On Second-Order Symmetric Duality in Nondifferentiable Programming*

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Submitted by William F. Ames

Received June 15, 2000

This article is concerned with a pair of second-order symmetric dual non-differentiable programs and second-order $F$-pseudo-convexity. We establish the weak and the strong duality theorems for the new pair of dual models under the $F$-pseudo-convexity assumption. Several known results including Mond and Schechter, as well as others are obtained as special cases.

1. INTRODUCTION

Symmetric duality in nonlinear programming was introduced by Dorn [3], who defined a mathematical programming problem and its dual to be symmetric if the dual of the dual is the original problem, that is, when the dual is recast in the form of primal, its dual is primal. Subsequently, Dantzig, Eisenberg, and Cottle [2] and Mond [5] formulated a pair of symmetric dual programs for scalar function $f(x, y)$ that is convex in the first variable and that is concave in the second variable. Then Mond and

* This research was partially supported by the National Natural Science Foundation of China (Grant 19771092 and 19401040) and the Croucher Foundation of Hong Kong.
Weir [8] gave another different pair of symmetric dual non-linear programs in which a weaker convexity assumption was imposed on \( f \).

On the other hand, Mond [6] and Bector and Chandra [1] studied second-order primal and dual non-linear programs and proved second-order symmetric duality results for these programs. Later on, Yang [10] generalized the results in Bector and Chandra [1] to non-linear programs involving second-order pseudo-convex functions. More recently, Mond and Schechter [7] constructed two new symmetric dual pairs in which the objectives contain a support function and are therefore nondifferentiable.

In this article, we are motivated by Mond and Schechter [7] to show that there is a new pair of second-order symmetric models for a class of nondifferentiable non-linear programs related by duality. To discuss the various formulations of the second-order symmetric dual models and their duality results, we introduce the concept of second-order \( F \)-pseudo-convexity, and we establish appropriate weak as well as strong duality theorems under \( F \)-pseudo-convexity assumptions. Our study naturally extends some of the known results in [1, 10, 7].

2. NOTATIONS AND PRELIMINARIES

Throughout this article, let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space and let \( \mathbb{R}^n_+ \) be its non-negative orthant.

Let \( K(x, y) \) be a real-valued twice continuously differentiable function defined on an open set in \( \mathbb{R}^n \times \mathbb{R}^m \). Let \( \nabla_x K(x, y) \) denote the gradient vector of \( K \) with respect to \( x \) at \( (x, y) \). Also let \( \nabla_{xx} K(x, y) \) denote the Hessian matrix with respect to \( x \) evaluated at \( (x, y) \). \( \nabla_y K(x, y) \) and \( \nabla_{yy} K(x, y) \) are defined similarly.

**Definition 2.1.** A functional \( F: X \times X \times \mathbb{R}^n \rightarrow \mathbb{R} \) (where \( X \subseteq \mathbb{R}^n \)) is sublinear if for all \( (x, u) \in X \times X \),

(i) \( F(x, u, a_1 + a_2) \leq F(x, u, a_1) + F(x, u, a_2) \) for all \( a_1, a_2 \in \mathbb{R}^n \),

(ii) \( F(x, u, \alpha a) = \alpha F(x, u, a) \) for all \( \alpha \in \mathbb{R}_+ \) and for all \( a \in \mathbb{R}^n \).

For notational convenience, we write \( F_{x,u}(a) = F(x, u, a) \).

Now we consider a sublinear function \( F: X \times X \times \mathbb{R}^n \rightarrow \mathbb{R} \) and a twice differentiable function \( f: X \rightarrow \mathbb{R} \).
**Definition 2.2.** $f$ is said to be second-order $F$-pseudo-convex at $u \in X$ if

$$(x, p) \in X \times \mathbb{R}^n, \quad F_{x, u} \left[ \nabla_u f(u) + \nabla_{uu} f(u) p \right] \geq 0$$

$$f(x) \geq f(u) - \frac{1}{2} p^T \nabla_{uu} f(u) p.$$ 

$f$ is second-order $F$-pseudo-concave if $-f$ is second-order $F$-pseudo-convex.

**Remark 2.1.** (i) The second-order $F$-pseudo-convexity reduces to the $F$-pseudo-convexity in Hanson and Mond [4] when $p = 0$.

(ii) For $F_{x, a}(a) := \eta(x, a)^T a$ where $\eta$ is a function from $X \times X$ to $\mathbb{R}^n$, the second-order $F$-pseudo-convexity reduces to the second-order $\eta$-pseudo-convexity in Yang [10].

