcal{O}(1)$ by $\mathrm{id} \otimes \sigma$. Choose on $\tilde{E}$ a metric $\tilde{h}$ for which the isomorphisms $i_0^* \tilde{E} \simeq E$ and $i_\infty^* \tilde{E} \simeq S \oplus Q$ are isometries ($S \oplus Q$ being equipped with the orthogonal direct sum $h' \oplus h''$). Then $\tilde{ch}(\bar{\mathscr{E}})$ is the class in $\tilde{A}(X)$ of

$$-\int_{\mathbb{P}^1} ch(\tilde{E}, \tilde{h}) \log |z|^2$$

(cf. *loc. cit.*)

Now consider the following commutative diagram of proper smooth analytic maps

$$\begin{array}{ccc} X \times \mathbb{P}^1 & \longrightarrow & X \\ \downarrow \tilde{f} & & \downarrow f \\ Y \times \mathbb{P}^1 & \longrightarrow & Y \end{array}$$

where $\tilde{f} = f \times id_{\mathbb{P}^1}$ and the horizontal maps are the first projections. We get

$$f_*(\tilde{ch}(\bar{\mathscr{E}}) Td(T_{X/Y}, h_{X/Y})) = -\int_{\mathbb{P}^1} \tilde{f}_*(ch(\tilde{E}, \tilde{h}) \log |z|^2 \, Td(T_{X/Y}, h_{X/Y})). \tag{48}$$

The pull back of $T''X$ and $h_{X/Y}$ by the projection $X \times \mathbb{P}^1 \to X$ define with $\tilde{f}$ a Kähler fibration in the sense of 3.1, and we have, for every $\eta \in \tilde{A}(X \times \mathbb{P}^1)$,

$$\tilde{f}_!(\eta) = \tilde{f}_*(\eta \, Td(T_{X/Y}, h_{X/Y})).$$

Furthermore $R^q \tilde{f}_* \tilde{E} = 0$ when $q > 0$ since $i_z^* \tilde{E}$ is either E or $S \oplus Q$. Therefore we may apply Theorem 3.1 to $\tilde{f}$ and $(\tilde{E}, \tilde{h})$. We get

$$\tilde{f}_!(ch(\tilde{E}, \tilde{h})) = dd^c \tau(\tilde{E}) + ch(f_* \tilde{E}, f_* \tilde{h}). \tag{49}$$

From (48) and (49) we deduce

$$\begin{aligned} f_! \tilde{ch}(\bar{\mathscr{E}}) &= -\int_{\mathbb{P}^1} dd^c \tau(\tilde{E}) \log |z|^2 - \int_{\mathbb{P}^1} ch(f_* \tilde{E}, f_* \tilde{h}) \log |z|^2 \\ &= -\int_{\mathbb{P}^1} \tau(\tilde{E}) dd^c \log |z|^2 - \int_{\mathbb{P}^1} ch(f_* \tilde{E}, f_* \tilde{h}) \log |z|^2. \end{aligned}$$

We now use the equation of currents

$$dd^c \log |z|^2 = \delta_0 - \delta_\infty,$$

where δ_z is the Dirac mass at $z \in \mathbb{P}^1$, and we obtain

$$f_! \tilde{ch}(\bar{\mathscr{E}}) = -i_0^* \tau(\tilde{E}) + i_\infty^* \tau(\tilde{E}) - \int_{\mathbb{P}^1} ch(f_* \tilde{E}, f_* \tilde{h}) \log |z|^2.$$

By definition of τ, $\tilde{E}$ and $\tilde{h}$ we get

$$i_0^* \tau(\tilde{E}) = \tau(i_0^* \tilde{E}) = \tau(E)$$

and

$$i_\infty^* \tau(\tilde{E}) = \tau(i_\infty^* \tilde{E}) = \tau(S \oplus Q) = \tau(S) + \tau(Q).$$

