Symplectic topology in the nineties*

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Communicated by M. Gotay
Received 18 January 1998
Revised 5 May 1998

Abstract: This is a survey of some selected topics in symplectic topology. In particular, we discuss low-dimensional symplectic and contact topology, applications of generating functions, Donaldson's theory of approximately complex manifolds and some other recent developments in the field.

Keywords: Symplectic and contact structure, Lagrangian submanifold, pseudo-holomorphic curves.

Mathematics Subject Classification: 53C15, 58F05.

Twenty years ago Symplectic Topology did not exist. When ten years ago I wrote a short survey [5], it was possible at least to indicate the main directions of the new field. The development of Symplectic Topology during the last decade can only be characterized as an explosion. Today it is a melting pot of ideas from Physics, Topology, Algebraic and Differential Geometry, Analysis, etc.

In the current survey we discuss some of the new developments without any attempt to be complete. The choice of subjects more reflects the author's current interests rather than the objective state of the art in the field.

I want to thank P. Biran for his help in the preparation of this survey. I am also grateful to the referee and the editor for helpful critical remarks.

1. Symplectic basics

1.1. Symplectic and contact manifolds

A symplectic structure on a 2k-manifold W is a closed non-degenerate differential 2-form ω. The non-degeneracy condition means that the formula Iω(τ) = ω(τ, ·), τ ∈ T(W), defines an isomorphism Iω : T(W) → T*(W) between the tangent and cotangent bundles of the manifold W. For a 2-dimensional W a symplectic form is just an area form of the surface. According to Darboux’ theorem any symplectic form is locally equivalent to the canonical form ω0 = ℓ dx_i ∧ dy_i. Thus, equivalently a symplectic manifold can be characterized by existence of local Darboux charts glued together by symplectomorphisms, i.e., diffeomorphisms which preserve the canonical form.

A 1-form α on a (2k − 1)-dimensional manifold V is called contact if the restriction of dα to the (2k − 2)-dimensional tangent distribution ξ = {α = 0} is non-degenerate (and hence
symplectic). Equivalently, we can say that a 1-form $\alpha$ is contact if $\alpha \wedge (d\alpha)^{k-1}$ does not vanish on $V$. A codimension 1 tangent distribution $\xi$ on $V$ is called a contact structure if it can be locally (and in the co-orientable case globally) defined by the Pfaffian equation $\alpha = 0$ for some choice of a contact form $\alpha$. The pair $(V, \xi)$ in this case is called a contact manifold. Note that according to Frobenius’ theorem the contact condition is a condition of maximal non-integrability of the tangent hyperplane field $\xi$. In particular, all integral submanifolds of $\xi$ have dimension $\leq k - 1$. On the other hand, $(k - 1)$-dimensional integral submanifolds, called Legendrian, always exist in abundance. Any non-co-orientable contact structure can be canonically double-covered by a coorientable one. If a contact form $\alpha$ is fixed then one can associate with it the Reeb vector field $R_\alpha$, which is transversal to the contact structure $\xi = \{\alpha = 0\}$. The field $R_\alpha$ is uniquely determined by the equations

$$R_\alpha \perp d\alpha = 0; \quad \alpha(R_\alpha) = 1.$$ 

A symplectic structure $\omega$ on $W$ defines a volume form $\omega^k$, and hence an orientation of $W$. In particular, if $W$ is closed then the cohomology class $[\omega] \in H^2(W; \mathbb{R})$ represented by the closed form $\omega$ satisfies the inequality $[\omega]^k \neq 0$.

If a contact structure $\xi$ is defined by a 1-form $\alpha$, then the conformal class of the symplectic structure $d\alpha|_\xi$ depends only on $\xi$ (because $d(\alpha)|_\xi = f d\alpha|_\xi$ for a function $V \to \mathbb{R}$); we denote it by $CS(\xi)$. For an even integer $k$ the contact structure defines an orientation of the $(2k - 1)$-dimensional manifold $V$; if $k$ is odd, it defines an orientation of $\xi$.

An important example of a symplectic manifold is provided by the cotangent bundle $T^*(M)$ of any smooth manifold $M$. The symplectic form $\omega$ on $T^*(M)$ is the differential of the famous 1-form $\rho dq$. Alternatively the symplectic structure on $T^*(M)$ can be described as follows. If $M = \mathbb{R}^k$ then $\mathbb{R}^{2k} = T^*(\mathbb{R}^k)$ is endowed with the canonical symplectic structure $\omega_0 = d(p dq) = \sum_i dq_i \wedge dp_i$, where the coordinates $q = (q_1, \ldots, q_k)$ and $p = (p_1, \ldots, p_k)$ are chosen in such a way that the projection $T^*(\mathbb{R}^k) \to \mathbb{R}^k$ is given by $(p, q) \mapsto q$. Let us observe that any diffeomorphism $f : \mathbb{R}^k \to \mathbb{R}^k$ lifts to a symplectomorphism $f_* : T^*(\mathbb{R}^k) \to T^*(\mathbb{R}^k)$ by the formula

$$f_*(p, q) = (f(q), (df^*)^{-1}(p)).$$

Thus a coordinate atlas $M = \bigcup_j U_j$ on $M$ lifts to a symplectic atlas $T^*(M) = \bigcup_j T^*(U_j)$ with gluing symplectomorphisms lifted by the above formula.

The standard contact structure $\xi_0$ on $\mathbb{R}^{2k-1}$ is defined by the contact 1-form $dz - \sum_{i=1}^{k-1} p_i dq_i$ in the coordinates $(q_1, \ldots, q_{k-1}, p_1, \ldots, p_{k-1}, z)$. More generally, the space $J^1(M) = T^*(M) \times \mathbb{R}$ of 1-jets of functions on $M$ has the canonical contact structure defined by the contact form $dz - p dq$ on $T^*(M) \times \mathbb{R}$, where the coordinate $z$ corresponds to the second factor, and where we identify the form $p dq$ on $T^*(M)$ with its pull-back on $J^1(M)$.

Another important example of a contact manifold is provided by the space of contact elements of a smooth manifold $M$, or in other words, by the projectivized cotangent bundle $PT^*(M)$. A point of $PT^*(M)$ is a tangent hyperplane to $M$ which can be identified with a line in $T^*(M)$. The canonical 1-form $p dq$ does not descend to $PT^*(M)$, but its kernel does and this defines a canonical contact structure on $PT^*(M)$. This contact structure is not co-orientable. The double cover of $PT^*(M)$, which carries a co-orientable contact structure, is the associated spherical
bundle $ST^*(M)$ which can be viewed as the space of co-oriented tangent hyperplanes (elements).

Similar to the symplectic case, contact manifolds have no local geometry: any $(2k-1)$-dimensional contact manifold is locally isomorphic to the standard contact $\mathbb{R}^{2k-1}$; moreover, any contact form is locally isomorphic to the standard contact form $dz - \sum_{i=1}^{k-1} p_i dq_i$.

Complex geometry serves as a rich source of examples of symplectic and contact manifolds. An imaginary part of a Kähler metric is a symplectic form. In particular, the standard symplectic structure on the complex projective space $\mathbb{C}P^n$, which in homogeneous coordinates $(z_0 : \ldots : z_n)$ equals to

$$\omega = \frac{i}{2\pi} \partial \overline{\partial} \log \sum_{k=0}^{n} |z_k|^2,$$

is the imaginary part of the Fubini–Study metric. Here the coefficient for $\omega$ is chosen to satisfy the condition $\int_{\mathbb{C}P^n} \omega = 1$.

A strictly pseudo-convex hypersurface in a complex manifold carries a canonical contact structure defined by the maximal complex subbundle of its tangent bundle. See Section 2 below for more details.

Contact and symplectic structures on closed manifolds satisfy the following stability property, which is due to J. Gray [29] in the contact case, and to J. Moser [36] in the symplectic one.

**Theorem 1.1.** Let $M$ be a closed manifold, $\xi_t$ a family of contact structures on $M$, or $\omega_t$ a family of symplectic structures on $M$ with $[\omega_t] = \text{const}$, $t \in [0, 1]$. Then there exists an isotopy $f_t : M \to M$, such that $df_t(\xi_0) = \xi_t$, or $f_t^*(\omega_0) = \omega_t$, $t \in [0, 1]$.

### 1.2. Lagrangian and Legendrian submanifolds

**Lagrangian submanifolds** of symplectic manifolds and **Legendrian submanifolds** of contact manifolds play the central role in Symplectic Topology.

A $k$-dimensional submanifold $L \subseteq W$ of a $2k$-dimensional symplectic manifold $(W, \omega)$ is called Lagrangian if the form $\omega|_L$ vanishes.

A $(k-1)$-dimensional submanifold $\Lambda \subseteq W$ of a $(2k-1)$-dimensional contact manifold $V$ is called Legendrian if $\Lambda$ is tangent to $\xi$. Equivalently, $\Lambda$ is Legendrian if for each point $q \in \Lambda$ the tangent space $T_p(\Lambda)$ is a Lagrangian subspace of $\xi_p$ with respect to a symplectic structure from $CS(\xi)$. Similarly, one can define Lagrangian and Legendrian embeddings and immersions.

