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Shock-peakon and shock-compacton solutions for $K(p, q)$ equation by variational iteration method

Lixin Tian*, Jiuli Yin

Nonlinear Scientific Research Center, Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212003, PR China

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Abstract

By variational iteration method, we obtain new solitary solutions for non-linear dispersive equations. Particularly, shock-peakon solutions in $K(2, 2)$ equation and shock-compacton solutions in $K(3, 3)$ equation are found by this simple method. These two types of solutions are new solitary wave solutions which have the shapes of shock solutions and compacton solutions (or peakon solutions).

MSC: 35Q51; 35Q53

Keywords: $K(p, q)$ equation; Variational iteration method; Shock-peakon; Shock-compacton

1. Introduction

In order to study the role of non-linear dispersion in the formation of patterns in liquid drops, Rosenau and Hyman [9] studied the genuinely non-linear dispersive equation $K(p, q)$ given by

$$u_t + a(u^p)_x + (u^q)_{3x} = 0. \tag{1}$$

Eq. (1) has been widely studied by many authors [11–18,20,21]. Recently, two special solitary wave solutions were discovered [1,9]. Tian and Yin [11–13] obtained compacton solutions for generalized Camassa–Holm equation, $K(m, n, 1)$ equation and fully non-linear sine-Gordon. Peakon solutions were found in the Camassa–Holm equation [1] which are called peakons because they have a discontinuous first-order derivative at the wave peak. Camassa–Holm equation was studied by different methods [10,2,8].

Peakon and compacton have the characters of soliton, and both are non-local. They are continuous waves and can be expressed by δ function. When waves appear discontinuously, there exists a shock wave (see [19]). It is a kind of wave which exists in many mathematical physics equations. In this paper, we will study the $K(p, q)$ equation and obtain its new shock solitary waves, having the characters of peakon and compacton. This is a new type of solitary wave. At the same time, we find that it is non-local as well. Hence, this solution is of very high significance.

The variational iteration method (see [3–7]) has been shown to solve effectively, easily, and accurately a large class of non-linear problems with approximations converging rapidly to accurate solutions. Particularly, He and Wu [7] obtained solitary solutions and compacton-like solutions for $K(3, 1)$ equation by variational iteration method.

* Corresponding author. Tel./fax: +86 5118791467.
E-mail address: tianlx@ujs.edu.cn (L. Tian).

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In the next section, we obtain new solitary solutions and many exact solutions for non-linear dispersive equations by the variational iteration method. Particularly, shock-peakon solutions in $K(2,2)$ equation and shock-compacton solutions in $K(3,3)$ equation are found by this simple method.

2. Exact solutions by variational iteration method

According to the variational iteration method [3–7], we construct a correction functional for Eq. (1), which reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\{(u_n)_t + a(\tilde{u}_n^p)_x + (\tilde{u}_n^q)_{3x}\}\,\mathrm{d}\tau = 0, \tag{2}$$

where λ is a general Lagrange multiplier, and $\tilde{u}_n$ denotes restricted variation [6], i.e., $\delta\tilde{u}_n = 0$.

With the help of the above correction functional stationary, we obtain the following stationary conditions

$$\lambda'(\tau) = 0, \tag{3}$$

$$1 + \lambda(\tau)|_{t=\tau} = 0. \tag{4}$$

The Lagrangian multiplier, therefore, can be identified as

$$\lambda = -1. \tag{5}$$

Substituting Eq. (5) into the correction functional (2) results in the following iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{(u_n)_t + a(u_n^p)_x + (u_n^q)_{3x}\}\,\mathrm{d}\tau. \tag{6}$$

As illustrating examples, we only consider $K(2,2)$ and $K(3,3)$ in the following sections.

