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Triality invariance in the $N = 2$ superstring

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ABSTRACT

We prove the discrete triality invariance of the $N = 2$ NSR superstring moving in a $D = 2 + 2$ target space. We find that triality holds also in the Siegel–Berkovits formulation of the selfdual superstring. A supersymmetric generalization of Cayley's hyperdeterminant, based on a quartic invariant of the $SL(2|1)^3$ superalgebra, is presented.

1. Introduction

Cayley's hyperdeterminant [1], the generalization to cubic $2 \times 2 \times 2$ matrices of the usual determinant of square 2×2 matrices, was recently recognized [2] to be at the basis of fascinating connections between black hole entropy in string theory and the quantum entanglement of qubits and qutrits in quantum information theory (see [3] and references therein).

The hyperdeterminant was also used in [4] to rewrite the Nambu–Goto Lagrangian for a $D = 4$ target space with signature $(2, 2)$ in a way that makes manifest a hitherto hidden discrete symmetry. The eight variables given by the world-sheet derivatives of the string coordinate functions $\partial_\alpha X^\mu$ are rearranged in a $2 \times 2 \times 2$ hypermatrix $X_{AA'A''}$, whose hyperdeterminant square root is shown to coincide with the Nambu–Goto action. The hyperdeterminant being invariant under interchange of the indices A, A', A'', the triality invariance of the Nambu–Goto Lagrangian becomes explicit. Moreover, the hyperdeterminant encodes in a symmetric way also the $[SL(2, R)]^3$ symmetry of the action, where the $SL(2, R)$ acting on the index A and the $SL(2, R)$ acting on the index A' are the $O(2, 2)$ spacetime symmetry, and the $SL(2, R)$ acting on A'' is the world-sheet symmetry.

In [5] the Green–Schwarz σ-model for the $N = 2$ superstring in $D = 2 + 2$ target space was re-expressed in terms of an hyperdeterminant, once the zweibein is eliminated via its (non-algebraic) field equation. The issue of quantum equivalence of the resulting action with the original GS $N = 2$ superstring, or with the NSR $N = 2$ superstring, is still not completely settled.

In this Letter we make manifest a discrete triality invariance of the NSR $N = 2$ superstring moving in a $D = 2 + 2$ target space, without direct recourse to Cayley's hyperdeterminant, but rearranging the fields in a way suggested by the hyperdeterminant. This triality could well be the origin of the triality observed in [6] between the worldsheet moduli, the complex moduli of the target, and the metric moduli of the target.

Moreover, considering the Siegel–Berkovits action for the selfdual superstring [7–9], we find that triality holds also in its matter part.

It is natural to ask whether the NSR $N = 2$ superstring in $D = 2 + 2$ target space could be expressed in terms of a supersymmetric generalization of the hyperdeterminant. We present such a generalization, based on a quartic invariant of the $SL(2|1)^3$ superalgebra.

2. The $N = 2$ superstring action

The $N = 2$ NSR superstring action [10] in a flat target space of signature $(2, 2)$[1] and in the conformal gauge is given by:

$$S_{N=2} = -\frac{1}{2\pi} \int d^2\sigma \, \big(\partial_\alpha X^\mu \partial^\alpha X^\nu + \partial_\alpha Y^\mu \partial^\alpha Y^\nu - i\bar{\psi}_i^\mu \gamma^\alpha \partial_\alpha \psi_i^\nu\big)\eta_{\mu\nu}, \tag{2.1}$$

where we have used the notations of [12]: $\eta_{\mu\nu} = (1, -1)$ is the two-dimensional Minkowski metric, $\mu = 0, 1$, $i = 1, 2$ and the γ^α are the two-dimensional Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.2}$$

The fermions ψ_i^μ are two-dimensional Majorana fermions, i.e.:

$$\bar{\psi}_i^\mu \equiv \big(\psi_i^\mu\big)^\dagger \gamma^0 = \big(\psi_i^\mu\big)^T \gamma^0, \tag{2.3}$$

[1] The $D = 2 + 2$ critical dimension for the $N = 2$ superstring was first found in [11].