A section $s : M \to T^*(M)$ is a Lagrangian embedding iff $s$ is a closed 1-form. In fact, any Lagrangian submanifold $L$ of $T^*(M)$ can be viewed as a multi-valued closed form, or better to say that the closed form $p \, dq|_L$ is defined on the submanifold $L$ itself rather than on its projection. The Lagrangian submanifold $L \subseteq T^*(M)$ is called exact if the closed 1-form $p \, dq|_L$ is exact. A section $\sigma : M \to J^1(M)$ is a Legendrian embedding iff $\sigma$ is the 1-jet extension $j^1(\sigma)$ of a function $f : M \to \mathbb{R}$.

Similarly to the case of Lagrangian submanifolds of the cotangent bundle, a general Legendrian submanifold of $J^1(M)$ corresponds to a graph ("wave front") of a multivalued function. A projection $J^1(M) = T^*(M) \times \mathbb{R} \to T^*(M)$ sends Legendrian submanifolds of $J^1(M)$ onto (impressed) exact Lagrangian submanifold of $T^*(M)$. Conversely, any exact Lagrangian
submanifolds of $T^*(M)$ lifts, uniquely up to a translation along the $\mathbb{R}$-factor, to a Legendrian submanifold of $J^1(M)$. The Legendrian submanifold $L_f = j^1(f)(M) \subset J^1(M)$, and its exact Lagrangian projection $L_f = df(M) \subset T^*(M)$ are called graphical Legendrian or exact Lagrangian submanifolds.

1.3. Hamiltonian functions, vector fields and diffeomorphisms

A vector field $X$ on a symplectic manifold $(M, \omega)$ is called symplectic if the Lie derivative $\mathcal{L}_X \omega$ vanishes, which is equivalent to the equation $d(X \lrcorner \omega) = 0$. If the form $X \lrcorner \omega$ is exact, i.e., $X \lrcorner \omega = dH$, then the vector field $X = X_H$ is called Hamiltonian, and the function $H$ is called the Hamiltonian function for the vector field $X$. If $M$ is non-compact then we will always assume that $X$ and $H$ have compact supports. This condition defines $H$ uniquely. If $M$ is compact then $H$ is defined up to adding a constant. A time-dependent Hamiltonian function $H_t : M \to \mathbb{R}$, $t \in [0, 1]$, defines a symplectic isotopy $\varphi_t : M \to M$, $t \in [0, 1]$. Namely, $\varphi_t$ is determined by the differential equation

$$\frac{d\varphi_t(x)}{dt} = X_{H_t}(\varphi_t(x)), \quad t \in [0, 1], x \in M,$$

with the initial condition $\varphi_0(x) = x$, $x \in M$. The isotopy $\varphi_t$ is called a Hamiltonian isotopy with the time-dependent Hamiltonian function $H_t$, $t \in [0, 1]$. A symplectomorphism $\varphi : M \to M$ is called Hamiltonian if it is the time 1 map of a Hamiltonian isotopy. The group Ham of Hamiltonian diffeomorphisms of $(M, \omega)$ is a normal subgroup of the identity component $\text{Diff}_\omega$ of the group of symplectomorphisms of $(M, \omega)$, and we have $\text{Ham} = [\text{Diff}_\omega, \text{Diff}_\omega]$, see [25].

Hamiltonian diffeomorphisms can be recognized among all symplectomorphisms with the help of the Flux, or Calabi homomorphism (see [28] and also [25, 13]), which is defined as follows. Let $\text{Diff}_\omega$ be the universal cover of the group $\text{Diff}_\omega$. Thus an element of $\text{Diff}_\omega$ is a homotopy class of a path $\varphi_t \in \text{Diff}_\omega$, $t \in [0, 1]$, where the homotopy fixes the ends of the path. Given a path $\varphi_t \in \text{Diff}_\omega$, $t \in [0, 1]$, we denote by $X_t$ the symplectic vector field

$$X_t(\varphi_t(x)) = \frac{d\varphi_t(x)}{dx}$$

and set

$$\text{Flux}([\varphi_t]) = \int_0^1 [X_t \lrcorner \omega] dt,$$

where $[X_t \lrcorner \omega] \in H^1(M; \mathbb{R})$ is the cohomology class of the closed 1-form $X_t \lrcorner \omega$, $t \in [0, 1]$. It is straightforward to check that $\text{Flux}([\varphi_t])$ depends only on the homotopy class of a path connecting $\text{Id}$ and $\varphi_1$, and thus $\text{Flux}$ is a homomorphism $\tilde{\text{Diff}}_\omega \to H^1(M; \mathbb{R})$. A symplectomorphism $\varphi \in \text{Diff}_\omega$ is Hamiltonian if and only if $\text{Flux}(\varphi) = 0$ for some lift $\tilde{\varphi} \in \tilde{\text{Diff}}_\omega$ of $\varphi \in \text{Diff}_\omega$.

A. Banyaga [25] proved that if the manifold $M$ is closed then the group $\text{Ham}(M, \omega)$ is simple. However, in the non-compact case this is no longer true. Namely, Ham has the commutator
subgroup \([\text{Ham}, \text{Ham}]\) as a proper normal subgroup which is equal to the kernel of the Calabi homomorphism, which can be defined as follows.

Let \(f \in \text{Ham}\) be the time 1 map of a Hamiltonian isotopy generated by a time-dependent Hamiltonian function \(H_t\). Then \(\text{Cal}(f) = \int_M H_t \omega^n dt\). In the general case this integral depends on the homotopy class of the path chosen to connect \(f\) with the identity, and thus \(\text{Cal}\) is a homomorphism \(\mathcal{H}am \to \mathbb{R}\), where \(\mathcal{H}am\) is the universal cover of the group \(\text{Ham}\). However, in the case when the symplectic structure \(\omega\) is exact, this homomorphism descends to the group \(\text{Ham}\) itself (see [28] and [25] for the details).

1.4. Symplectic and almost complex structures

Let \(\omega\) be a symplectic bilinear form on a real vector space \(V\). A complex structure \(J : V \to V\), \(J^2 = -\text{Id}\), on \(V\) is called compatible with \(\omega\) if the form \(\omega\) is invariant with respect to \(J\) and \(\omega(X, JX) > 0\) for any non-zero vector \(X \in V\). In other words, this can be expressed by saying that

\[
\omega(X, JY) = i \omega(X, Y), \quad X, Y \in V,
\]

is a positive definite Hermitian form on the complex vector space \((V, J)\). If just a weaker condition

\[
\omega(X, JX) > 0 \quad \text{for any tangent vector } X \neq 0
\]

is satisfied, than according to Gromov’s definition (see [48]) the complex structure \(J\) is tamed by \(\omega\). The group \(\text{Sp}(V, \omega)\) of symplectic automorphisms acts on the space \(\mathcal{J}\) of compatible complex structures with the subgroup \(U(V, J)\) of unitary transformations as the stabilizer subgroup. Thus \(\mathcal{J}\) can be identified with the homogeneous space \(\text{Sp}(V, \omega)/U(V, J)\) which is contractible, as can be seen, for instance, from the polar decomposition. The space of complex structures tamed by \(\omega\) has \(\mathcal{J}\) as its deformation retract, and hence is also contractible.

Coming to a non-linear situation let \((W, \omega)\) be a symplectic manifold. We denote by \(\mathcal{J}(W, \omega)\) the space of all almost complex structures \(J : T(W) \to T(W)\) compatible with \(\omega\) on every tangent space \(T_x(W), x \in W\). The space \(\mathcal{J}(W, \omega)\) is non-empty and contractible, as it is the space of sections of a fibration with contractible fibers. It is also important to notice that the space of symplectic structures compatible with a given almost complex structure is a convex subset of a vector space, and therefore also contractible. This space is non-empty if \(W\) is open (see [7]). In the case of a closed manifold \(W\) existence of an almost complex structure is far from being sufficient for existence of a symplectic structure on \(W\) (see Problem 2 in Section 7).

A generic almost complex manifold has no complex submanifolds of (complex) dimension \(> 1\). However, it was observed already by A. Nijenhuis and W. Wolf (see [222]) that locally any almost complex manifold has as many holomorphic curves as one has in the integrable case. M. Gromov (see [48]) was the first who understood that in the presence of a taming symplectic form one can develop a global theory of holomorphic curves, similar to the case of (integrable) Kähler manifolds. His introduction of tamed (or compatible) almost complex structures into Symplectic Geometry revolutionized this area. The theory of pseudo-holomorphic curves, as introduced by Gromov, forms the foundation for most recent developments in Symplectic
Topology. In this survey we will omit the prefix “pseudo” and use the term “holomorphic” for curves in almost complex manifolds.

1.5. Symplectic rigidity

One of the basic facts, which Gromov proved [48] using his theory of holomorphic curves in symplectic manifolds is the following

**Theorem 1.2.** The groups of symplectic and contact diffeomorphisms are $C^0$-closed in the groups of all diffeomorphisms.

This result, which establishes the existence of symplectic topology, was proved earlier by the author (see [47]) using a combinatorial theorem about the structure of wave fronts, but the proof of this combinatorial theorem was not published. Symplectic rigidity could also be deduced from Conley–Zehnder’s work on Arnold’s conjecture for $T^{2n}$ (see [45]).