Finally

$$f_*\tilde{E}=(f_*E\oplus f_*(S)(1))/f_*(S),$$

$$i_0^*(f_*\tilde{h})=f_*h \quad \text{and} \quad i_\infty^*(f_*\tilde{h})=f_*h'\oplus f_*h'',$$

therefore

$$\int_{\mathbf{P}^1} ch(f_*\tilde{E},f_*\tilde{h})\log|z|^2=c\tilde{h}(f_*\bar{\mathscr{E}}).$$

We conclude that

$$f_!(c\tilde{h}(\bar{\mathscr{E}}))=-\tau(E)+\tau(S)+\tau(Q)-c\tilde{h}(f_*\bar{\mathscr{E}})$$

as stated in (47). q.e.d.

3.3. A Conjecture.

We keep the notations of 3.2. The Conjecture 1.3 may be extended to higher degrees as follows.

Conjecture 3.3. *For any* $\alpha\in\hat{K}_0(X)$, *the following holds in* $\widehat{CH}(Y)_{\mathbf{Q}}$:

$$\widehat{ch}(f_!(\alpha))=f_*(\widehat{ch}(\alpha)Td^A(T_{X/Y},h_{X/Y})). \tag{50}$$

From Theorem 3.2, the Grothendieck-Riemann-Roch Theorem in Chow groups, and the exact sequence (3), we know that the difference between both sides of (50) lies in the image of a.

The Conjecture 1.3 is a special case of Conjecture 3.3 since

$$\hat{c}_1(f_!(\bar{E}))=\hat{c}_1(\lambda(E),h_Q)$$

(using Theorem 3.1(ii))

APPENDIX BY D. ZAGIER: PROOF OF THE IDENTITY (28)

§ 1. PRELIMINARIES

We will consistently use the notation, T, y for two variables related by

$$T=1-e^{-y}=\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l!}y^l,\quad y=\log\frac{1}{1-T}=\sum_{n=1}^{\infty}\frac{1}{n}T^n.$$

Define coefficients $S_1(n,l)$, $S_2(l,n)$ $(n,l\geq 0)$ by the generating functions

$$y^l=\sum_{n=0}^{\infty}S_1(n,l)T^n,\quad T^n=\sum_{l=0}^{\infty}S_2(l,n)y^l,$$

so that $\{S_1(n,l)\}_{n,l\geq 0}$ and $\{S_2(l,n)\}_{l,n\geq 0}$ are mutually inverse infinite triangular matrices. Define coefficients β_l, s_n, λ_n by the generating functions

$$\sum_{l=0}^{\infty}\beta_l y^l=\frac{1-T}{T}y,\quad \sum_{n=0}^{\infty}\sigma_n T^n=\frac{1}{1-T}y,\quad \sum_{n=0}^{\infty}\lambda_n T^n=\frac{T}{1-T}y^{-1}.$$

Alternatively, we can define these numbers by the recursions

$$nS_1(n,l)=lS_1(n-1,l-1)+(n-1)S_1(n-1,l),\quad lS_2(l,n)=nS_2(l-1,n-1)-nS_2(l-1,n)$$

$$\beta_l=-\sum_{k=0}^{l-1}\frac{\beta_k}{(l+1-k)!},\quad \sigma_n=\sigma_{n-1}+\frac{1}{n},\quad \lambda_n=1-\sum_{m=0}^{n-1}\frac{\lambda_m}{n+1-m}\quad (n,l\geq 1)$$

with the initial conditions

$$S_1(r,0)=S_1(0,r)=S_2(r,0)=S_2(0,r)=\delta_{r,0},\ \beta_0=1,\ \sigma_0=0,\ \lambda_0=1.$$

Thus $l!\beta_l$ is the lth Bernoulli number, $\sigma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$, and $\frac{n!}{l!}S_1(n, l)$ and $\frac{l!}{n!}S_2(l, n)$ are (up to sign) the integers known as Stirling numbers of the first and second kind (= number of permutations of $\{1, 2, \ldots, n\}$ having exactly l cycles and number of partitions of $\{1, 2, \ldots, l\}$ into exactly n non-empty subsets, respectively). The numbers $S_1(n, l)$ are also the Taylor coefficients of binomial coefficients:

$$\binom{x}{n} = \sum_{l=0}^{n} \frac{(-1)^{n-l}}{l!} S_1(n, l) x^l, \quad \binom{x+n-1}{n} = \sum_{l=0}^{n} \frac{1}{l!} S_1(n, l) x^l.$$