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γ^0 being the charge conjugation matrix (see Appendix A for conventions). The action (2.1) is invariant under the supersymmetry variations:

$$\delta X = \bar{\epsilon}_i \psi_i, \tag{2.4}$$

$$\delta Y = \epsilon_{ij} \bar{\epsilon}_i \psi_j, \tag{2.5}$$

$$\delta \psi_i = -i\gamma^\alpha \partial_\alpha X \epsilon_i + i\epsilon_{ij}\gamma^\alpha \partial_\alpha Y \epsilon_j. \tag{2.6}$$

We can rearrange the bosonic and fermionic degrees of freedom (respectively X^μ, Y^μ and $\psi_i{}^\mu$) in the 2×2 matrices:

$$X_{AA'} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -X^0 + X^1 & Y^0 - Y^1 \\ -Y^0 - Y^1 & -X^0 - X^1 \end{pmatrix} \tag{2.7}$$

and

$$\psi_{AA'} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -\psi_1^0 + \psi_1^1 & \psi_2^0 - \psi_2^1 \\ -\psi_2^0 - \psi_2^1 & -\psi_1^0 - \psi_1^1 \end{pmatrix}. \tag{2.8}$$

Using these notations, the Lagrangian in (2.1) can be recast in the form:

$$\mathcal{L}_{N=2} = (X_{AA'A''} X_{BB'B''} - i\bar{\psi}_{AA'}\gamma_{A''}\partial_{B''}\psi_{BB'}) \times \left(\eta^{A''B''}\epsilon^{AB}\epsilon^{A'B'}\right), \tag{2.9}$$

with $X_{AA'A''} \equiv \partial_{A''} X_{AA'}$.

The supersymmetry variations (2.6) become:

$$\delta X_{AA'} = \bar{\epsilon}_i \rho_{iA'}{}^{B'} \psi_{AB'},$$
$$\delta \psi_{AA'} = -i\partial_{A''} X_{AB'} \rho^t{}_i{}^{B'}{}_{A'} \gamma^{A''} \epsilon_i, \tag{2.10}$$

where the 2×2 matrices ρ_i are:

$$\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.11}$$

3. Triality invariance

Consider now the world-sheet metric:

$$G''_{A''B''} \equiv \partial_{A''} X^\mu \partial_{B''} X^\nu \eta_{\mu\nu} + \partial_{A''} Y^\mu \partial_{B''} Y^\nu \eta_{\mu\nu} = X_{AA'A''} X_{BB'B''} \epsilon^{AB} \epsilon^{A'B'}. \tag{3.1}$$

In terms of $G''_{A''B''}$ the bosonic part of the Lagrangian (2.9) is given by:

$$\mathcal{L}_b = \eta^{A''B''} G''_{A''B''}. \tag{3.2}$$

As shown by Duff in [4], the Nambu–Goto Lagrangian

$$\mathcal{L}_{\mathrm{NG}} = \sqrt{-\det(G''_{A''B''})} \tag{3.3}$$

is invariant under the discrete triality transformations interchanging the three indices of $X_{AA'A''}$. In fact, the usual determinant of the world-sheet metric G'' can be reexpressed as (minus) the Cayley's hyperdeterminant of the cubic matrix $X_{AA'A''}$, which is explicitly triality invariant [4]:

$$\operatorname{Det} X \equiv -\frac{1}{2}\epsilon^{AB}\epsilon^{A'B'}\epsilon^{CD}\epsilon^{C'D'}\epsilon^{A''D''}\epsilon^{B''C''} \times X_{AA'A''} X_{BB'B''} X_{CC'C''} X_{DD'D''}. \tag{3.4}$$