A related fact, also proven in Gromov’s seminal paper [48], is the existence of a first specifically symplectic invariant, which is now called Gromov’s width. This invariant can be defined as follows. Let $(M, \omega)$ be a symplectic manifold (for instance, a domain in the standard symplectic $\mathbb{R}^{2n}$). Fix a point $p \in M$. Given an almost complex structure $J$ on $M$ tamed by $\omega$, and a $J$-holomorphic curve $C \subset M$ which is a closed subset of $M$ and passes through the point $p$, set

$$A(C, J, p) = \int_C \omega$$

and define Gromov’s width as

$$w(M, \omega) = \sup \inf_j A(C, J, p).$$

Notice that $w(M, \omega)$ is independent of the choice of the point $p$ because the symplectomorphism group of $(M, \omega)$ acts transitively on $M$.

The following theorem (see [48]) summarizes the properties of Gromov’s width.

**Theorem 1.3.** 1. For any symplectic $(M, \omega)$ we have $0 < w(M, \omega)$.

2. $w(M, \lambda \omega) = \lambda^2 w(M, \omega)$ for any $\lambda > 0$.

3. $w(D^{2n}(r)) = w(D^2(r) \times \mathbb{R}^{2n-2}) = \pi r^2$. Here $D^{2n}(r)$ is the ball of radius $r$ in $\mathbb{R}^{2n}$, which is endowed with the standard symplectic structure.

As a corollary we get the following famous Gromov non-squeezing theorem

**Corollary 1.4.** If there exists a symplectic embedding $D^{2n}(1) \to D^2(r) \times \mathbb{R}^{2n-2}$ then $r \geq 1$.

The theory of symplectic invariants pioneered by Gromov was later developed by Ekeland–Hofer [162] and Hofer–Zehnder [169], and culminated in Floer–Hofer symplectic homology theory, which they developed jointly with K. Cieliebak and K. Wysocki (see [165, 160, 167, 161]). C. Viterbo (see [196]) found an alternative finite-dimensional approach to the theory of symplectic invariants (see also Section 6 below).
2. Contact 3-manifolds

A contact structure on a 3-dimensional manifold is easy to visualize. However, contact geometry in dimension 3 is a great source of interesting and difficult symplectic geometric problems.

To fix the stage we restrict our discussion to co-orientable \textit{positive} contact structures $\xi = \{\alpha = 0\}$, where the positivity means that the contact orientation defined by the form $\alpha \wedge d\alpha$ coincides with an a priori given orientation of the manifold.

It is known since the work of J. Martinet ([35]) and R. Lutz ([33]) that any orientable contact manifold admits a contact structure in every homotopy class of tangent plane fields. However, as was first discovered by D. Bennequin ([44]), even on $S^3$ there exists a homotopy class of plane fields which can be represented by non-isomorphic contact structures. The phenomenon which causes this non-uniqueness is called \textit{overtwisting} and was studied in [54]. It turned out that it is useful to distinguish two complementary classes of contact structures. A contact structure $\xi$ on a 3-manifold $M$ is called \textit{over-twisted} if there exists an embedded 2-disc $D \subset M$ bounded by a Legendrian curve, and which is transversal to $\xi$ along $\partial D$. A non-over-twisted contact structure is called \textit{tight}. It was shown in [54] that classification of overtwisted contact structures up to isotopy coincides with their homotopical classification as plane fields. Therefore, it seems that overtwisted contact structures do not exhibit enough rigidity to make them useful for serious geometric applications.

Tight contact structures are much more rigid. They also seem to be much more useful (see, for instance, proof of Cerf's theorem "$\Gamma_4 = 0$" in [55]), and thus more difficult to understand.

2.1. Classification results

The following theorem gives a nearly exhaustive description of classification results for tight contact structures, known at this moment.

\textbf{Theorem 2.1.} 1. The following manifolds:

$$S^3, \quad \mathbb{R}P^3, \quad S^2 \times S^1,$$

and each of these manifolds minus a finite number of points, have unique, up to isotopy, tight contact structures (see [55]).

2. Let $\xi_1$ be the canonical contact structure on $T^3 = S^1 \times T^2 = T^2 \times S^1$, and $\xi_n, n = 2, \ldots$ be the pull-back of the structure $\xi_1$ by the covering map $\text{Id} \times \pi_n : T^3 = T^2 \times S^1 \rightarrow T^2 \times S^1 = T^3$, where $\pi_n(u) = nu, u \in \mathbb{R}/\mathbb{Z} = S^1$. The structures $\xi_n, n = 1, \ldots$, are tight, pairwise non-isomorphic and comprise the complete list of tight contact structures on $T^3$, up to isomorphism (see [64, 67]).

3. A connected sum $M_1 \# M_2$ of two tight contact manifolds $(M, \xi_1)$ and $(M, \xi_2)$ admits a unique tight contact structure $\xi = \xi_1 \# \xi_2$ compatible with the contact structures of the summands. Conversely, if a manifold $M$ splits as a connected sum $M = M_1 \# M_2$ then any contact structure splits uniquely up to isotopy into the connected sum $\xi = \xi_1 \# \xi_2$ (see [55]).
Recently, E. Giroux extended the classification result in Theorem 2.1.2 from $T^3$ to other toric fibrations over $S^1$, and J. Etnyre (see [60]) classified tight contact structures on certain Lens spaces. In particular, he proved that any Lens space $L(p, q)$ admits only finitely many non-isomorphic tight contact structures. Notice also that 2.1.2 classifies contact structures on $T^3$ only up to an isomorphism, i.e., a preserving contact structure diffeomorphism which need not to be isotopic to the identity. The classification of contact structures on $T^3$ up to isotopy is also known (see [64] and [178]).

As Theorem 2.1.2 shows, the number of isomorphism classes of tight contact structures on a given manifold need not be finite, even in the same homotopy class of plane fields. However, this phenomenon could be related to the presence of incompressible tori: I do not know any atoroidal manifold with infinitely many non-isomorphic contact structures. (A surface in a 3-manifold is called incompressible if its fundamental group injects into the fundamental group of the ambient manifold.) It is also likely that on any 3-manifold only finitely many homotopy classes of tangent plane fields can be realized by tight contact structures. Although this is still unknown, a result of P. Kronheimer and T. Mrowka (see [68]) establishes the finiteness of homotopy classes of plane fields realizable by symplectically semifillable structures (see the next section for the definition of semi-fillability).

Let us also mention here a recent result by V. Colin (see [53]) which states that $C^0$-close tight contact structures are isotopic.

2.2. Recognizing and constructing tight contact structures

It is not easy to verify whether a contact structure is tight. Even the tightness of the standard contact structure on $S^3$ is a highly non-trivial fact which was proven by D. Bennequin in 1982 (see [44]).

A contact 3-manifold $(M, \xi)$ is called symplectically fillable if there exists a compact symplectic manifold $(W, \omega)$ with boundary, such that

- $\partial W = M$;
- the form $\omega|_\xi$ does not vanish;
- the contact orientation of $(M, \xi)$ coincides with the orientation of $M$ as the boundary of the symplectically oriented manifold $(W, \omega)$.

The manifold $(M, \xi)$ is called symplectically semi-fillable if it is a connected component of a symplectically fillable contact manifold. The following theorem (see [48] and [59]) provides an effective criterion for tightness.

**Theorem 2.2.** A symplectically semi-fillable contact structure is tight.

An important class of symplectically fillable manifolds consists of holomorphically fillable ones. Any contact plane bundle $\xi = \{\alpha = 0\}$ admits a structure of a complex bundle, such that the complex structure $J : \xi \to \xi$ is compatible with the symplectic form $d\alpha|_\xi$ (see Section 1.4 above). Moreover, in the 3-dimensional case this $J$, called a CR-structure, always can be chosen integrable and extendable as a complex structure to $W = M \times (\epsilon, \epsilon) \supset M \times 0 = M$. We assume here that the splitting of $W$ is chosen in such a way that the vector field $JR_\alpha$, where $R_\alpha$ is the Reeb vector field determined by the 1-form $\alpha$, is an inward transversal to the boundary $M \times 0$. 

of the domain $W_+ = M \times (-\varepsilon, 0]$. From the complex analytic point of view this means that $M \times 0$ is a strictly pseudo-convex boundary of the domain $W_+ = M \times (-\varepsilon, 0]$. If the complex manifold $(\mathcal{W}, J)$ extends to a compact complex manifold $\mathcal{W}$ with boundary $M$, then the contact structure $\xi$ is called holomorphically fillable. In this case the manifold $\mathcal{W}$ is actually Kähler, and hence symplectic, and the contact manifold $(M, \xi)$ serves as a contact boundary of the symplectic manifold $\mathcal{W}$. All the structures on closed 3-manifolds mentioned in Theorem 2.1 from Section 2.1, except the structures $\xi_n$ on $T^3$ for $n > 1$ are holomorphically fillable. On the torus $T^3$ the standard structure $\xi$ is a unique, up to a contactomorphism, holomorphically fillable contact structure (see [56]).

Theorem 2.3 below (see [203]) gives a complete, although not very constructive description of all holomorphically fillable contact structures.