The first few values are as follows (note $S_1(n, l) = 0$ if $n < l$, $S_2(l, n) = 0$ if $l < n$):

r	β_r	s_r	λ_r
0	1	0	1
1	$-\frac{1}{2}$	1	$\frac{1}{2}$
2	$\frac{1}{12}$	$\frac{3}{2}$	$\frac{5}{12}$
3	0	$\frac{11}{6}$	$\frac{3}{8}$
4	$-\frac{1}{720}$	$\frac{25}{12}$	$\frac{251}{720}$
5	0	$\frac{137}{60}$	$\frac{95}{288}$
6	$\frac{1}{30240}$	$\frac{49}{20}$	$\frac{19087}{60480}$

l \ n	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
2	0	$-\frac{1}{2}$	1	1	$\frac{11}{12}$	$\frac{5}{6}$	$\frac{137}{180}$
3	0	$\frac{1}{6}$	-1	1	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{15}{8}$
4	0	$-\frac{1}{24}$	$\frac{7}{12}$	$-\frac{3}{2}$	1	2	$\frac{17}{6}$
5	0	$\frac{1}{120}$	$-\frac{1}{4}$	$\frac{5}{4}$	-2	1	$\frac{5}{2}$
6	0	$-\frac{1}{720}$	$\frac{31}{360}$	$-\frac{3}{4}$	$\frac{13}{6}$	$-\frac{5}{2}$	1

$S_1(n, l)$ (above the diagonal), $S_2(l, n)$ (below the diagonal)

For $n \geq 1$, $l \geq 1$, and $\nu \in \{-1, 0, 1\}$ define

$$\alpha_\nu(n, l) = \sum_{m=1}^{n} m^\nu S_1(n, m) S_2(l+m, n).$$

We will need the following proposition, which says that for fixed l the function $\alpha_\nu(n, l)$ becomes constant (respectively zero, respectively linear) for $n > l$ and $\nu = 0$ (respectively $\nu = -1$, respectively $\nu = 1$), and also gives the values for $n = l, l - 1$.

PROPOSITION. *For $1 \leq l \leq n+1$ and $-1 \leq \nu \leq 1$, $\alpha_\nu(n, l)$ is given by*

$$\alpha_0(n, l) = \beta_l + \begin{cases} 0 & (l \leq n), \\ \lambda_n - \lambda_{n+1} & (l = n+1), \end{cases}$$

$$\alpha_{-1}(n, l) = \begin{cases} 0 & (l < n), \\ -\frac{1}{n}\lambda_n & (l = n), \\ \lambda_{n+1} - \frac{1}{2}\lambda_n & (l = n+1), \end{cases}$$

$$\alpha_1(n, l) = -n[(l-1)\beta_l + \beta_{l-1}] - l\beta_l + \begin{cases} 0 & (l \leq n), \\ \lambda_{n+1} - \lambda_n & (l = n+1). \end{cases}$$

Proof. Form the generating function $A_\nu(x, y) = \sum_{n \geq 1, l \geq 0} \alpha_\nu(n, l) x^n y^l$. Then the definitions of $S_1(n, l)$ and $S_2(l, n)$ give

$$\begin{aligned} A_\nu(x, y) &= \sum_{m=1}^{\infty} m^\nu \sum_{n=m}^{\infty} S_1(n, m) \left(\sum_{l=0}^{\infty} S_2(l+m, n) y^l \right) x^n \\ &= \sum_{m=1}^{\infty} m^\nu \sum_{n=m}^{\infty} S_1(n, m) y^{-m} T^n x^n = \sum_{m=1}^{\infty} m^\nu \left(\frac{1}{y} \log \frac{1}{1-xT} \right)^m \end{aligned}$$

or

$$A_0(x,y)=H(x,y)-1,\quad A_{-1}(x,y)=\log H(x,y),\quad A_1(x,y)=H(x,y)^2-H(x,y)$$

with

$$H(x,y)=\left(1+\frac{1}{y}\log(1-xT)\right)^{-1}=y\left(\log\frac{1-xT}{1-T}\right)^{-1}.$$