(iii) If $F_{x, a}(a) := (x - u)^T a$, the second-order $F$-pseudo-convexity reduces to the second-order pseudo-convexity in Bector and Chandra [1].

Let $C$ be a compact convex set in $\mathbb{R}^n$. The support function of $C$ is defined by

$$s(x | C) := \max \{ x^T y : y \in C \}.$$ 

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists $z$ such that $s(y | C) \geq s(x | C) + z^T (y - x)$ for all $y \in C$. The subdifferential of $s(x | C)$ is given by

$$\partial s(x | C) := \{ z \in C : z^T x = s(x | C) \}.$$ 

For any set $S \subset \mathbb{R}^n$ the normal cone to $S$ at a point $x \in S$ is defined by

$$N_S(x) \equiv \{ y \in \mathbb{R}^n : y^T (z - s) \leq 0 \text{ for all } z \in S \}.$$ 

It is readily verified that for a compact convex set $C$, $y$ is in $N_C(x)$ if and only if $s(y | C) = x^T y$, or equivalently, $x$ is in the subdifferential of $s$ at $y$.

We now state the following pair of non-differentiable programs and we discuss their duality results in subsequent sections.

**Primal Problem:**

Minimize $f(x, y) + s(x | C) - y^T z - \frac{1}{2} p^T \nabla_{yy} f(x, y) p$

subject to:

$$\nabla_y f(x, y) - z + \nabla_{yy} f(x, y) p \leq 0, \quad (1)$$

$$y^T \left[ \nabla_y f(x, y) - z + \nabla_{yy} f(x, y) p \right] \geq 0, \quad (2)$$

$$z \in D, \quad (3)$$
Dual Problem:
Maximize \( f(u, v) - s(v \mid D) + u^T w - \frac{1}{2} r^T \nabla_{uv} f(u, v) r \)
subject to:
\[
\nabla_u f(u, v) + w + \nabla_{uv} f(u, v) r \geq 0, \quad (4)
\]
\[
u^T \left[ \nabla_u f(u, v) + w + \nabla_{uv} f(u, v) r \right] \leq 0, \quad (5)
\]
\[
w \in C, \quad (6)
\]

where

(i) \( f \) is a differentiable function from \( \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \);

(ii) \( r, w \) are vectors in \( \mathbb{R}^n \) and \( p, z \) are vectors in \( \mathbb{R}^m \); and

(iii) \( C \) and \( D \) are compact convex sets in \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively.

For the sake of convenience, we denote the primal problem by \( (P) \) and we denote the dual problem by \( (D) \). Clearly, \( (P) \) and \( (D) \) belong to a special class of nonlinear programming problems. The non-differentiability terms in the form of support functions were included in the objective function of each problem. It can be easily seen that if we write the dual problem as a minimization problem, then its dual is just the primal problem written as a maximization problem. This enables us to establish the symmetry of the two problems \( (P) \) and \( (D) \).

3. SYMMETRIC DUALITY

We prove the following duality results for the pairs \( (P), (D) \).

**Theorem 3.1 (Weak Duality).** Let \( (x, y, z, p) \) be feasible for the primal problem \( (P) \) and let \( (u, v, w, r) \) be feasible for the dual problem \( (D) \). Suppose there exists sublinear functionals \( F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) and \( G: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \) satisfying
\[
F_{x,y} (a) + a^T u \geq 0 \quad \text{for all } a \in \mathbb{R}^n_+, \quad (7)
\]
\[
G_{v,y} (b) + b^T y \geq 0 \quad \text{for all } b \in \mathbb{R}^m_+. \quad (8)
\]

Furthermore, assume that \( f(\cdot, v) + (\cdot)^T w \) is second-order \( F \)-pseudo-convex at \( u \) and \( f(x, \cdot) - (\cdot)^T z \) is second-order \( G \)-pseudo-concave at \( y \). Then,
\[
\inf (P) \geq \sup (D).
\]

**Proof.** Let \( (x, y, z, p) \) be feasible for \( (P) \) and let \( (u, v, w, r) \) be feasible for \( (D) \). By the dual constraint \( (4) \), the vector \( \bar{a} := \nabla_u f(u, v) + w + \)
\[ \nabla_{uu} f(u, v) r \text{ belongs to } \mathbb{R}_+^n, \text{ and so by (7),} \]
\[ F_{x, u}(\bar{a}) + \bar{a}^T u \geq 0. \]

This together with (5) implies \( F_{x, u}(\bar{a}) \geq 0. \) Thus, by the second-order pseudo-convexity of \( f(\cdot, v) + (\cdot)^T w \) at \( u, \) we have
\[ f(x, v) + x^T w \geq f(u, v) + u^T w - \frac{1}{2} r^T \nabla_{u, u} f(u, v) r. \tag{9} \]