We prove now that also the bosonic Lagrangian $\mathcal{L}_b$ in (3.2) is invariant under triality, up to total divergence terms. To show this, we need the metrics G and G' defined by:

$$G_{AB} \equiv X_{AA'A''} X_{BB'B''} \epsilon^{A'B'} \epsilon^{A''B''}, \tag{3.5}$$

$$G'_{A'B'} \equiv X_{AA'A''} X_{BB'B''} \epsilon^{A''B''} \epsilon^{AB}. \tag{3.6}$$

With these metrics, we can write the "triality symmetrized" bosonic Lagrangian, explicitly invariant under triality:

$$\mathcal{L}_b = \left(\eta^{AB} G_{AB} + \eta^{A'B'} G'_{A'B'} + \eta^{A''B''} G''_{A''B''}\right) = \left(g^{AA',BB'}\eta^{A''B''} + B^{AA',BB'}\epsilon^{A''B''}\right) X_{AA'A''} X_{BB'B''} \tag{3.7}$$

where

$$g^{AA',BB'} = \epsilon^{AB}\epsilon^{A'B'} = g^{BB',AA'} \tag{3.8}$$

plays the role of a 4-dimensional flat metric, and

$$B^{AA',BB'} = \left(\epsilon^{AB}\eta^{A'B'} + \epsilon^{A'B'}\eta^{AB}\right) = -B^{BB',AA'} \tag{3.9}$$

plays the role of a 4-dimensional constant B-field.

The triality-invariant Lagrangian (3.7) differs from $\mathcal{L}_b$ in (3.2) by the B-term: this term is a total divergence, since it is equal to

$$B^{AA',BB'}\epsilon^{A''B''}\partial_{A''} X_{AA'} \partial_{B''} X_{BB'} = \partial_{A''}\left(B^{AA',BB'}\epsilon^{A''B''} X_{AA'} \partial_{B''} X_{BB'}\right). \tag{3.10}$$

This result can be generalized to the supersymmetric case. The Lagrangian

$$\mathcal{L}_{N=2} = \left(g^{AA',BB'}\eta^{A''B''} + B^{AA',BB'}\epsilon^{A''B''}\right) \times (X_{AA'A''} X_{BB'B''} - i\bar{\psi}_{AA'}\gamma_{A''}\partial_{B''}\psi_{BB'}) \tag{3.11}$$

is explicitly invariant under (2.10) and triality transformations, and differs from the original $N=2$ superstring Lagrangian in (2.9) only by the B-terms. Again these terms are a total divergence. This has already been proven for the BXX term; to show that also the $B\psi\psi$ term is a total divergence we just have to use the antisymmetry of B and the equality:

$$\bar{\psi}_{AA'}\gamma_{A''}\partial_{B''}\psi_{BB'} = -\partial_{B''}\bar{\psi}_{BB'}\gamma_{A''}\psi_{AA'} \tag{3.12}$$

due to ψ being a $D=2$ Majorana fermion.

4. Siegel–Berkovits formulation

As shown by Siegel [7,8] one can describe self-dual super-Yang–Mills in superspace by extending the bosonic coordinates $X_{AA'}$ to $(X_{AA'}, \Theta_{Aj})$ where $j = 1,\ldots,\mathcal{N}$ (we do not include the antichiral coordinates as in [9]). It is convenient to cast the $(X_{AA'}, \Theta_{Aj})$ into a supercoordinate $Y_{AJ} = (X_{AA'}, \Theta_{Aj})$ with $J = (A', j)$, which is a vector representation of the supergroup $OSp(\mathcal{N}|2)$. In order to implement the triality we choose the real form $OSp(2,2|2)$ which has the subgroups $SO(2,2)\times Sp(2) \sim SL(2,R)\times SL(2,R)\times SL(2,R)$. Therefore, the supercoordinates are labelled by $(X_{AA_1}, \Theta_{AA_2A_3})$ where the $SO(2,2)$ acts on the A_2 and A_3 indices, $Sp(2)$ acts on A_1 and the supersymmetry generators Q_{A_1,A_2A_3} act as follows:

$$Q_{A_1,A_2A_3} X_{BB_1} = \epsilon_{A_1B_1}\Theta_{BA_2A_3},$$
$$Q_{A_1,A_2A_3}\Theta_{BB_2B_3} = \epsilon_{A_2B_2}\epsilon_{A_3B_3} X_{BA_1}. \tag{4.1}$$