First notice that any embedded Legendrian circle $S$ in a contact 3-manifold $(M, \xi = \{\alpha = 0\})$ admits a canonical framing. Namely, choose an orientation of $S$ by a tangent vector field $\tau$ (the final result of the construction will be independent of the choice of the orientation), and take the framing $(R_\alpha, J_T)$. It is independent, up to homotopy, of the choice of the contact form $\alpha$ and a compatible complex structure $J : \mathbb{C} \to \mathcal{W}$. It is shown in [203] (see also [236]) that the Morse surgery along $S$ with respect to a framing which differs from the canonical one by the rotation by $-2\pi$, can be performed, in a unique way, in the category of holomorphically fillable contact manifolds.

**Theorem 2.3.** Any holomorphically fillable contact manifold can be obtained from the standard contact structure on the connected sum of $k$ copies of $S^2 \times S^1$ by a sequence of Legendrian surgeries.

Notice that any knot is isotopic to a Legendrian one. Moreover, if one gets a Legendrian realization of a knot with certain framing then all framings which differ by a rotation by a negative multiple of $2\pi$ can also be realized. This observation shows that there are a lot of 3-manifolds which carry holomorphically fillable contact structures. In [66] R. Gompf systematically studied what kind of manifolds one can obtain by this construction. In particular, he proved

**Theorem 2.4.** a) Any oriented Seifert fiber space carries a holomorphically fillable, possibly negative contact structure.

b) An oriented Seifert fiber space carries a positive holomorphically fillable contact structure, except possibly the case when the base of the fibration is $S^2$.

Gompf’s actual theorem provides more constructions of holomorphically fillable contact structures, and more detailed description of exceptional cases in 2.4a.

The theory of 2-dimensional foliations on 3-manifolds provides a rich source of constructions of symplectically semifillable contact structures. In the discussion below we do not distinguish between codimension one foliations and integrable plane fields on 3-manifolds. Let us recall that an important class of foliations on 3-manifolds is formed by taut foliations. A foliation $\xi$ is called taut if there exists a closed transversal curve which intersects all leaves of the foliation. Equivalently, taut foliations can be characterized by existence of a closed 2-form $\omega$ such that $\omega|_\xi$ nowhere vanishes.
Taut foliations are contained in a larger class of Reebless foliations, i.e., foliations without Reeb components. The following theorem is proved in [6].

**Theorem 2.5.**  
(a) Any co-orientable foliation, except the foliation of the product \(S^2 \times S^1\) by the horizontal 2-spheres, can be \(C^0\)-approximated by contact structures, both positive and negative.

(b) A contact structure, \(C^0\)-close to a taut foliation is symplectically semi-fillable.

(c) A contact structure, \(C^0\)-close to a Reebless foliation is tight.

In [207] D. Gabai constructed a lot of taut foliations on 3-manifolds. In particular, he proved that if a surface in an irreducible 3-manifold has minimal genus \(> 0\) in its homology class then it can be included as a leaf into a taut foliation.

It is possible that Theorems 2.3 and 2.5 provide enough tools to construct tight, or symplectically (semi-)fillable contact structures on all irreducible orientable 3-manifolds. Notice, however, that recently P. Lisca (see [7]) proved that the Poincaré homology 3-sphere \(P\) with one of its orientations has no positive symplectically semi-fillable contact structure. It follows then from Theorem 2.1.3 that \(P\#(-P)\) has no symplectically semi-fillable contact structure at all.

### 3. Hofer geometry

One of the most remarkable manifestations of symplectic rigidity is the existence of a bi-invariant metric on the group of Hamiltonian symplectomorphisms. The existence of a bi-invariant metric is highly unusual for non-compact groups of transformations. For instance, the group of volume preserving diffeomorphisms of a manifold of dimension \(\geq 3\) does not admit such a metric, as there exist volume preserving diffeomorphisms whose conjugacy classes contain the identity in their \(C^\infty\)-closure.

This metric was first discovered by H. Hofer in his seminal paper (see [106]) which laid the foundation of what is now called Hofer geometry.

Let \(\text{Ham} = \text{Ham}(M, \omega)\) be the group of (compactly supported) Hamiltonian symplectomorphisms of a symplectic manifold \((M, \omega)\). Given a Hamiltonian isotopy \(\varphi_t\), defined by a time-dependent Hamiltonian function \(H_t : M \to \mathbb{R}, t \in [0, 1]\), we set

\[
\|\{\varphi_t\}\| = \sup_{x, t} H_t(x) - \inf_{x, t} H_t(x),
\]

where the sup and inf are taken over all \((x, t) \in M \times [0, 1]\). We further set for \(\varphi \in \text{Ham}\)

\[
\|\varphi\| = \inf \|\{\varphi_t\}\|,
\]

where the inf is taken over all compactly supported Hamiltonian isotopies \(\{\varphi_t\}\) with \(\varphi_1 = \varphi\). It is easy to check that \(\|\cdot\|\) is a seminorm, and that \(\rho(f, g) = \|fg^{-1}\|\) is a bi-invariant pseudo-metric on the group \(\text{Ham}\). However, it is highly non-trivial to show that

**Theorem 3.1.** The semi-norm \(\|\cdot\|\) is non-degenerate, i.e., defines a norm, which is called theHofer norm, on the group of Hamiltonian symplectomorphisms.

The non-triviality of the Hofer metric is a corollary of the following statement, called the energy-capacity inequality, which is very interesting by itself. Given a compact subset \(A \subset M\)
(the manifold $M$ is assumed to be without boundary) we define the *displacement energy* of $A$ as

$$e(A) = \inf \{ \| \varphi \| ; \varphi \in \text{Ham}(M), \varphi(A) \cap A = \emptyset \}.$$ 

We also define a *capacity* of $A$ as

$$c(A) = \sup \{ \pi r^2 ; \text{there exists a symplectic embedding } B^{2n}(r) \text{ into } \text{Int}A \}$$

where $B^{2n}(r)$ is the ball of radius $r$ in the standard symplectic $\mathbb{R}^{2n}$.

**Theorem 3.2.** For any compact set $A \subset M$ one has the inequality

$$e(A) \geq \frac{1}{2} c(A).$$

Theorems 3.1 and 3.2 were first established by H. Hofer (see [106]), and independently but slightly later by C. Viterbo (see [196]) for the case when $(M, \omega)$ is the standard symplectic $\mathbb{R}^{2n}$. Notice that in this case the coefficient $\frac{1}{2}$ can be dropped from the inequality in Theorem 3.2. These results were later generalized by the efforts of many people to the case of a general symplectic manifold, and in the final form were proven by F. Lalonde and D. McDuff in [110].

It is not difficult (see, for instance, [113]) to check that

$$\| \varphi \| = \inf_{H_t} \max_{t \in [0,1]} \max_{x \in M} \left( H_t(x) - \min_{x \in M} H_t(x) \right)$$

$$= \inf_{H_t} \int_0^1 \left( \max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) \right) dt.$$ 

The latter equality shows that the Hofer metric $\rho$ defines a *length structure* in the sense of Gromov (see [212]) on the group Ham. In other words, one gets the Hofer metric starting with the $L^\infty$-type norm on the Lie algebra of the group Ham, which is just the space $C^\infty(M)$ of smooth functions on $M$. Next, one computes the length of a path by integrating the length of its velocity vector, and finally defines the distance between two diffeomorphisms as the infimum of lengths of connecting paths. A natural question is, what is the general class of norms on $C^\infty$-functions which produces via this construction bi-invariant metrics on the group of symplectomorphisms. The bi-invariance condition just means that the norm is invariant under symplectic changes of coordinates. However, as it is shown in [105], most norms generate only pseudo-metrics, usually identically equal to 0, i.e., the Hofer metric is essentially unique. However, the problem of the complete description of bi-invariant non-degenerate (Finsler) metrics on the group Ham is still open.

Hofer geometry is an alive and active area. An interesting question studied by many mathematicians concerns the diameter of the group $\text{Ham}(M, \omega)$ in Hofer metric. Is diam$(M, \omega)$ finite or infinite? It is infinite in all known cases. In the case when $(M, \omega)$ is non-compact it is easy to see that diam$(M, \omega) = \infty$, because the Hofer norm $\| f \|$ is bounded below by the absolute value of the Calabi invariant Cal$(f)$ (see 1.3 above) which obviously can take arbitrarily large values. For closed manifolds this is more subtle. F. Lalonde and D. McDuff (see [108]) proved the infiniteness of the diameter for all surfaces of genus $> 0$, as well as for high-dimensional tori. This result was extended by M. Schwarz [175] to many symplectically aspherical manifolds.
(i.e., manifolds satisfying the condition $\omega|_{\tau_2(M)} = 0$). Recently L. Polterovich [113] developed new techniques for dealing with the spherical case and proved the infiniteness of the diameter for a large class of manifolds, including $S^2$ and $\mathbb{C}P^2$.

Let me also mention here the results of M. Bialy and L. Polterovich ([156, 157, 158]) relating Hofer geometry and Hamiltonian dynamics, and the results of I. Ustilovsky [114], M. Bialy and L. Polterovich [104] and F. Lalonde and D. McDuff [108] on the structure of geodesics in Hofer geometry.