We now develop everything in powers of $1-x$, obtaining

$$\log\frac{1-xT}{1-T}=\log\left(1+\frac{T}{1-T}(1-x)\right)=\frac{T}{1-T}(1-x)\cdot\sum_{r=0}^{\infty}\frac{1}{r+1}\left(\frac{T}{1-T}\right)^r(x-1)^r$$

and hence

$$A_0(x,y)=y\frac{1-T}{T}\cdot\frac{1}{1-x}-1-\sum_{r=0}^{\infty}\mu_{r+1}y\left(\frac{T}{1-T}\right)^r(x-1)^r,$$

$$A_{-1}(x,y)=\log\left(y\frac{1-T}{T}\right)+\log\left(\frac{1}{1-x}\right)-\sum_{r=1}^{\infty}\kappa_r\left(\frac{T}{1-T}\right)^r(x-1)^r,$$

$$A_1(x,y)=\left(y\frac{1-T}{T}\right)^2\frac{1}{(1-x)^2}-y(1-y)\frac{1-T}{T}\cdot\frac{1}{1-x}+\sum_{r=0}^{\infty}(\mu_{r+1}y+\mu'_{r+2}y^2)\left(\frac{T}{1-T}\right)^r(x-1)^r,$$

where μ_r, κ_r, and μ'_r are defined by the generating functions

$$\left(\sum_{r=0}^{\infty}\frac{u^r}{r+1}\right)^{-1}=\sum_{r=0}^{\infty}\mu_r u^r,\quad \log\left(\sum_{r=0}^{\infty}\frac{u^r}{r+1}\right)=\sum_{r=1}^{\infty}\kappa_r u^r,\quad \left(\sum_{r=0}^{\infty}\frac{u^r}{r+1}\right)^{-2}=\sum_{r=0}^{\infty}\mu'_r u^r.$$

Comparing the coefficients of x^n $(n\geqq 1)$ gives

$$\sum_{l=0}^{\infty}\alpha_0(n,l)y^l=y\frac{1-T}{T}-\mu_{n+1}y^{n+1}+O(y^{n+2}),$$

$$\sum_{l=0}^{\infty}\alpha_{-1}(n,l)y^l=\frac{1}{n}-\kappa_n y^n+\left((n+1)\kappa_{n+1}-\frac{n}{2}\kappa_n\right)y^{n+1}+O(y^{n+2}),$$

$$\sum_{l=0}^{\infty}\alpha_1(n,l)y^l=(n+1)\left(y\frac{1-T}{T}\right)^2-y(1-y)\frac{1-T}{T}+\mu_{n+1}y^{n+1}+O(y^{n+2}).$$

The proposition now follows if we note that $\mu_{n+1}=\lambda_{n+1}-\lambda_n$ (from the definitions), $\kappa_n=\frac{1}{n}\lambda_n$ (by differentiation), and

$$y(1-y)\frac{1-T}{T}=\sum_{l=0}^{\infty}(\beta_l-\beta_{l-1})y^l,\quad \left(y\frac{1-T}{T}\right)^2=-y^2\left(1+\frac{d}{dy}\right)\left(\frac{1-T}{T}\right)=-\sum_{l=0}^{\infty}((l-1)\beta_l+\beta_{l-1})y^l.$$