Notice that there are effectively four $SL(2,R)$ groups. Let us denote them by $SL_0(2,R)\times SL_1(2,R)\times SL_2(2,R)\times SL_3(2,R)$. In addition, we add the $SL(2,R)$ of the worldsheet and we denote it by $SL_w(2,R)$. We have denoted by A_0, A_1, A_2, A_3, A_w the indices for each of them. Thus, for example, the bosonic coordinates $X_{A_0A_1A_w}$ transform under $SL_0(2,R)\times SL_1(2,R)\times SL_w(2,R)$.

In the formulation of [9], the matter part of the action reads

$$S_m = \int d^2z \left(\partial Y_{AJ} \bar{\partial} Y^{AJ}\right) = \int d^2x \left(\eta^{A_wB_w}\epsilon^{A_0B_0}\epsilon^{A_1B_1} X_{A_0A_1A_w} X_{B_0B_1B_w} + \eta^{A_wB_w}\epsilon^{A_0B_0}\epsilon^{A_2B_2}\epsilon^{A_3B_3}\Theta_{A_0A_2A_3A_w}\Theta_{B_0B_2B_3B_w}\right), \tag{4.2}$$

where $\Theta_{A_0A_2A_3A_w} \equiv \partial_{A_w}\Theta_{A_0A_2A_3}$.

The contraction of indices is performed with the invariant tensors of $SL_0(2,R) \times SL_1(2,R) \times SL_2(2,R) \times SL_3(2,R)$ (except for the worldsheet indices, contracted with the metric $\eta^{A_wB_w}$). The bosonic term is manifestly invariant under the triality exchange of the three groups in $SL_0(2,R) \times SL_1(2,R) \times SL_w(2,R)$: it means that any permutation of the A_0, A_1, A_w indices leaves the action invariant.

It is easy to verify that the action is invariant under the supersymmetry transformations (4.1).

Notice however, that the bosonic term and the fermionic term are separately invariant under the reshuffling of the $SL(2)$ indices. For the action to be invariant under the same triality, we have to identify the groups $SL_i(2,R)$, $i = 1,2,3$. Namely, the action is invariant only under the small triality reshuffling and not under the big pentality reshuffling of the fermionic terms. To see this, consider the bosonic coordinates $X_{A_0A_1A_w}$. The action is invariant, for example, under the reshuffling $X_{A_0A_1A_w} \to X_{A_1A_wA_0}$ as discussed above. However, if we are reshuffling the indices as $X_{A_0A_1A_w} \to X_{A_0A_2A_w}$, where we exchange $SL_1(2)$ with $SL_2(2)$, we have to define the new quantities $X_{A_0A_2A_w}$ since they are now charged under a new $SL(2)$. So, in order to complete the triality, we have to identify $X_{A_0A_2A_w}$ with $X_{A_0A_1A_w}$, which means that they transform only under the diagonal subgroup of $SL_1(2) \times SL_2(2)$. Adding also $X_{A_0A_3A_w}$ we obtain an action invariant under $SL_0 \times SL_{\text{diag}} \times SL_w$, SL_{diag} being the diagonal subgroup of $SL_1(2) \times SL_2(2) \times SL_3(2)$. In the same way we proceed for the fermions.