4. Donaldson’s theory of approximately complex submanifolds

Symplectic manifolds have a lot of symplectic submanifolds of codimension $> 2$. As was shown by M. Gromov (see [30] and [7]) one has the following $h$-principle for symplectic embeddings:

**Theorem 4.1.** Let $(M_0, \omega_0)$ and $(M_1, \omega_1)$ be two symplectic manifolds of dimension $2n$ and $2(n + k)$, $k \geq 2$, respectively. Suppose that an embedding $f : M_0 \to M_1$ satisfies the following two conditions:
- the cohomology classes of the forms $\omega_0$ and $f^* \omega_1$ coincide;
- the differential $df : T(M_0) \to T(M_1)$ is homotopic through injective homomorphisms to a symplectic homomorphism.

Then $f$ is isotopic to a symplectic embedding.

This $h$-principle fails completely for symplectic embeddings in codimension two. However, S. Donaldson proved recently a surprising theorem, analogous to the Kodaira embedding theorem in Kähler geometry. In particular, Donaldson’s result provides us with an effective tool for the construction of codimension two symplectic submanifolds, which are analogous to hyperplane sections of complex projective varieties.

Let $(M, \omega)$ be a closed symplectic manifold and suppose that the cohomology class $[\omega]$ of its symplectic form is integral. Let $L_k \to M$ be a complex line bundle whose first Chern class is equal to $k[\omega]$, $k \in \mathbb{Z}$. The bundle $L_k$ admits an Hermitian metric and a compatible connection $c$, which is also compatible with the complex structure on $L_k$, and whose curvature form equals $k\omega$. The connection $c$, together with the complex structure of the bundle and an almost complex structure on $M$, compatible with $\omega$, defines an almost complex structure $\bar{J}$ on the total space $L_k$ of the line bundle. This almost complex structure is non-integrable in general, and in particular, one cannot hope to have holomorphic sections $M \to L_k$. However, Donaldson proved that one can still construct *approximately holomorphic* sections, and the larger $k$ is, the better approximation can be obtained. Approximate holomorphicity of a section $s : M \to L_k$ means that the differential $df : T(M) \to T(L_k)$ does not deviate much from a complex linear homomorphism, or more precisely, that $|\bar{\partial}s| < |\partial s|$. If one can make $s$ sufficiently transversal to the 0-section, so that the angle between the section and the 0-section is much larger than the deviation of $s$ from a holomorphic section, then $s^{-1}(0)$ will be an approximately holomorphic, and hence symplectic submanifold of codimension 2. Donaldson realized this approach using Yomdin’s refinement of Sard’s theorem.

Here is Donaldson’s result (see [138]).
Theorem 4.2. Let \((M, \omega)\) and \(L_k\) be as above. Then for a sufficiently large \(k\) there exists an approximately holomorphic section \(s : M \to L_k\) sufficiently transversal to the 0-section. In particular, the homology class from \(H_{2n-2}(M, \mathbb{R})\), dual to \(k[\omega]\) can be represented by a symplectic submanifold. Moreover, all symplectic submanifolds obtained by this construction are Hamiltonian isotopic. The real-valued function \(-\log|s|\) on \(\widetilde{M} = M \setminus s^{-1}(0)\) has critical points of index \(\leq n\), and in particular \(\widetilde{M}\) is homotopy equivalent to an \(n\)-dimensional cell complex.

Very recently S. Donaldson ([139]) announced further developments in his theory of approximately holomorphic sections. In particular, one of his new results asserts existence of symplectic singular fibrations, similar to Lefschetz pencils in Algebraic Geometry.

Here is one of Donaldson's new results in this direction.

Theorem 4.3. Let \((M, \omega)\) be a closed symplectic 4-manifold whose symplectic form represents an integral cohomology class. Then for a sufficiently large \(k > 0\) one can blow-up the manifold \(M\) at \(d = k^2 \int_M \omega^2\) points (see [7, 127, 213]), so that the resultant manifold \((\widetilde{M}, \tilde{\omega})\) admits a smooth map \(p : \widetilde{M} \to S^2\) with the following properties:

a) there exists a finite set of points \(\Sigma = \{z_1, \ldots, z_l\}, z_i \in S^2\), such that \(p|_{p^{-1}(S^2 \setminus \Sigma)} : p^{-1}(S^2 \setminus \Sigma) \to S^2 \setminus \Sigma\) is a fibration with symplectic fibers \(E_1, \ldots, E_l\);

b) each exceptional fiber \(E_i = p^{-1}(z_i), i = 1, \ldots, l\), is an immersed symplectic surface with a single transversal double point \(d_i \in E_i\);

c) in a neighborhood \(U_i\) of each of the points \(d_i, i = 1, \ldots, l\), there exists an integrable complex structure, compatible with \(\omega\), and such that the map \(p|_{U_i} : U_i \to S^2\) is holomorphic and \(d_i\) is a unique, non-degenerate critical point of the holomorphic function \(p|_{U_i}\).

It is likely that Theorem 4.3 will play an important role for understanding the topology of symplectic (4-dimensional?) manifolds. In fact, Theorems 4.2 and 4.3 suggest that symplectic structures may be more important and interesting than the smooth ones. For instance, the differential topology in dimension 4 is much richer than the higher-dimensional one. On the other hand, Theorem 4.2 shows that the symplectic topology of higher-dimensional manifolds is at least as rich as the symplectic topology of 4-dimensional manifolds.

In dimension 4 the difference between smooth and symplectic structures does not look so dramatic. For instance, it is possible that the real difference in dimension 4 is not between diffeomorphism and homeomorphism but rather between symplectic and non-symplectic. One may expect that if two symplectic manifolds are homeomorphic but not diffeomorphic, then actually there is no symplectic homeomorphism (see [10]) between them. This mysterious notion is yet to be properly defined and understood.

5. Symplectic 4-manifolds

In dimension 4 an alternative technique for finding codimension 2 symplectic submanifolds is provided by C. Taubes' discovery of relations between \(J\)-holomorphic curves and solutions to the Seiberg–Witten equations.
Notice that in all dimensions a symplectic submanifold is a complex submanifold for a suitable choice of a compatible complex structure, tamed by the symplectic form. However, in higher dimension these complex submanifolds are highly unstable: the subspace of almost complex structures which have any, even local complex submanifolds of complex dimension $> 1$ has an infinite codimension in the space of all almost complex structures.

On the other hand all almost complex manifolds have, at least locally, plenty of 1-dimensional complex submanifolds, or (pseudo-)holomorphic curves. It was a major discovery of M. Gromov, that in the presence of a taming symplectic structure one has a powerful global theory of holomorphic curves. Because of the positivity of intersections, the holomorphic curves technique is especially effective in the 4-dimensional symplectic topology.

We begin with the following theorem by Gromov (see [48]) which has already become classical.

**Theorem 5.1.** Let $(M, \omega)$ be a connected symplectic 4-manifold which coincides with the standard symplectic $\mathbb{R}^4$ outside a compact set. Then if $M$ contains no symplectic 2-spheres (for instance if $\pi_2(M) = 0$) then $(M, \omega)$ is symplectomorphic to the standard symplectic $\mathbb{R}^4$.

D. McDuff extended (see [125]) this theorem to the case when symplectic 2-spheres may be present. In this case $(M, \omega)$ is symplectomorphic to $\mathbb{R}^4$ with a few points blown up.

Gromov's theorem, together with McDuff's extension can be reformulated as follows:

A closed symplectic 4-manifold, which contains a symplectic 2-sphere with self-intersection index $+1$, is symplectomorphic to $\mathbb{C}P^2$ with the standard (Fubini–Study) symplectic form, possibly blown up at a few points.

Removing the assumption about the existence of a symplectic 2-sphere with the self-intersection index $+1$ had seemed to be out of reach of the methods of Symplectic Topology until Taubes found a link between the theory of holomorphic curves and the Seiberg–Witten theory.

We cannot discuss in the framework of this survey neither Seiberg–Witten theory, nor the subtle precise definition of Gromov invariants counting the number of holomorphic curves, which is needed for the complete formulation of Taubes' result. Thus we restrict ourself only to some corollaries of Taubes' theory related to the problem of existence of holomorphic curves, and refer the reader to Taubes' papers [134, 135, 136, 133] and McDuff's survey [15] and the bibliography therein for the detailed account of the story.

In what follows we use the same notations for a homology class and its Poincaré dual cohomology class. The distinction should be clear from the context. Given a symplectic manifold $(M, \omega)$ we denote by $K$ its canonical class $-c_1(T(M), J)$, where $J$ is any almost complex structure compatible with $\omega$, and $c_1(T(M), J)$ is the first Chern class of the complex bundle $(T(M), J)$.

Here are two theorems of Taubes (see [134, 135]).

**Theorem 5.2.** Let $(M, \omega)$ be a closed symplectic 4-dimensional manifold with $b_2^+ > 1$ ($b_2^+$ is the positive index of the intersection form of the 4-manifold $M$). Suppose that $K \neq 0$. Then for a generic almost complex structure $J$ compatible with $\omega$
the homology class dual to the canonical class $K$ can be represented by an embedded holomorphic curve;

b) if a homology class $A \in H_2(M)$ with $A^2 = -1$ and $K \cdot A \neq 0$ can be represented by an embedded 2-sphere then either $A$, or $-A$ can be represented by an embedded holomorphic sphere.