In a similar fashion,
\[ G_{x, y}(\bar{b}) \geq 0 \]
for the vector \( \bar{b} := \nabla_y f(x, y) - z + \nabla_{yy} f(x, y) p \) in \( \mathbb{R}_+^m, \) and so by the second-order \( G \)-pseudo-concavity of \( f(x, \cdot) - (\cdot)^T z \) at \( y \) we have
\[ f(x, v) - v^T z \leq f(x, y) - y^T z - \frac{1}{2} p^T \nabla_{yy} f(x, y) p. \tag{10} \]

Now it follows from (9) and (10) that
\[ f(u, v) - v^T z + u^T w - \frac{1}{2} r^T \nabla_{u, u} f(u, v) r \leq f(x, y) + x^T w - y^T z - \frac{1}{2} p^T \nabla_{yy} f(x, y) p. \]

Finally, using \( x^T w \leq s(x \mid C) \) and \( v^T z \leq s(v \mid D) \) we obtain
\[ f(x, y) + s(x \mid C) - y^T z - \frac{1}{2} p^T \nabla_{yy} f(x, y) p \geq f(u, v) - s(v \mid D) + u^T w - \frac{1}{2} r^T \nabla_{uu} f(u, v) r, \]
and the theorem follows.

**Remark 3.1.** The significance of Theorem 3.1 is obvious when the objective values of \( P \) and \( D \) are equal for some \((x^0, y^0, z^0, p^0)\) and \((u^0, v^0, w^0, r^0),\) respectively; in that case \((x^0, y^0, z^0, p^0)\) and \((u^0, v^0, w^0, r^0)\) are global optimal solutions of the primal and dual problems, respectively. This is illustrated in the theorem below.

**Theorem 3.2 (Strong Duality).** Let the function \( f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) be three times differentiable. Suppose that the hypotheses of Theorem 3.1 are satisfied, and \((\bar{x}, \bar{y}, \bar{z}, \bar{p})\) is a local optimal solution of the primal problem \( P \) such that
\[ \nabla_{y} f(\bar{x}, \bar{y}) + \nabla_{yy} f(\bar{x}, \bar{y}) \bar{p} \neq \bar{z}. \tag{11} \]
Assume either (a) $\nabla_{yy} f(\bar{x}, \bar{y})$ is positive definite and $\bar{p}^T [\nabla_y f(\bar{x}, \bar{y}) - \bar{z}] \geq 0$ or (b) $\nabla_{yy} f(\bar{x}, \bar{y})$ is negative definite and $\bar{p}^T [\nabla_y f(\bar{x}, \bar{y}) - \bar{z}] \leq 0$. Then,

(i) $\bar{p} = 0$;

(ii) $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$ is a global optimal solution for (P);

(iii) there exists $\bar{w} \in C$ such that $(u, v, w, r) = (\bar{x}, \bar{y}, \bar{w}, 0)$ is a global optimal solution for (D); and

(iv) $\min(P) = \max(D)$.

Proof. Since $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$ is an optimum solution of (P), there exist $\alpha \in \mathbb{R}_+$, $\beta \in \mathbb{R}_+^n$, and $\mu \in \mathbb{R}_+$ such that the following Fritz John conditions [9] are satisfied at $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$,

$$\alpha(\nabla_x f + \gamma) + (\nabla_{xy} f)(\beta - \mu \bar{y}) + \nabla_y [\nabla_{yy} f \bar{p}](\beta - \frac{1}{2} \alpha \bar{p} - \mu \bar{y}) = 0,$$

$$\quad (\alpha - \mu)(\nabla_y f - \bar{z}) + (\nabla_{yy} f)(\beta - \mu \bar{y} - \mu \bar{p})$$

$$+ \nabla_y [\nabla_{yy} f \bar{p}](\beta - \frac{1}{2} \alpha \bar{p} - \mu \bar{y}) = 0,$$

