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Note

Influences of monotone Boolean functions

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ABSTRACT

Recently, Keller and Pilpel conjectured that the influence of a monotone Boolean function does not decrease if we apply to it an invertible linear transformation. Our aim in this short note is to prove this conjecture.

1. Introduction

Given a positive integer n, a *Boolean function* on n variables is a function $f : \{0,1\}^n \to \{0,1\}$. The function is called *monotone* if for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \{0,1\}^n$ satisfying $x_i \leqslant y_i$ for each $1 \leqslant i \leqslant n$, we have $f(x) \leqslant f(y)$.

For an n-variable Boolean function f, the *influence of the ith variable on* f is defined to be

$$I_i(f) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} |f(x+e_i) - f(x)|,$$

where e_i denotes the element of $\{0,1\}^n$ whose only non-zero coordinate is in the ith position, and addition is done coordinate-wise modulo two. The *total influence of* f is defined to be

$$I(f) = \sum_{i=1}^{n} I_i(f).$$

For the proof of our result it will be convenient to introduce the following definition: Given $y \in \{0,1\}^n$ we define the *influence of* y *on* f to be

$$I_y(f) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} |f(x+y) - f(x)|.$$

[We remark that if we consider the correspondence between the elements of $\{0,1\}^n$ and the subsets of $\{1, \dots, n\}$ then the influence of y on f is not the same as the usual definition of the influence of the set Y (corresponding to y) over f. Since we will not be using the latter definition, we hope that no confusion arises.]

The notion of influence of a variable on a Boolean function was introduced by Ben-Or and Linial [1]. It has since found many applications in discrete mathematics, theoretical computer science and social choice theory. We refer the reader to [2] for a survey of some of these applications. In this note we study the effect on the influence after applying an invertible linear transformation on a monotone Boolean function.

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Given an n-variable Boolean function f and an invertible linear transformation $L \in GL_n(\mathbb{F}_2)$, the function Lf is defined by $Lf(x) = f(Lx)$. In [3] Keller and Pilpel raised the following conjecture.

Conjecture 1 (*Keller and Pilpel [3]*). *If f is an n-variable monotone Boolean function and $L \in GL_n(\mathbb{F}_2)$ then $I(f) \leqslant I(Lf)$.*

We prove this conjecture in the next section.

2. Proof of the conjecture

Since L is invertible, its determinant is non-zero and thus by the formula for the determinant there must be a permutation π of $\{1, \dots, n\}$ such that $L_{i\pi(i)}$ is non-zero for each i. In particular, if P is the permutation matrix which maps e_i to $e_{\pi(i)}$, then each diagonal entry of $L' := LP$ is non-zero. Moreover, it is immediate that $I_i(L'f) = I_{\pi(i)}(Lf)$ for each $1 \leqslant i \leqslant n$ and thus the total influences of L' and L are equal. Thus, we may assume that the diagonal entries of L are non-zero. In this case, we will prove the stronger assertion that $I_i(f) \leqslant I_i(Lf)$ for each $1 \leqslant i \leqslant n$. We claim that $I_i(Lf) = I_{Le_i}(f)$. Indeed,

$$\begin{aligned} I_i(Lf) &= \frac{1}{2^n}\sum_x |Lf(x+e_i) - Lf(x)| \\ &= \frac{1}{2^n}\sum_x |f(Lx+Le_i) - f(Lx)| \\ &= \frac{1}{2^n}\sum_y |f(y+Le_i) - f(y)| \\ &= I_{Le_i}(f). \end{aligned}$$

Splitting the sum in the definition of $I_{Le_i}(f)$ into two parts depending on whether the ith coordinate is equal to zero or not we obtain that

$$\begin{aligned} I_{Le_i}(f) &= \frac{1}{2^n}\sum_y |f(y+Le_i) - f(y)| \\ &= \frac{1}{2^n}\left(\sum_{\{y:y_i=0\}} |f(y+Le_i) - f(y)| + \sum_{\{y:y_i=1\}} |f(y) - f(y+Le_i)|\right) \\ &= \frac{1}{2^n}\sum_{\{z:z_i=0\}} (|f(z+Le_i) - f(z)| + |f(z+e_i) - f(z+e_i+Le_i)|) \\ &\geqslant \frac{1}{2^n}\sum_{\{z:z_i=0\}} |f(z+e_i) + f(z+Le_i) - f(z) - f(z+e_i+Le_i)|. \end{aligned}$$

Observe that since each diagonal entry of L is non-zero, the ith coordinate of Le_i is equal to one and so if the ith coordinate of z is zero, then the ith coordinate of $z + e_i + Le_i$ is also zero and so by the monotonicity of f we have $f(z) \leqslant f(z + e_i)$ and $f(z + e_i + Le_i) \leqslant f(z + Le_i)$. It follows that

$$\begin{aligned} I_{Le_i}(f) &\geqslant \frac{1}{2^n}\sum_{\{z:z_i=0\}} |f(z+e_i) + f(z+Le_i) - f(z) - f(z+e_i+Le_i)| \\ &= \frac{1}{2^n}\sum_{\{z:z_i=0\}} |f(z+e_i) - f(z)| + \frac{1}{2^n}\sum_{\{z:z_i=0\}} |f(z+Le_i) - f(z+e_i+Le_i)| \\ &= \frac{1}{2^n}\sum_{\{z:z_i=0\}} |f(z+e_i) - f(z)| + \frac{1}{2^n}\sum_{\{w:w_i=1\}} |f(w) - f(w+Le_i)| \\ &= I_i(f), \end{aligned}$$

as required. This completes the proof of Conjecture 1.

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