§2. PROOF OF THE IDENTITY

Define an operator $f\to f^*$ on polynomials by

$$f(x)=\sum_{n=0}^{N}c_n x^n\to f^*(x)=\sum_{n=0}^{N}c_n\sigma_n\frac{x^{n+1}}{n+1},$$

and for integers $1\leqq r\leqq n$ define a polynomial $\delta_{n,r}(x)$ of degree $2n-1$ by

$$\delta_{n,r}(x)=\frac{n!}{(n-r)!(r-1)!}\left(\frac{1}{x}+\frac{1}{x+r}\right)\binom{x+n}{n}\binom{x+r-1}{n}.$$

We wish to evaluate the expression

$$L(n)=\frac{1}{2}\sum_{r=1}^{n}(-1)^{n-r}\delta^*_{n,r}(-r).$$

We first observe that

$$\delta_{n,r}(x)+\delta_{n-1,r}(x)=\binom{n-1}{r-1}\left[\binom{x+r}{n}\binom{x+n-1}{n-1}+\binom{x+r-1}{n-1}\binom{x+n-1}{n}\right]$$

and hence, denoting the expression in square brackets by $\phi_{n,r}(x)$,

$$L(n)-L(n-1)=\frac{1}{2}\sum_{r=1}^{n}(-1)^{n-r}\binom{n-1}{r-1}\phi^*_{n,r}(-r).$$

Substituting the polynomial expansions of binomial coefficients in terms of Stirling numbers of the first kind, we find

$$\phi_{n,r}(x)=n\sum_{l,m=1}^{n}(-1)^{n-m}\frac{S_1(n,l)S_1(n,m)}{l!\quad m!}[x^{l-1}(x+r)^m+x^l(x+r)^{m-1}].$$

LEMMA. *Denote by $f_{l,m}(x)$ the polynomial $x^l(x+r)^m (l,m\geqq 0)$. Then*

$$f^*_{l,m}(-r)=\frac{(-1)^{l+1}l!\,m!}{(l+m+1)!}(\sigma_l-\sigma_m)r^{l+m+1}.$$

Proof. If $f(x)$ is the derivative of a polynomial $g(x)$ with $g(0)=0$, then

$$f^*(x)=\int_0^1\frac{g(x)-u^{-1}g(ux)}{1-u}du.$$

$\left(\text{To see this, take } f(x)=x^n,\ g(x)=\frac{x^{n+1}}{n+1}.\right)$ Applying this to $g=f_{l,m}$ $(m\geqq 1)$ gives

$$lf^*_{l-1,m}(-r)+mf^*_{l,m-1}(-r)=(-1)^{l+1}r^{l+m}\int_0^1 u^{l-1}(1-u)^{m-1}du \quad (m\geqq 1).$$

The integral equals $\frac{(l-1)!(m-1)!}{(l+m-1)!}$ (beta integral). The lemma now follows by induction on m, the case $m=0$ being trivial.

Now apply the lemma to $\phi_{n,r}(x)$ to get

$$\begin{aligned}\phi^*_{n,r}(-r)&=n\sum_{l,m=1}^{n}\frac{(-1)^{n-m-l}}{(l+m)!}S_1(n,l)S_1(n,m)\left[\frac{1}{l}(\sigma_{l-1}-\sigma_m)-\frac{1}{m}(\sigma_l-\sigma_{m-1})\right]r^{l+m}\\&=2(-1)^n n\sum_{l,m=1}^{n}\frac{S_1(n,l)S_1(n,m)}{(l+m)!}\left[\frac{\sigma_{l-1}}{l}-\frac{\sigma_l}{m}\right](-r)^{l+m},\end{aligned}$$

where to get the second line we have interchanged the roles of l and m in two of the terms. This gives

$$L(n)-L(n-1)=\sum_{l,m=1}^{n}\frac{S_1(n,l)S_1(n,m)}{(l+m)!}\left(\frac{\sigma_l}{m}-\frac{\sigma_{l-1}}{l}\right)\left[\sum_{r=0}^{n}(-1)^r\binom{n}{r}(-r)^{l+m+1}\right].$$