2.1. Shock-peakon solution in $K(2,2)$ equation

Now we consider $K(2,2)$ equation:

$$u_t + a(u^2)_x + (u^2)_{3x} = 0 \tag{7}$$

where α plays an important role in determining the shape of solutions. In view of the iteration formula, Eq. (6), we have

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{(u_n)_t + a(u_n^2)_x + (u_n^2)_{3x}\}\,\mathrm{d}\tau. \tag{8}$$

To search for its compacton solution, we can assume an initial condition in the form

$$u(x,0) = -\frac{4c}{3a}\cos^2\frac{\sqrt{a}}{4}(x+x_0), \tag{9}$$

where x_0 and c are constants. With the initial approximation given by (9) and using the above iteration formula (8), we can obtain the rest of the other components

$$u_0 = -\frac{4c}{3a}\cos^2\frac{\sqrt{a}}{4}(x+x_0),$$

$$u_1(x,t) = -\frac{4c}{3a}\cos^2\frac{\sqrt{a}}{4}(x+x_0) + \frac{c^2}{3\sqrt{a}}t\sin\frac{\sqrt{a}}{2}(x+x_0),$$

$$u_2(x,t) = -\frac{4c}{3a}\cos^2\frac{\sqrt{a}}{4}(x+x_0) + \frac{c^2}{3\sqrt{a}}t\sin\frac{\sqrt{a}}{2}(x+x_0) - \frac{c^3}{12}t^2\cos\frac{\sqrt{a}}{2}(x+x_0)$$

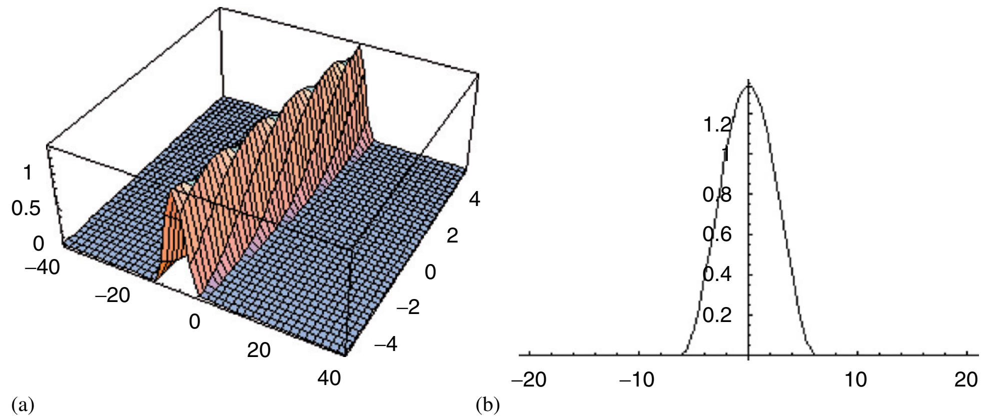


Fig. 1. Compacton solution.

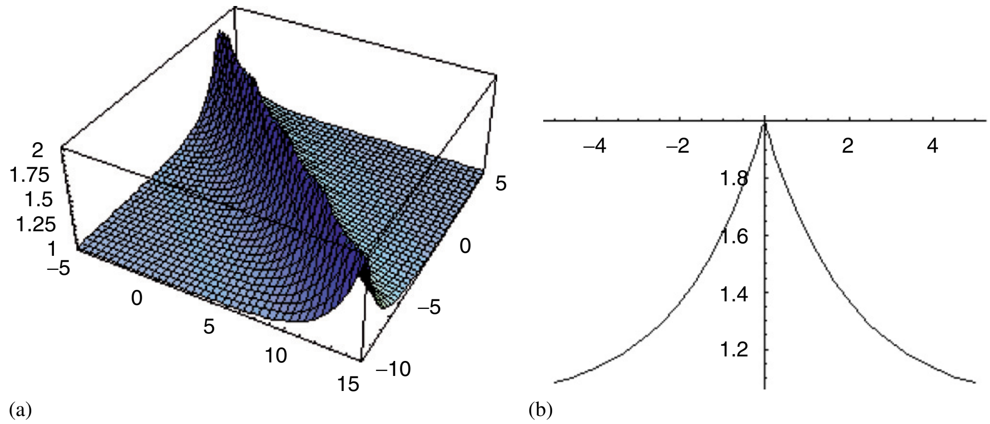


Fig. 2. Flowing peakon solution.