**Theorem 5.3.** The complex projective space $\mathbb{C}P^2$ (considered as a smooth manifold) endowed with a generic almost complex structure compatible with an arbitrary symplectic structure $\omega$ has an embedded 2-sphere with the self-intersection index $+1$.

Theorem 5.3, together with Gromov's Theorem 5.1, implies uniqueness (up to diffeomorphism) of a symplectic structure on $\mathbb{C}P^2$ compatible with the standard smooth structure.

For symplectic manifolds with $b_2^+ = 1$ and $b_1 = 0$ with the help of the "wall crossing formula" of Kronheimer–Mrowka (see [120], and also [16, 126] for applications) Taubes' theory implies:

**Theorem 5.4.** Let $(M, \omega)$ be a symplectic 4-manifold with $b_2^+ = 1$ and $b_1 = 0$. Then for a generic compatible almost complex structure $J$, and any homology class $A \in H_2(M)$, $A \neq 0$, $K$, for which $d(A) = \frac{1}{2} (A \cdot A - K \cdot A) \geq 0$, one of the classes $A$, or $K - A$ can be represented by a holomorphic curve $C$ passing through any given $d(A)$ points of $M$.

Theorem 5.4 can be generalized to a larger class of symplectic manifolds, of so-called non-simple Seiberg–Witten type. The generalization requires a more advanced wall-crossing formula (see [129, 123, 130]).

For a generic $J$, a symplectic manifold $M$ has no connected $J$-holomorphic curves of negative self-intersections, except for smooth holomorphic spheres with self-intersection $-1$. These spheres can always be blown down (see [125]), so one can assume that the manifold $M$ is minimal in the sense that it does not have any holomorphic curves of negative self-intersection. In this case, we have the following result, see [15].

**Theorem 5.5.** For a generic choice of $d(A)$ points, the holomorphic curve $C$ provided by Theorem 5.4 is a smooth curve which consists of several, possibly multiply covered, disjoint components $\Sigma_i$, $i = 1, \ldots, p$, with $\Sigma_i^2 \geq 0$. Multiply covered components can only be tori. If $C^2 > 0$ then $p = 1$, i.e., the curve $C$ is irreducible. If $C^2 = 0$ then $C$ may have several components $A_i$, $i = 1, \ldots, k$, with $A_i^2 = 0$. All these components $A_i$ are either spheres representing the same homology class (and in this case $M$ is a ruled surface), or all the $A_i$'s are tori, representing proportional homology classes.

Due to efforts of several mathematicians the theory of holomorphic curves in symplectic 4-manifolds found a lot of new applications, in particular for the topology of symplectic 4-manifolds and contact 3-manifolds (see [68, 118, 117] et al.). Let me recall that M. Gromov proved in [48] that the group $\text{Diff}_\omega$ of symplectomorphisms of $S^2 \times S^2$ with the split form $\omega = \sigma \oplus \sigma$, where $\sigma$ is an area form on $S^2$, deformation retracts to the subgroup of orientation-preserving orthogonal transformations. It was also indicated in [48] that this result is no longer true when the factors have different symplectic area. M. Abreu (see [115])
and M. Abreu–D. McDuff (see [116]) completely described the homological type of the group \( \text{Diff}_\omega(S^2 \times S^2) \) for the general symplectic form \( \omega \).

D. McDuff and F. Lalonde used Taubes’ and Gromov’s theorems to obtain a complete classification of symplectic structures on ruled and rational symplectic manifolds (see [125] and [122]). D. McDuff’s classification of symplectic structures on rational surfaces implies the following striking result (see [126, 127], and also Biran’s and Lalonde’s papers [201] and [121]). For positive real numbers \( r_1, \ldots, r_k \) we denote by \( \text{Emb}(r_1, \ldots, r_k) \) the space of symplectic embeddings of the disjoint union of balls \( B(r_1), \ldots, B(r_k) \) of radii \( r_1, \ldots, r_k \) into the unit ball \( B(1) \subset \mathbb{R}^4 \) with the standard symplectic structure.

**Corollary 5.6.** For any positive real numbers \( r_1, \ldots, r_k \) the space \( \text{Emb}(r_1, \ldots, r_k) \) is connected.

Note that for certain choices of \( r_1, \ldots, r_k \) the space \( \text{Emb}(r_1, \ldots, r_k) \) can be empty (see [48] and [128]).

This result sounds to me as counter-intuitive. Indeed, the non-squeezing theorem of Gromov (see Corollary 1.4 above) and different packing inequalities (see [48, 128] et al.) lead to believe that symplectic embeddings of balls behave more like isometric embeddings, than volume preserving ones. On the other hand, an analog of 5.6 is obviously wrong for isometric embeddings.

Using Taubes’ theory together with Donaldson’s theory described in Section 4, P. Biran obtained a nearly complete solution to the symplectic packing problem (see [199] and [200]).

### 6. Generating functions and their applications

We will discuss in this section a finite-dimensional approach, called the **method of generating functions**, which in certain cases allows us to get results which seemed to be currently unaccessible by holomorphic methods.

#### 6.1. The direct image construction

The **direct image construction**, which we describe below, is a partial case of a **Lagrangian correspondence**, see [31, 187].

As it was mentioned above, a function \( f : M \to \mathbb{R} \) generates an exact graphical Lagrangian submanifold \( L_f \subset T^*(M) \) and a graphical Legendrian submanifold \( \mathcal{L}_f \subset J^1(M) \).

Given a smooth map \( h : M \to N \), one can define, under certain transversality assumptions, the direct-image construction which allows us to transport Lagrangian submanifolds of \( T^*(M) \) into immersed Lagrangian submanifolds of \( T^*(N) \), and Legendrian submanifolds of \( J^1(M) \) into Legendrian submanifolds of \( J^1(N) \). In the Lagrangian case, for instance, it can be done as follows. Consider the manifold \( W = T^*(M) \times T^*(N) \) with the symplectic structure \( \Omega = -dp \wedge dq + d\tilde{p} \wedge d\tilde{q} \). Set

\[
H = \{(d_{\tilde{h}(q)}h)^*\tilde{p}, q, \tilde{p}, h(q)) ; q \in M, \tilde{p} \in T^*_{\tilde{h}(q)}(N) \}.
\]
Then $H$ is a Lagrangian submanifold of $(W, \Omega)$. Suppose that for a Lagrangian submanifold $L \subset T^*(M)$ the product $L \times N \subset T^*(M) \times T^*(N) = W$ is transverse to $H$. It is then straightforward to see that the restriction of the projection $\pi : W \to T^*(N)$ to $L \cap H$ is a Lagrangian immersion $L \cap H \to T^*(N)$. The image $L_h = \pi(L \cap H) \subset T^*(N)$ is the required direct image of $L$.

If $L = L_f$ is a graphical Lagrangian submanifold in $T^*(M)$, i.e.,

$$L = \left\{ p = \frac{\partial f}{\partial q} \right\},$$

then we say that $L_h$ is a subgraphical Lagrangian submanifold generated by the function $f$ with respect to the map $h : M \to N$. When $h$ is a diffeomorphism then $L_h = h_*(L)$, where $h_* : T^*(M) \to T^*(N)$ is the symplectic lift of the diffeomorphism $h$, as in Section 1.1 above.

The same holds when $h$ is an equidimensional immersion. When $h : M \to N$ is an embedding (or immersion) into a manifold of a larger dimension, then $L_h$ can be described as follows. Set $M' = h(M) \subset N$ and consider the restriction map $pr : T^*(N)_{|M'} \to T^*(M')$. Then $L_h = pr^{-1}(h_*(L))$, where $h_* : T^*(M) \to T^*(M')$ is the symplectomorphism induced by the embedding $h$.

The most interesting examples of the direct image construction are when $h : M \to N$ is a submersion and, in particular, a fibration. Suppose, for instance, $M = N \times F$ and $h : M \to N$ is the projection to the second factor. Let $(q, \eta), q \in N, \eta \in F$, be coordinates in $M$, and $(p, \xi)$ dual coordinates in the cotangent bundle, so that the canonical symplectic form in $T^*(M)$ is given by the form $dp \wedge dq + d\xi \wedge d\eta$. Let $L = L_f$ be a graphical Lagrangian submanifold. In the coordinates $(q, \eta, \phi, \xi)$ the manifold $L$ is defined by the equations

$$p = f_q(q, \eta),$$
$$\xi = f_\eta(q, \eta).$$

Then the subgraphical submanifold $L_h$ in $T^*(N)$ is defined by the equations

$$L_h = \left\{ p = f_q(q, \eta), f_\eta(q, \eta) = 0. \right\}$$

Let us consider another extreme case of the direct image construction. Suppose that we are still in the situation when $M = N \times F$ and $h : M \to N$ is the projection to the second factor. Notice that $T^*(M)$ symplectically splits into the product $T^*(N) \times T^*(F)$. Suppose that a Lagrangian submanifold $L \subset T^*(M)$ is the product $L = L_1 \times L_2$, where $L_1, L_2$ are Lagrangian submanifolds of $T^*(N)$ and $T^*(F)$, respectively, and $L_2$ intersects the zero-section in $T^*(F)$ transversally at $k$ points. Then $L_h$ consists of $k$ copies of $L_1$. In particular, when $F = \mathbb{R}^k$ and $L_2$ is a Lagrangian plane in $T^*(\mathbb{R}^k) = \mathbb{R}^{2k}$, transversal to the zero-section, then $L_h = L_1$. In the language of generating functions this can be expressed as follows. Suppose that $L = L_f$ is a graphical Lagrangian submanifold in $T^*(M)$ and $Q : \mathbb{R}^k \to \mathbb{R}$ a non-degenerate quadratic form. Given a map $h : M \to N$, the subgraphical submanifold $L \subset T^*(N)$ generated by the function $f$ with respect to the map $h$ coincides with the subgraphical submanifold generated
by the function $f \oplus Q: M \times \mathbb{R}^k \to \mathbb{R}$ with respect to the composition $M \times \mathbb{R}^k \xrightarrow{\text{proj}} M \xrightarrow{h} N$.