The expression in square brackets is the coefficient of $y^{l+m+1}/(l+m+1)!$ in $(1-e^{-y})^n$, i.e., it equals $(l+m+1)!\,S_2(l+m+1,n)$. Thus

$$L(n)-L(n-1)=\sum_{l=1}^{n}S_1(n,l)\left[-\frac{1}{l}(\sigma_{l-1}-1)\alpha_0(n,l+1)+(l+1)\sigma_l\alpha_{-1}(n,l+1)-\frac{1}{l}\sigma_{l-1}\alpha_1(n,l+1)\right]$$

with $\alpha_v(l,n)$ as in §1. The Proposition of §1 now gives

$$\begin{aligned}L(n)-L(n-1)=&\sum_{l=1}^{n}\frac{S_1(n,l)}{l}[\beta_{l+1}+\sigma_{l-1}((n+1)l\beta_{l+1}+n\beta_l)]\\&+\frac{1}{n}(\lambda_n-\lambda_{n+1})-\frac{n-1}{2}\sigma_{n-1}\lambda_n+(n+1)\sigma_n\left(\lambda_{n+1}-\frac{1}{2}\lambda_n\right).\end{aligned}$$

We are now ready to prove the main identity.

THEOREM. *Define rational numbers* s_n, $\tilde{Td}_n$, *and* t_n $(n \geq 1)$ *by*

$$\sum_{n\geq 1} \frac{s_n}{n+1} T^n = \frac{1}{1-T} \sum_{k\geq 2} \sigma_{k-1} \beta_k y^{k-1}, \qquad \sum_{n\geq 1} \frac{\tilde{Td}_n}{n+1} T^{n+1} = -\sum_{k\geq 2} \frac{\beta_k y^k}{k(k-1)},$$

$$t_n = (n+1)\lambda_{n+1}(\sigma_{n+1} - 1).$$

Then $L(n) = s_n + \tilde{Td}_n + t_n$. (See Table below.)

Proof. Let $R(n)$ denote $s_n + \tilde{Td}_n + t_n$; we will write $R(n)$ in terms of Stirling numbers and then show that $R(n) - R(n-1)$ agrees with the above expression for $L(n) - L(n-1)$, establishing the result by induction. The generating function for s_n is equivalent to $\sum_{n\geq 1} \frac{s_n}{(n+1)^2} T^{n+1} = \sum_{k\geq 2} \sigma_{k-1} \beta_k \frac{y^k}{k}$, as we see by integration. Hence from the definition of $S_1(n, k)$ we get

$$s_n = (n+1)^2 \sum_{k=2}^{n+1} \sigma_{k-1} \frac{\beta_k}{k} S_1(n+1, k),$$

$$\tilde{Td}_n = -(n+1) \sum_{k=2}^{n+1} \frac{\beta_k}{k(k-1)} S_1(n+1, k)$$

and therefore, using the recursion satisfied by $S_1(n, k)$,

$$s_n - s_{n-1} = \sum_{l=1}^{n} \left[\sigma_{l-1} \frac{\beta_l}{l} n + \sigma_l \beta_{l+1}(n+1) \right] S_1(n, l), \qquad \tilde{Td}_n - \tilde{Td}_{n-1} = -\sum_{l=1}^{n} \frac{\beta_{l+1}}{l} S_1(n, l).$$

Also,

$$\lambda_n = \text{coefficient of } T^n \text{ in } \frac{e^y - 1}{y} = \sum_{l=1}^{n} \frac{1}{(l+1)!} S_1(n, l),$$

so – using the recursion of $S_1(n, l)$ again –

$$(n+1)\lambda_{n+1} = \sum_{l=1}^{n} \left[\frac{n+1}{(l+1)!} - \frac{1}{(l+2)!} \right] S_1(n, l).$$

Combining these formulas and the formula for $L(n) - L(n-1)$, we find after some work

$$R(n) - R(n-1) - L(n) + L(n-1) = \frac{n-1}{n} \sum_{l=1}^{n} \left[\frac{n}{l} \beta_{l+1} - \frac{l}{2(l+2)!} \right] S_1(n, l).$$