We can therefore rewrite the action as

$$\begin{aligned}
S_m = \frac{1}{3}\int d^2x \big[&\eta^{A_wB_w}\epsilon^{A_0B_0}\big(\epsilon^{A_1B_1}X_{A_0A_1A_w}X_{B_0B_1B_w} \\
&+ \epsilon^{A_2B_2}X_{A_0A_2A_w}X_{B_0B_2B_w} + \epsilon^{A_3B_3}X_{A_0A_3A_w}X_{B_0B_3B_w}\big) \\
&+ \eta^{A_wB_w}\epsilon^{A_0B_0}\big(\epsilon^{A_2B_2}\epsilon^{A_3B_3}\Theta_{A_0A_2A_3A_w}\Theta_{B_0B_2B_3B_w} \\
&+ \epsilon^{A_1B_1}\epsilon^{A_2B_2}\Theta_{A_0A_1A_2A_w}\Theta_{B_0B_1B_2B_w} \\
&+ \epsilon^{A_1B_1}\epsilon^{A_3B_3}\Theta_{A_0A_1A_3A_w}\Theta_{B_0B_1B_3B_w}\big)\big].
\end{aligned}$$

The triality under the exchange of SL_0, SL_{diag} and SL_w becomes manifest after adding some boundary terms, as we did in the case of the NSR action. This means adding to the metric the B term as in (3.7), replacing the invariant tensors $\eta^{A_wB_w}\epsilon^{A_0B_0}\epsilon^{A_iB_i}$ with $\frac{1}{3}(\eta^{A_wB_w}\epsilon^{A_0B_0}\epsilon^{A_iB_i} + \eta^{A_0B_0}\epsilon^{A_iB_i}\epsilon^{A_wB_w} + \eta^{A_iB_i}\epsilon^{A_wB_w}\epsilon^{A_0B_0})$, $\eta^{A_wB_w}\epsilon^{A_0B_0}\epsilon^{A_2B_2}\epsilon^{A_3B_3}$ with $\frac{1}{3}(\eta^{A_wB_w}\epsilon^{A_0B_0}\epsilon^{A_2B_2} + \eta^{A_0B_0}\epsilon^{A_2B_2}\epsilon^{A_wB_w} + \eta^{A_2B_2}\epsilon^{A_wB_w}\epsilon^{A_0B_0})\epsilon^{A_3B_3}$ and similarly for the terms $\eta^{A_wB_w}\epsilon^{A_0B_0}\epsilon^{A_1B_1}\epsilon^{A_3B_3}$ and $\eta^{A_wB_w}\epsilon^{A_0B_0}\epsilon^{A_1B_1}\epsilon^{A_2B_2}$. In this way we add only boundary terms.

For the ghost field, the situation is more involved, but since this is a consequence of the specific gauge choice that reduces the Green–Schwarz action to the Siegel–Berkovits action, the possible violation of the triality is only through BRST exact terms which do not affect the physical amplitudes.

There are two important aspects that we have to point out. The first one is that the boundary terms for the fermionic pieces work as in the case of the bosonic terms, and therefore the action is manifestly invariant under the triality that exchanges the three groups $SL_0(2)$, $SL_w(2)$ and $SL_{\text{diag}}(2)$. The second aspect, as was noted in [9], is that the choice $\mathcal{N} = 2+2$ is mandatory to cancel the BRST anomaly. Here we found that the triality – which is only present in the case of the supergroup $OSp(2,2|2)$ – implies that cancellation of the anomalies. This is a confirmation of previous work and an unexpected present from the triality. There is an additional minor point: the fermionic terms display an additional $SL(2,R)$ symmetry which implies a tetrality instead of a triality. We do not have any interpretation, but it might refer to a twist between the R-symmetry and the triality.

The present formulation is suitable for computations of amplitudes and the manifest duality should show up in the computations. This will be explored in a separate work.

5. Super-hyper-det based on $SL(2|1)^3$ algebra

Given the results of the previous sections it is natural to try and generalize the hyperdeterminant, invariant under $SL(2)^3$, to a supersymmetric object, invariant under a superalgebra which contains $SL(2)^3$ as a bosonic subalgebra. In fact we can build a quartic (bosonic) supersymmetric object based on the $SL(2|1)^3$ superalgebra which, by setting a suitable set of fields to zero, precisely reproduces the hyperdeterminant of [1].