$$\quad (\nabla_{yy} f)(\alpha \bar{p} - \beta + \mu \bar{y}) = 0,$$

$$\quad \beta^T (\nabla_y f - \bar{z} + \nabla_{yy} f \bar{p}) = 0,$$

$$\quad \mu \bar{y}^T (\nabla_y f - \bar{z} + \nabla_{yy} f \bar{p}) = 0,$$

$$\quad \alpha \bar{y} - \beta + \mu \bar{y} \in N_D(\bar{z}),$$

$$\quad \gamma \in C, \quad \gamma^T \bar{x} = s(\bar{x} | C),$$

$$\quad (\alpha, \beta, \mu) \neq 0.$$  

Since $\nabla_{yy} f$ is non-singular by hypothesis (a) or (b), (14) yields

$$\beta = \alpha \bar{p} + \mu \bar{y}.$$  

We claim that $\alpha \neq 0$. Indeed, if $\alpha = 0$, then (20) gives

$$\beta = \mu \bar{y}.$$  

This together with (13) yields

$$\mu (\nabla_y f - \bar{z} + \nabla_{yy} f \bar{p}) = 0.$$  

Since by (11) $\nabla_y f - \bar{z} + \nabla_{yy} f \bar{p} \neq 0$, this implies $\mu = 0$ and whence $\beta = 0$, contradicting (19). Therefore, $\alpha \neq 0.$
Subtracting (16) from (15) yields
\[
(\beta - \mu \bar{y})^T (\nabla_y f - \bar{z} + \nabla_{yy} f \bar{p}) = 0,
\]
and noting (20), we have
\[
\bar{p}^T [\nabla_y f - \bar{z} + \nabla_{yy} f \bar{p}] = 0. \tag{24}
\]
We now prove that \( \bar{p} = 0 \). Otherwise, either hypothesis (a) or (b) implies that \( \bar{p}^T [\nabla_y f - \bar{z} + \nabla_{yy} f \bar{p}] \neq 0 \), contradicting (24). Hence, \( \bar{p} = 0 \). In particular, (20), (11), and (13) give \( \bar{y} \geq 0 \). Also, it follows from (12) (because \( \bar{p} = 0 \) and \( \beta = \mu \bar{y} \)) that
\[
(\nabla_y f + \gamma) = 0.
\]
Now, taking \( \bar{w} := \gamma \in C \), we find that \( (u, v, w, r) = (\bar{x}, \bar{y}, \bar{w}, \bar{r} = 0) \) satisfies (4)-(6), the constraint conditions of (D), and is therefore a feasible solution for the dual problem (D).
Moreover, since \( \beta = \mu \bar{y} \) and \( \alpha > 0 \), (17) implies \( \bar{y} \in N_{\beta}(\bar{x}) \), so that
\[
y^T \bar{z} = s(\bar{y} | D). \tag{25}
\]
Therefore, using (18) and (25), we get
\[
f(\bar{x}, \bar{y}) + s(\bar{x} | C) - \bar{y}^T \bar{z} - \frac{1}{\beta} \bar{p}^T \nabla_{yy} f(\bar{x}, \bar{y}) \bar{p} \\
= f(\bar{x}, \bar{y}) - s(\bar{y} | D) + \bar{x}^T \bar{w} - \frac{1}{\beta} \bar{p}^T \nabla_{xx} f(\bar{x}, \bar{y}) \bar{r},
\]
that is, the objective values of (P) at \( (\bar{x}, \bar{y}, \bar{z}, \bar{p}) \) and the objective values of (D) at \( (\bar{x}, \bar{y}, \bar{w}, \bar{r}) \) are equal. Finally, from Theorem 3.1 we get that \( (\bar{x}, \bar{y}, \bar{z}, \bar{p}) \) and \( (\bar{x}, \bar{y}, \bar{w}, \bar{r}) \) are global optimal solutions for (P) and (D), respectively. This completes the proof of the theorem.

4. SPECIAL CASES

In this section, we consider some special cases of the problem (P) and (D) by choosing particular forms of the sublinear functional \( F \) and the compact convex sets \( C \) and \( D \).

(i) If \( C = \{0\} \), \( D = \{0\} \), and \( F_{x,a}(a) := (x - a)^T a \), then (P) and (D) reduce to programs studied in Bector and Chandra [1].
(ii) If $C = \{0\}$, $D = \{0\}$, and $F_{x,u}(a) := \eta(x,u)^{T}a$ where $\eta$ is a function from $X \times X$ to $\mathbb{R}^{n}$, then (P) and (D) reduce to programs considered in Yang [10].

(iii) If $p = 0$, $r = 0$, then (P) and (D) become a pair of symmetric dual non-differentiable programs considered in Mond and Schechter [7].

**Remark 4.1.** From the symmetric dual models (P) and (D), we can construct other new symmetric dual pairs. For example, if we take $C = \{Ay : y^{T}Ay \leq 1\}$ and $D = \{Bx : x^{T}Bx \leq 1\}$ where $A$ and $B$ are positive semidefinite, then it can be readily verified that $(x^{T}Ax)^{1/2} = s(x \mid C)$ and $(y^{T}By)^{1/2} = s(y \mid D)$, and thus a number of new symmetric dual pairs and duality results are obtained.

**REFERENCES**