But this is zero because

$$\sum_{n=1}^{\infty} \sum_{l=1}^{n} \left[\frac{n}{l} \beta_{l+1} S_1(n, l) - \frac{l}{2(l+2)!} S_1(n, l) \right] T^n = T \frac{d}{dT} \left(\sum_{l=1}^{\infty} \beta_{l+1} \frac{y^l}{l} \right) - \frac{1}{2} \sum_{l=0}^{\infty} \frac{l}{(l+2)!} y^l$$

$$= \frac{T}{1-T} \sum_{l=1}^{\infty} \beta_{l+1} y^{l-1} - \frac{y}{2} \frac{d}{dy} \left(\frac{e^y - 1 - y}{y^2} \right) = \frac{e^y - 1}{y} \left(\frac{1}{e^y - 1} - \frac{1}{y} + \frac{1}{2} \right) - \frac{y}{2} \frac{d}{dy} \left(\frac{e^y - 1 - y}{y^2} \right) = 0.$$

This completes the proof of the theorem.

n	1	2	3	4	5	6
s_n	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{649}{1080}$	$\frac{1445}{1728}$	$\frac{162871}{151200}$	$\frac{171311}{129600}$
$\tilde{Td}_n$	$-\frac{1}{12}$	$-\frac{1}{8}$	$-\frac{329}{2160}$	$-\frac{149}{864}$	$-\frac{56947}{302400}$	$-\frac{1933}{9600}$
t_n	$\frac{5}{12}$	$\frac{15}{16}$	$\frac{3263}{2160}$	$\frac{7315}{3456}$	$\frac{553523}{201600}$	$\frac{1172311}{345600}$
$L(n)$	$\frac{1}{2}$	$\frac{19}{16}$	$\frac{529}{270}$	$\frac{3203}{1152}$	$\frac{2198159}{604800}$	$\frac{4678657}{1036800}$