The $SL(2|1)^3$ superalgebra is made out of three copies of the following:

$$\begin{aligned}
&\{Q_A, Q_B\} = P_{AB}, \\
&[P_{AB}, Q_C] = -\epsilon_{C(A}Q_{B)}, \\
&[P_{AB}, P_{CD}] = 2\epsilon_{(A(C}P_{D)B)}.
\end{aligned} \tag{5.1}$$

The indices A, A', A'' label the three $SL(2|1)$ factors of the superalgebra. It is possible to construct a $SL(2|1)^3$ representation with 27 fields, 14 of which, $X_{AA'A''}$, Y_A, $Y_{A'}$ and $Y_{A''}$, are bosonic while the remaining 13, $\psi_{AA'}$, $\psi_{A'A''}$, $\psi_{A''A}$ and η are fermionic. The action of the algebra on the fields is given by:

$$\begin{aligned}
&Q_AX_{BB'B''} = \frac{1}{2}\epsilon_{AB}\psi_{B'B''}, \qquad Q_{A'}X_{BB'B''} = \frac{1}{2}\epsilon_{A'B'}\psi_{B''B}, \\
&Q_{A''}X_{BB'B''} = \frac{1}{2}\epsilon_{A''B''}\psi_{BB'}, \qquad Q_A\psi_{BB'} = \epsilon_{AB}Y_{B'}, \\
&Q_{A'}\psi_{BB'} = -\epsilon_{A'B'}Y_B, \qquad Q_{A''}\psi_{BB'} = X_{BB'A''}, \\
&Q_A\psi_{B'B''} = X_{AB'B''}, \qquad Q_{A'}\psi_{B'B''} = \epsilon_{A'B'}Y_{B''}, \\
&Q_{A''}\psi_{B'B''} = -\epsilon_{A''B''}Y_{B'}, \qquad Q_A\psi_{B''B} = -\epsilon_{AB}Y_{B''}, \\
&Q_{A'}\psi_{B''B} = X_{BA'B''}, \qquad Q_{A''}\psi_{B''B} = \epsilon_{A''B''}Y_B, \\
&Q_AY_B = \frac{1}{2}\epsilon_{AB}\eta, \qquad Q_{A'}Y_B = -\frac{1}{2}\psi_{BA'}, \\
&Q_{A''}Y_B = \frac{1}{2}\psi_{A''B}, \qquad Q_AY_{B'} = \frac{1}{2}\psi_{AB'}, \\
&Q_{A'}Y_{B'} = \frac{1}{2}\epsilon_{A'B'}\eta, \qquad Q_{A''}Y_{B'} = -\frac{1}{2}\psi_{B'A''}, \\
&Q_AY_{B''} = -\frac{1}{2}\psi_{B''A}, \qquad Q_{A'}Y_{B''} = \frac{1}{2}\psi_{A'B''}, \\
&Q_{A''}Y_{B''} = \frac{1}{2}\epsilon_{A''B'}\eta, \qquad Q_A\eta = Y_A, \\
&Q_{A'}\eta = Y_{A'}, \qquad Q_{A''}\eta = Y_{A''}.
\end{aligned}$$

In the quartic invariant, only the following bilinear building blocks contribute:

$$\begin{aligned}
&X_{(AB)} = X_{AA'A''}X_{BB'B''}\epsilon^{A'B'}\epsilon^{A''B''}, \qquad A_{(AB)} = \psi_{AA'}\epsilon^{A'B'}\psi_{BB'}, \\
&B_{(AB)} = \psi_{A''A}\epsilon^{A''B''}\psi_{B''B}, \qquad W_{(AB)} = Y_AY_B, \\
&\omega_A = Y_{A''}\epsilon^{A''B''}\psi_{B''A}, \qquad \nu_A = Y_{A'}\epsilon^{A'B'}\psi_{AB'}, \\
&\Delta_A = X_{AA'A''}\psi_{B'B''}\epsilon^{A'B'}\epsilon^{A''B''}, \qquad \chi_A = Y_A\eta,
\end{aligned}$$

together with their prime and double prime counterparts; notice that the building blocks with two indices are bosonic and those with one index are fermionic. These blocks can be rearranged in the combinations:

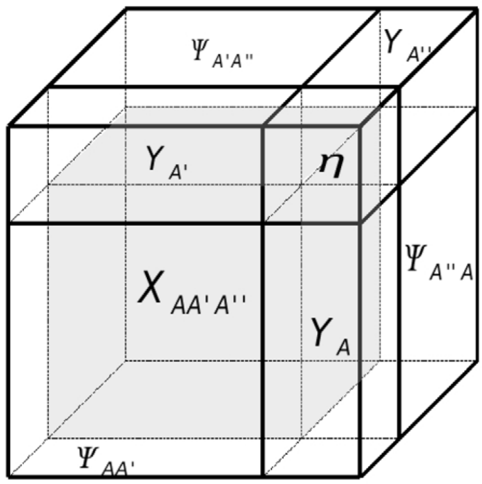

Fig. 1. The $3 \times 3 \times 3$ cubic supermatrix.

$$\mathcal{Z}_{AB} \equiv 2X_{AB} - A_{AB} - B_{AB} - 2W_{AB},$$
$$\Phi_A \equiv 2(\Delta_A - \nu_A + \omega_A - \chi_A),$$

which obey very simple supersymmetry relations:

$$Q_A \mathcal{Z}_{BC} = \epsilon_{A(B} \Phi_{C)},$$
$$Q_A \Phi_B = \mathcal{Z}_{AB},$$
$$Q_{A'} \mathcal{Z}_{AB} = Q_{A''} \mathcal{Z}_{AB} = 0,$$
$$Q_{A'} \Phi_A = Q_{A''} \Phi_A = 0.$$

Then one easily checks that

$$H = -\frac{1}{48}\big(\mathcal{Z}_{AB}\mathcal{Z}^{AB} + \mathcal{Z}_{A'B'}\mathcal{Z}^{A'B'} + \mathcal{Z}_{A''B''}\mathcal{Z}^{A''B''} + \Phi_A \Phi^A + \Phi_{A'}\Phi^{A'} + \Phi_{A''}\Phi^{A''}\big)$$

is invariant under the action of the superalgebra. The indices are raised/lowered with the use of the $SL(2)$-invariant epsilon tensors according to the rule given in (A.5), and the factor $-\frac{1}{48}$ has been chosen to reproduce the hyperdeterminant once all the fields but $X_{AA'A''}$ are set to zero.

Note 1. H can be seen as the definition of the super-Cayley determinant of the cubic supermatrix given in Fig. 1.

Note 2. H is also equal to the sum of the Berezinians of the three 3×3 supermatrices

$$\begin{pmatrix} \mathcal{Z}_{AB} & \frac{1}{\sqrt{2}}\Phi_A \\ \frac{1}{\sqrt{2}}\chi_B & 1 \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{Z}_{A'B'} & \frac{1}{\sqrt{2}}\Phi_{A'} \\ \frac{1}{\sqrt{2}}\chi_{B'} & 1 \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{Z}_{A''B''} & \frac{1}{\sqrt{2}}\Phi_{A''} \\ \frac{1}{\sqrt{2}}\chi_{B''} & 1 \end{pmatrix}, \tag{5.2}$$

with $\chi_B \equiv -\Phi^C \mathcal{Z}^{-1}_{BC}$, etc.

6. Conclusions and outlook

We have constructed the supersymmetric generalization of the triality invariance first found by Duff in the Nambu–Goto string moving in a flat $D = 2 + 2$ target space. This we achieve by adding boundary terms in the NSR superstring action, and in the Siegel–Berkovits formulation of the selfdual superstring. Moreover, we have proposed a supersymmetric generalization of the Cayley hyperdeterminant, based on a quartic invariant of the $SL(2|1)^3$ superalgebra. It may be intriguing to speculate on its possible applications in quantum information or in the description of black holes in string/brane theory.