The definition of the direct image construction in the Legendrian case is similar.

6.2. Quadratic stabilization

To make functions on non-compact manifolds amenable to Morse theory we will assume that all considered functions are fibrations at infinity (see [7, 187]).

A function $f: M \to \mathbb{R}$ is called a fibration at infinity, if there exist a finite segment $[-a, a] \subset \mathbb{R}$ and a compact subset $K \subset f^{-1}[-a, a] \subset M$ such that the restriction of $f$ to the following three subsets fibers them over their respective images

1. $f^{-1}(-\infty, -a) \to (-\infty, -a]$,
2. $f^{-1}[a, \infty) \to [a, \infty)$,
3. $(f^{-1}[-a, a]) - K \to [-a, a]$.

Let us fix a smooth map $h: M \to N$ and a fibration at infinity $f: M \to \mathbb{R}$. Given a Lagrangian submanifold $L \subset T^*(M)$ (or a Legendrian submanifold $L \subset J^1(M)$) we denote by $\mathcal{F}(L, h, f)$ (resp. $\mathcal{F}(L, h, f)$) the space of all functions $f: M \to \mathbb{R}$ which coincide with $f$ at infinity and transversally generate $L$ (resp. $L$) with respect to $h$. Let us stabilize the spaces $\mathcal{F}(L, h, f)$ and $\mathcal{F}(L, h, f)$ as follows. Consider a sequence of spaces $\mathcal{F}(L, h_k, f \oplus Q_k)$ and $\mathcal{F}_k(L, h_k, f \oplus Q_k), k = 1, \ldots$, where

$$Q_k(\eta) = \eta_1\eta_2 + \ldots + \eta_{2k-1}\eta_{2k}, \quad \eta = (\eta_1, \ldots, \eta_{2k}) \in \mathbb{R}^{2k},$$

is a non-degenerate quadratic form on $\mathbb{R}^{2k}$, and $h_k$ is the composition $M \times \mathbb{R}^{2k} \xrightarrow{\text{proj}} M \xrightarrow{h} N$. Now take the direct limits

$$\mathcal{F}(L, h, f) \hookrightarrow \mathcal{F}(L, h_1, f \oplus Q_1) \hookrightarrow \ldots \hookrightarrow \mathcal{F}(L, h_k, f \oplus Q_k) \hookrightarrow \ldots,$$

and similarly in the Legendrian case, defined by the above described stabilization construction.

6.3. The covering homotopy property

We denote by $\mathcal{Lag} = \mathcal{Lag}(N, L_0)$ the space of exact embedded Lagrangian submanifolds of $T^*(N)$, which coincide with a fixed Lagrangian submanifold $L_0$ at infinity. Let us set

$$\mathcal{F}^\text{st}_{\mathcal{Lag}} = \mathcal{F}^\text{st}_{\mathcal{Lag}}(L_0, h, f) = \bigcup_{L \in \mathcal{Lag}(N, L_0)} \mathcal{F}^\text{st}(L, h, f),$$

and denote by $\text{Gen}$ the projection $\mathcal{F}^\text{st}_{\mathcal{Lag}} \to \mathcal{Lag}$ which associates with a function from $\mathcal{F}^\text{st}_{\mathcal{Lag}}$ the Lagrangian submanifold from $\mathcal{Lag}$ which it generates.

Similarly we define spaces $\mathcal{F}^\text{st}_{\mathcal{Leg}} = \mathcal{F}^\text{st}_{\mathcal{Leg}}(L_0, h, f)$ and $\mathcal{Leg} = \mathcal{Leg}(N, L_0)$ and the projection $\text{Gen} : \mathcal{F}^\text{st}_{\mathcal{Leg}} \to \mathcal{Leg}$ in the Legendrian case.

The following Theorem 6.1 (see [185, 194, 192, 186, 196, 195, 187]) is the main result of the theory of generating functions.
**Theorem 6.1.** The maps \( \text{Gen} : \mathcal{F}^*_{\text{Lag}} \to \text{Lag} \) and \( \text{Gen} : \mathcal{F}^*_{\text{Leg}} \to \text{Leg} \) are Serre fibrations over connected components which intersect the image of the map \( \text{Gen} \).

In particular, if a Legendrian submanifold \( \mathcal{L}_0 \) can be generated by a function \( f : M \to \mathbb{R} \) with respect to a map \( h : M \to N \), and a Legendrian submanifold \( \mathcal{L} \) is isotopic to \( \mathcal{L}_0 \) via a compactly supported Legendrian isotopy then \( \mathcal{L} \) can be generated by a function \( g : M \times \mathbb{R}^{2k} \to \mathbb{R} \) with respect to the map \( h_k : M \times \mathbb{R}^{2k} \to N \), such that \( g \) coincides with \( f \oplus Q_k \) at infinity.

Here are few corollaries of Theorem 6.1. Suppose that \( N \) is a closed manifold. Let us denote by \( \text{stabMor}(N) \) and \( \text{stabLuS}(N) \) the stable Morse and Lusternik–Shnirelman numbers of the manifold \( N \). These are the minimal number of critical points of a function \( f(x, y), x \in N, y \in \mathbb{R}^{2k}, k = 0, \ldots \), which is equal to \( Q_k(y) \) outside a compact set. In the Morse case, the critical points are required to be non-degenerate. The lower bounds on the numbers \( \text{stabMor}(N) \) and \( \text{stabLuS}(N) \) in terms of topology of \( N \) is the subject of stable Morse–Lusternik–Shnirelman theory (see [231] and [187]). The most commonly used bounds are the Morse inequality

\[
\text{stabMor}(N) \geq \text{rank} \, H_*(N),
\]

and the Lusternik–Shnirelman inequality

\[
\text{stabLuS}(N) \geq \text{cuplength}(N).
\]

However, when \( N \) is not simply-connected these inequalities can be essentially improved. See [231] and [187] for discussions of the corresponding results.

Let us denote by \( L_0 \) the zero-section in \( T^*(N) \). Suppose that \( L \) is an exact Lagrangian submanifold of \( T^*(N) \) which is Lagrangian isotopic to \( L_0 \). We denote by \( \cap(L, L_0) \) the cardinality of the intersection \( L \cap L_0 \), and by \( \hat{\cap}(L, L_0) \) the same number in the case when the intersection is transversal.

Then Theorem 6.1 implies

**Corollary 6.2.**

\[
\hat{\cap}(L, L_0) \geq \text{stabMor}(N);
\]

\[
\cap(L, L_0) \geq \text{stabLuS}(N).
\]

The corresponding bounds in terms of Betti numbers and the cup-length constitute one of Arnold’s symplectic conjectures and were first proven by M. Chaperon (for the case of \( N = T^{2n} \), see [185]), F. Laudenbach and J.-C. Sikorav (see [192]) and H. Hofer (see [92]).

Corollary 6.2 uses only the existence of generating functions, but not the full strength of Theorem 6.1. The next Corollary 6.3 uses the multi-parametric part of this theorem.

Suppose that \( V = N \times \mathbb{R} \) for a closed manifold \( N \) and denote by \( \pi \) the projection \( V = N \times \mathbb{R} \to \mathbb{R} \). Let \( \text{Lag}_0 \) be the space of exact Lagrangian submanifolds of \( T^*(V) \) which are isotopic via a compactly supported Lagrangian isotopy to the graphical submanifold \( L_\pi \). Similarly, the notation \( \text{Leg}_0 \) stands for the space of Legendrian submanifolds of \( J^1(V) \) which are Legendrian isotopic to \( L_\pi \) via a compactly supported Legendrian isotopy. Any Lagrangian submanifold from \( \text{Lag}_0 \) uniquely lifts to a Legendrian submanifold from \( \text{Leg}_0 \). This defines an inclusion \( i : \text{Lag}_0 \hookrightarrow \text{Leg}_0 \).
Let $\mathcal{P}(N)$ be the identity component of the \textit{pseudoisotopy group} of $N$, i.e., the group of diffeomorphisms $V \rightarrow V$, which preserve the function $\pi$ outside of a compact set, and which are equal to the identity on $N \times (-\infty, -1]$. The group $\mathcal{P}(N)$ acts on the space $\mathcal{Lag}_0$, by lifting diffeomorphisms of $V$ to symplectomorphisms of $T^*(V)$ (see Section 1.1 above). We denote by $j$ the inclusion of $\mathcal{P}(N)$ into $\mathcal{Lag}_0$ as the orbit of $L_\pi$.