and so on, the rest of components of the iteration formula of (8) can be deduced by Mathematical package. The solutions $u(x,t)$ are readily found in a closed form

$$u(\xi)=\begin{cases}-\dfrac{4c}{3a}\cos^2\dfrac{\sqrt{a}}{4}\xi, & |\xi|\leqslant\dfrac{\pi}{2},\\ 0 & \text{otherwise,}\end{cases} \tag{10}$$

where $\xi = x + ct + x_0$. The obtained compacton solution, Eq. (10), has the same expression with that in Ref. [1]. The solution is shown in Fig. 1 with $a = 1$, $c = -1$, $x_0 = 0$.

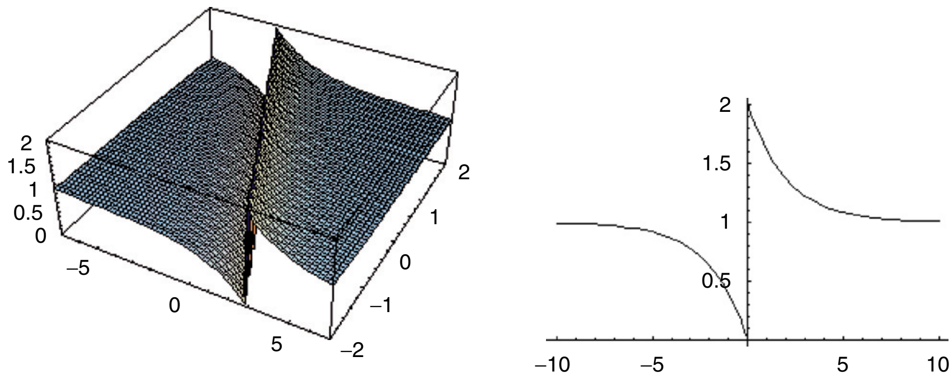
From (10), we can find compacton solutions arise as $a > 0$. Therefore, we pay more attention to what happens to the solution when $a < 0$. Assuming another initial condition as $u(x,0) = A\mathrm{e}^{\pm(\sqrt{-a}/2)(x+x_0)} + c_0$, we can then obtain the solutions $u(x,t)$ in a closed form as

$$u(x,t) = A\mathrm{e}^{-(\sqrt{-a}/2)|(x-(3/2)ac_0t+x_0)|} + c_0, \tag{11}$$

where A, x_0 and c_0 are arbitrary constants, which are flowing peakon solutions as shown in Fig. 2 with $A = 1$, $a = -1$, $c_0 = 1$, $x_0 = 0$.

Note that A in (11) is an arbitrary constant, hence we can obtain a new solitary solution called *shock-peakon* solution which can be written in the form

$$u(\xi) = A\,\mathrm{sign}(\xi)\mathrm{e}^{-(\sqrt{-a/2})|\xi|} + c_0, \tag{12}$$



(a) (b)

Fig. 3. Shock-peakon solution.

where $\xi = x - (3/2)ac_0 t + x_0$ and $\mathrm{sign}(\xi) = \xi/|\xi|$. The *shock-peakon* solution is illustrated in Fig. 3 with $A = 1$, $a = -1$, $c_0 = 1$, $x_0 = 0$. This is a new type of solitary waves and is a discontinuous wave. Hence it is a shock wave. At the same time, it is a peakon as well. In fact, from the graphs and computation we can find that it has a discontinuous first-order derivative at $\xi = 0$. But, this solitary wave is non-local. It can be expressed by δ function. Note that the fact

$$e^{-(\sqrt{-a}/2)|\xi|} = e^{-(\sqrt{-a}/2)\mathrm{sign}(\xi)\xi}, \quad \frac{d}{d\xi}\mathrm{sign}(\xi) = \delta(\xi).$$

Hence

$$\frac{d}{d\xi}u(\xi) = A\delta(\xi)e^{(-\sqrt{-a}/2)\mathrm{sign}(\xi)\xi} - \frac{\sqrt{-a}}{2}A[\mathrm{sign}(\xi)\delta(\xi)\xi + 1]e^{-(\sqrt{-a}/2)\mathrm{sign}(\xi)\xi},$$

where $\delta(\xi)$ is δ function.