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REFERENCES

1. C. ALVAREZ-GAUMÉ, J.-B. BOST, G. MOORE, P. NELSON and C. VAFA: Bosonization on higher genus Riemann surfaces, *Comm. in Math. Physics* **12** (1987), 503–552.
2. S. J. ARAKELOV: An intersection theory for divisors on an arithmetic surface, *Izv. Akad. Nauk SSSR*, Ser. mat., 38.6 (1974), 1179–1192.
3. S. J. ARAKELOV: Theory of intersections on arithmetic surfaces, *Proc. Int. Cong. Math.*, Vancouver Vol. 1 (1974), 405–408.
4. A. A. BEILINSON, YU. I. MANIN: The Mumford form and Polyakov measure in string theory, *Comm. in Math. Physics* **107** (1986), 359–376.
5. A. A. BELAVIN and V. G. KNIZHNIK: Algebraic geometry and the geometry of quantum strings, *Physics Letters B*, **168** (1986), 201–206.
6. N. BERLINE and M. VERGNE: A proof of Bismut local index theorem for a family of Dirac operators, *Topology* **26** (1987), 435–464.
7. J.-M. BISMUT: The Atiyah–Singer index theorem for families of Dirac operators: two heat equation proofs, *Invent. Math.* **83** (1986), 91–151.
8. J.-M. BISMUT, H. GILLET and C. SOULÉ: Analytic torsion and holomorphic determinant bundles I: Bott Chern forms and analytic torsion, *Comm. in Math. Physics* **115** (1988), 49–78.
9. J.-M. BISMUT, H. GILLET and C. SOULÉ: Analytic torsion and holomorphic determinant bundles II: Direct images and Bott Chern forms, *Comm. in Math. Physics* **115** (1988), 79–126.
10. J.-M. BISMUT, H. GILLET and C. SOULÉ: Analytic torsion and holomorphic determinant bundles III: Quillen metrics on holomorphic determinants, *Comm. in Math. Physics* **115** (1988), 301–351.
11. J.-B. BOST: Conformal and holomorphic anomalies on Riemann surfaces and determinant line bundles, in *VIIIth Cong. on Math. Physics*, Marseille (1986), 768–775, World Scientific Ed..
12. J.-B. BOST: Fibrés déterminants, déterminants régularisés et mesures sur les espaces de modules des courbes complexes, in *Sémin. Bourbaki* **676** (1987).
13. J.-B. BOST: Le déterminant régularisé du Laplacien scalaire sur la sphère S^d, private notes.
14. J.-B. BOST and T. JOLICOEUR: A holomorphic property and the critical dimension in a string theory from an index theorem, *Phys. Lett. B*, **174** (1974), 273–276.
15. R. BOTT and S. S. CHERN: Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic cross-sections, *Acta Math.* **114** (1968), 71–112.
16. J. CHEEGER: Analytic torsion and the heat equation, *Ann. Math.* **109** (1979), 259–322.
17. P. DELIGNE: Le déterminant de la cohomologie, in Current trends in Arithmetical Algebraic Geometry, *Contemp. Math.* **67** (1987), 93–178.
18. S. DONALDSON: Anti-self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles, *Proc. London Math. Soc.* **50**.
19. G. FALTINGS: Calculus on arithmetic surfaces, *Ann. Math.* **119** (1984) 387–424.
20. H. GILLET and C. SOULÉ: Arithmetic intersection theory, preprint (1988).
21. H. GILLET and C. SOULÉ: Characteristic classes for algebraic vector bundles with hermitian metric, to appear in *Ann. Math.*
22. P. B. GILKEY: Invariance theory, the heat equation and the Atiyah–Singer index theorem, *Math. Lecture Series* **11** (1984), Publish or Perish, Inc..
23. P. GREINER: An asymptotic expansion for the heat equation, *Arch. ration. Mech. Anal.* **41** (1971), 163–218.
24. A. GROTHENDIECK, P. BERTHELOT and L. ILLUSIE: SGA6, Théorie des intersections et théorème de Riemann–Roch, *Lect. Notes in Math.* **225** (1971), Springer-Verlag.
25. F. HIRZEBRUCH: Topological methods in algebraic geometry, *Grundleh. Math.* **131** (1966), Springer-Verlag.
26. J. E. HUMPHREYS: Introduction to Lie algebras and Representation Theory, *Graduate Texts in Maths.* **9** (1978), Springer-Verlag.
27. A. IKEDA and Y. TANIGUCHI: Spectra and eigenforms of the Laplacian on S^n and $P^n(C)$, *Osaka J. Math.* **15** (1978), 515–546.
28. F. F. KNUDSEN and D. MUMFORD: The projectivity of the moduli space of stable curves I: Preliminaries on "det" and "div", *Math. Scand.* **39** (1976), 19–55.
29. YU. I. MANIN: New dimensions in geometry, in *Lect. Notes in Maths.* **1111** (1985), 59–101, Springer-Verlag.
30. YU. I. MANIN: The partition function of the Polyakov string can be expressed through theta functions, *Phys. Lett. B* **172** (1986), 184–185.
31. YU. I. MANIN: Quantum strings and algebraic curves, talk at ICM 86, Berkeley.
32. D. MUMFORD: On the stability of projective varieties, *l'Ens. Math.* **23** (1977), 39–110.
33. D. QUILLEN: Higher K-theory I, in *Lecture Notes* 341, H. Bass ed., (1973), 85–147, Springer-Verlag.
34. D. QUILLEN: Superconnections and the Chern character, *Topology* **24** (1985), 89–95.

35. D. Quillen: Determinants of Cauchy-Riemann operators over a Riemann surface. *Funct. Anal. Appl.* (1985), 31–34.
36. D. B. Ray and I. M. Singer: Analytic torsion for complex manifolds. *Ann. Math.* **98** (1973), 154–177.
37. I. Vardi: Determinants of Laplacians and multiple gamma functions. *SIAM J. Math. Anal.* **19** (1988), 493–507.
38. W. I. Weisberger: Normalization of the path integral measure and the coupling constants for bosonic strings. *Nucl. Phys. B* **284** (1984), 171–200.

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