Appendix A. $D = 2$ gamma matrices

We use the representation:

$$\gamma^0 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{A.1}$$

for the two-dimensional γ-matrices, satisfying the usual relations

$$\{\gamma^\alpha, \gamma^\beta\} = -\eta^{\alpha\beta} \quad \text{and} \quad \gamma^\alpha \gamma^\beta = -\eta^{\alpha\beta}\mathbb{1} + \epsilon^{\alpha\beta}\gamma_3,$$

where the metric is $\eta = (-, +)$, ϵ is the usual Levi-Civita symbol and

$$\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The charge conjugation matrix is $C = \gamma^0$, so that all the spinors are real and the following relations hold:

$$\gamma^0 (\gamma^\alpha)^\dagger \gamma^0 = \gamma^\alpha, \qquad \gamma^0 (\gamma^\alpha)^t \gamma^0 = -\gamma^\alpha.$$

Finally, for Majorana fermions the currents satisfy:

$$\bar{\xi}\zeta = \bar{\zeta}\xi, \tag{A.2}$$
$$\bar{\xi}\gamma_3\zeta = -\bar{\zeta}\gamma_3\xi, \tag{A.3}$$
$$\bar{\xi}\gamma^\alpha\zeta = -\bar{\zeta}\gamma^\alpha\xi. \tag{A.4}$$

The $SL(2)$-invariant tensor $\epsilon^{\alpha\beta}$ is used to raise and lower the indices according to:

$$V_\alpha = \epsilon_{\alpha\beta} V^\beta, \qquad V^\alpha = -\epsilon^{\alpha\beta} V_\beta. \tag{A.5}$$

Appendix B. Some notes on $OSp(2, 2|2)$

The supergroup is characterized by the following superalgebra generated by the bosonic generators P_{AB}, $P'_{A'B'}$, $P''_{A''B''}$ and by the fermionic generators $Q_{AA'A''}$:

$$\begin{aligned}
&\{Q_{AA'A''}, Q_{BB'B''}\} \\
&\quad = \frac{1}{2}\epsilon_{AB}\epsilon_{A'B'}P_{A''B''} + \frac{1}{2}\epsilon_{AB}P'_{A'B'}\epsilon_{A''B''} - P_{AB}\epsilon_{A'B'}\epsilon_{A''B''}, \\
&[P_{AB}, P_{CD}] = 2\epsilon_{(A(C}P_{D)B)}, \\
&[P'_{A'B'}, P'_{C'D'}] = 2\epsilon_{(A'(C'}P'_{D')B')}, \\
&[P''_{A''B''}, P''_{C''D''}] = 2\epsilon_{(A''(C''}P''_{D'')B'')}, \\
&[P_{AB}, Q_{CC'C''}] = -\epsilon_{C(A}Q_{B)C'C''}, \\
&[P'_{A'B'}, Q_{CC'C''}] = -\epsilon_{C'(A'}Q_{C|B')C''}, \\
&[P''_{A''B''}, Q_{CC'C''}] = -\epsilon_{C''(A''}Q_{CC'|B'')}.
\end{aligned} \tag{B.1}$$

They provide the adjoint representation of the superalgebra. Denoting by $T_{\mathcal{M}}$ the supergenerators of $OSp(2, 2|2)$, by $V_{\mathcal{M}}$ the components of the supermultiplet and by $f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{R}}$ the super-structure constants, we set

$$T_{\mathcal{M}} V_{\mathcal{N}} = f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{R}} V_{\mathcal{R}}, \tag{B.2}$$

and it is obvious to see that it forms a representation. Notice that since the representation is linear, there is no problem to set either $X_{AA'A''}$ as a fermion or as a boson.

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