Assuming different initial conditions, we may obtain different exact solutions in a closed form as follows:

$u(x,0)$	$-\frac{4c}{3a}\sin^2\frac{\sqrt{a}}{4}(x+x_0)$	$-\frac{4c}{3a}\cosh^2\frac{\sqrt{-a}}{4}(x+x_0)$	$\frac{4c}{3a}\sinh^2\frac{\sqrt{-a}}{4}(x+x_0)$
$u(x,t)$	$-\frac{4c}{3a}\sin^2\frac{\sqrt{a}}{4}(x+ct+x_0)$	$-\frac{4c}{3a}\cosh^2\frac{\sqrt{-a}}{4}(x+ct+x_0)$	$\frac{4c}{3a}\sinh^2\frac{\sqrt{-a}}{4}(x+ct+x_0)$

2.2. *Shock-compacton solutions in $K(3,3)$ equation*

Now we consider $K(3,3)$ equation:

$$u_t + a(u^3)_x + (u^3)_{3x} = 0. \tag{13}$$

Its correction iteration formula reads

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{(u_n)_t + a(u_n^3)_x + (u_n^3)_{3x}\}\,d\tau = 0. \tag{14}$$

Its initial condition is assumed to have the form

$$u(x,0) = \sqrt{-\frac{3c}{2a}}\cos\frac{\sqrt{a}}{3}(x+x_0), \tag{15}$$

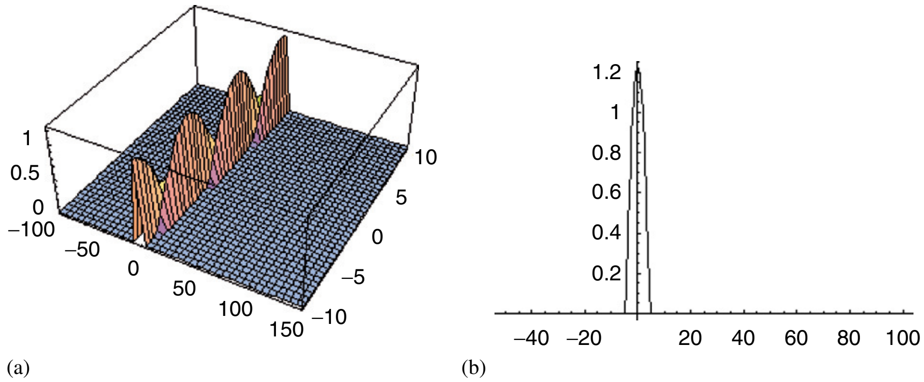


Fig. 4. Compacton solution.

where x_0 and c are constants. The rest of the other components are deduced as follows

$$u_0=\sqrt{-\frac{3c}{2a}}\cos\frac{\sqrt{a}}{3}(x+x_0),$$

$$u_1(x,t)=\sqrt{-\frac{3c}{2a}}\cos\frac{\sqrt{a}}{3}(x+x_0)-\sqrt{-\frac{c^3}{6}}t\sin\frac{\sqrt{a}}{3}(x+x_0),$$

$$u_2(x,t)=\sqrt{-\frac{3c}{2a}}\cos\frac{\sqrt{a}}{3}(x+x_0)-\sqrt{-\frac{c^3}{6}}t\sin\frac{\sqrt{a}}{3}(x+x_0)-\sqrt{-\frac{ac^5}{216}}t^2\cos\frac{\sqrt{a}}{3}(x+x_0)$$

and so on. The solutions $u(x,t)$ in a closed form are obtained by the Mathematica package

$$u(\xi)=\begin{cases}\sqrt{-\frac{3c}{2a}}\cos\frac{\sqrt{a}}{3}\xi, & |\xi|\leqslant\frac{\pi}{2},\\ 0 & \text{otherwise},\end{cases}\tag{16}$$

where $\xi=x+ct+x_0$. The compacton solution is shown in Fig. 4 with $a=1$, $c=-1$, $x_0=0$.