\textbf{Corollary 6.3.} The inclusions $j : \mathcal{P}(N) \rightarrow \mathcal{Lag}_0$ and $i \circ j : \mathcal{P}(N) \rightarrow \mathcal{Leg}_0$ are injective on the homotopy groups of $\mathcal{P}(N)$ of dimension $\leq \frac{1}{2}n - 2$.

This corollary (see [187]), together with known information about the homotopy type of pseudoisotopy spaces provide non-trivial elements in homotopy groups of spaces of Lagrangian and Legendrian embeddings. See [187] for a detailed discussion of the subject, as well as for other applications of the method of generating functions.

7. Old and new open problems

In this section I will review the status of some basic open problems which were discussed in my survey [5]. As the reader can observe, despite all the progress most of the basic problems remain wide open.

7.1. Existence and uniqueness of contact and symplectic structures

In dimension $> 4$ we still do not have any counter-examples to the following “soft conjectures”:

1. Does any odd-dimensional manifold with a stable almost complex structure have a contact structure?

1'. Does any contact structure, which is given near the boundary of the ball $B^{2n+1}$ and which extends to the ball as a stable almost complex structure, extend to the ball as a contact structure? (A similar symplectic question has the negative answer in all dimensions $> 2$). Does any closed odd-dimensional, stably almost complex manifold admit a contact structure?

2. Does any closed $2n$-dimensional, $2n > 4$, almost complex manifold $M$ with a cohomology class $u \in H^2(M; \mathbb{R})$ with $u^n \neq 0$ has a symplectic structure (in the cohomology class $u$)?

3. Is any symplectic, or contact structure on $\mathbb{R}^n$, which is standard at infinity, isotopic to the standard contact structure via a compactly supported isotopy?

The answer to Question 2 is negative in dimension 4. Indeed, a theorem of Taubes (see [131]) implies that a symplectic 4-manifold cannot split into a connected sum of two manifolds with $b_2^+ > 0$. In particular, $\mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2$ has no symplectic structure despite that it has an almost complex structure and a 2-dimensional cohomology class $u$ with $u^2 \neq 0$. The answer to Question 3 is “yes” in dimension 4 ([48]) and “no” in dimension 3 ([44]), but “yes” for tight contact structures in dimension 3 ([55]).
7.2. Lagrangian and Legendrian embeddings

4. Is any exact embedded Lagrangian submanifold \( L \subset T^*(M) \) Hamiltonian isotopic to the 0-section? In case \( M \) and \( L \) are non-compact we assume that \( M \) and \( L \) coincide at infinity and want the isotopy to be compactly supported.

5. Let \( f : S^n \to S^n \) be a diffeomorphism, non-isotopic to the identity. Let \( i \) and \( j \) be the inclusions of \( S^n \) (as the 0-sections) into \( T^*(S^n) \) and \( J^1(S^n) \), respectively. Is the Lagrangian (resp. Legendrian) embedding \( i \circ f : S^n \to T^*(M) \) (resp. \( j \circ f : S^n \to J^1(M) \)) Lagrangian (resp. Legendrian) isotopic to the corresponding inclusion? Notice that \( i \circ f \) and \( j \circ f \) are isotopic to \( i \) and \( j \) as smooth embeddings.

5'. More globally, let \( S^n \subset S^{2n+1} \) be the Legendrian sphere obtained by intersecting \( S^{2n+1} \) with the Lagrangian plane \( \mathbb{R}^{n+1} \subset \mathbb{C}^{n+1} \), and \( k : S^n \to S^{2n+1} \) be the inclusion. Let \( f : S^n \to S^n \) be as in Problem 5. Are \( j \) and the composition \( k \circ f : S^n \to S^{2n+1} \) Legendrian isotopic?

Problems 4, 5 and 5' are related to the following question which I first heard from Jeff Mess about 9 years ago.

6. Let \( S^n \) be an exotic sphere. Are \( T^*(S^n) \) and \( T^*(\Sigma^n) \) symplectomorphic?
Some progress towards this problem was achieved in [206].

7.3. Topology of the groups of symplectic and contact diffeomorphisms

Let \( \mathcal{D}_n \) denote the group of compactly supported symplectic or contact (depending on the parity of the dimension) diffeomorphisms of the standard symplectic or contact space \( \mathbb{R}^n \). For \( n \leq 4 \) the group \( \mathcal{D}_n \) is contractible. For \( n = 2 \) this is, essentially, a theorem of S. Smale (see [233]), for \( n = 3 \) this is proven in [55] and for \( n = 4 \) this is a result of M. Gromov, see [48].

7. Suppose that \( n > 4 \). Is the group \( \mathcal{D}_n \) contractible?

Nothing is known about the topology of the group \( \mathcal{D}_n \) for \( n > 4 \). However it is likely that the methods of [187] may provide non-trivial elements in higher homotopy groups of \( \mathcal{D}_n \) in the contact case.

Let \( \text{Diff}_n \) denote the group of compactly supported diffeomorphisms of \( \mathbb{R}^n \).

8. What are the homotopical properties of the inclusion \( \mathcal{D}_n \hookrightarrow \text{Diff}_n \)?

For \( n \leq 3 \) this inclusion is a homotopy equivalence. But already in the 4-dimensional case nothing is known. In view of Gromov's result for \( \mathcal{D}_4 \) Problem 84 is equivalent to the problem about the topology of the group \( \text{Diff}_4 \). Notice that according to theorems of J. Moser and J. Gray (see 1.1 above) the factor-space \( \text{Diff}_n/\mathcal{D}_n \) is homeomorphic to the space of symplectic forms on \( \mathbb{R}^n \), which are standard at infinity.

8. Other developments

In this section we just mention few other important developments during the last decade.

After a sequence of successive improvements Arnold's conjecture about the number of fixed points of a Hamiltonian symplectomorphism is now proven for a general symplectic manifold,
at least as far it concerns with a lower bound by rational Betti numbers. The final step was made independently by several groups of authors: K. Fukaya and K. Ono, G. Liu and G. Tian, H. Hofer and D. Salamon. (see [90,96,94]). K. Fukaya and K. Ono were the first who announced the solution.

F. Laudenbach partially realized his "engulfing program" (see [215]).

A lot of progress has been achieved towards the Weinstein conjecture about periodic orbits of a Hamiltonian system (see [176,166,171,172,173]). H. Hofer adapted the technique of holomorphic curves in compact symplectic manifolds, or manifolds with finite geometry at infinity (see [48]), for use in symplectizations of contact manifolds (see [75,74]). This technique has proven to be extremely powerful and useful for results about the Weinstein conjecture and many other applications. For instance, in the 3-dimensional case it allowed one to go significantly further in understanding the topology of contact manifolds (see [72,76,77,78,79,9,80]).

Holomorphic curves in symplectizations were used by Hofer and the author [73] for constructing new invariants of contact manifolds and their Legendrian submanifolds, called contact homology theory. For the case of Legendrian knots in $\mathbb{R}^3$ a similar theory was independently developed by Yu. Chekanov [50] via a purely combinatorial approach.

A new phenomenon of unknottedness of Lagrangian submanifolds of symplectic 4-manifolds (comp. Problem 4 in the previous section) was discovered in the works of L. Polterovich and the author, and K. Luttinger (see [180,179,181,182,184]). Recently H. Hofer and K. Luttinger (unpublished) obtained new strong results in this direction using Hofer’s method of holomorphic curves in symplectizations. Yu. Chekanov [177] found new invariants of Lagrangian tori in $\mathbb{R}^{2n}$ with respect to Hamiltonian isotopy.

R. Gompf (see [209]), rediscovered Gromov’s fibered connected sum construction and transformed it into a powerful machine for constructing symplectic 4-manifolds with prescribed properties. This construction, together with its generalization by J. McCarthy and J. Wolfson (see [217,216]) and M. Symington (see [234]) were used to disprove many too optimistic conjectures in 4-dimensional symplectic topology (see [210,211,235]).

V.I. Arnold suggested, and partially proved a symplectic-geometric conjecture generalizing the classical 4-vertices theorem in Differential Geometry. This generated a lot of works studying Vassiliev type invariants of Legendrian and Lagrangian wave fronts and caustics (see [197] and the bibliography therein). Despite a lot of interesting new results in this direction Arnold’s conjecture generalizing 4-vertices theorem still remains wide open.

Besides the results discussed in Section 5 there were several other exciting developments in the theory of holomorphic curves in symplectic manifolds. Let us mention here the foundations of the theory of Gromov (and Gromov–Witten) invariants, quantum cohomology theory and its relations with enumerative Algebraic Geometry and mirror symmetry. These led recently to a partial solution of the Mirror conjecture by A. Givental (see [143,140,142]), see also Lian–Liu–Yau’s preprint [148]. P. Seidel found a remarkable application of quantum cohomology for his study of the effect of the generalized Dehn twist (see [100]).

Despite the large number of references, the bibliography below is far from being complete, especially in those areas of symplectic topology which are not discussed in this survey. However, the size of the bibliography below should give the reader an idea of the intensity of research in this field.
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Other references