By the same method, we can obtain the solution $u(x,t)$ in a closed form as

$$u(\xi)=\begin{cases}-\sqrt{-\frac{3c}{2a}}\cos\frac{\sqrt{a}}{3}\xi, & |\xi|\leqslant\frac{\pi}{2},\\ 0 & \text{otherwise},\end{cases}\tag{17}$$

So we can obtain a new solitary solution called the *shock-compacton* solution as follows

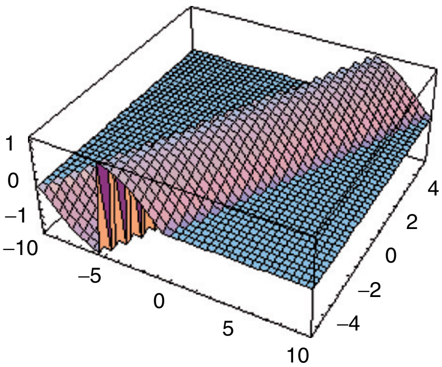
$$u(\xi)=\begin{cases}\sqrt{-\frac{3c}{2a}}\operatorname{sign}(\xi)\cos\frac{\sqrt{a}}{3}\xi, & |\xi|\leqslant\frac{\pi}{2},\\ 0 & \text{otherwise}\end{cases}\tag{18}$$

which is shown in Fig. 5 with $a=1$, $c=-1$, $x_0=0$.

From the graphs and computation we can find that it has a discontinuous first-order derivative at $\xi=0,\pm\pi/2$. Note that the fact

$$u(\xi)=\sqrt{-\frac{3c}{2a}}\operatorname{sign}(\xi)A(\xi)\cos\frac{\sqrt{a}}{3}\xi,$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}A(\xi)=\delta\left(\xi+\frac{\pi}{2}\right)-\delta\left(\xi-\frac{\pi}{2}\right)\quad\text{and}\quad\frac{\mathrm{d}}{\mathrm{d}\xi}\operatorname{sign}(\xi)=\delta(\xi),$$



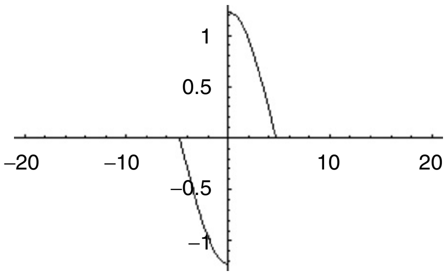


Fig. 5. Shock-compacton solution.

we have

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u(\xi) = \sqrt{-\frac{3c}{2a}}\delta(\xi)A(\xi)\cos\frac{\sqrt{a}}{3}\xi + \sqrt{-\frac{3c}{2a}}\,\mathrm{sign}(\xi)\left[\delta\left(\xi+\frac{\pi}{2}\right) - \delta\left(\xi-\frac{\pi}{2}\right)\right]\cos\frac{\sqrt{a}}{3}\xi$$
$$-\sqrt{-\frac{c}{6}}\,\mathrm{sign}(\xi)A(\xi)\sin\frac{\sqrt{a}}{3}\xi,$$

where

$$A(\xi) = \begin{cases} 1, & |\xi| \leqslant \dfrac{\pi}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

$\delta(\xi)$ is δ function. Hence shock-compacton solution is non-local and new type of solitary wave. It has the characters of shock and compacton.

3. Conclusion

In this paper, we obtain two new types of solitary wave solution: shock-peakon and shock-compacton for $K(p, q)$ equation. They are non-local shock wave solutions, having the characters of peakon and compacton.

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