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**A Continuum Modeling Approach for Network Vulnerability Analysis at Regional Scale**

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**Abstract**

Vulnerability analysis is crucial in strategic planning and highway maintenance to ensure a robust transportation system. Owing to the characteristics of network-modeling framework, traditional vulnerability analyses may not be able to realistically model the impacts of network degradations. This paper presents an application of the continuum traffic equilibrium model for network vulnerability analysis that aims to resolve the critical issues faced by the network-modeling framework. The continuum traffic equilibrium model treats the road system as a continuum over which the demands are continuously dispersed. In this study, a bi-level model is set up for finding the most vulnerable location(s) in the study region. At the lower-level model, a set of differential equations is constructed to describe the traffic equilibrium problem under capacity degradation. In the upper-level model, a constrained minimization problem is set up to find the most vulnerable location(s) that minimizes the accessibility index of the study region. A sensitivity-based solution algorithm that adopts the finite element method (FEM) is proposed to solve the bi-level model. Numerical examples are presented to demonstrate the characteristics of the proposed continuum vulnerability analysis and the efficiency of the proposed solution algorithm.

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**Keywords:** Vulnerability analysis; Regional model; Continuum modeling approach

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1. **Introduction**

Mobility is the key to success in the economic and social development of a city, while transport network infrastructure is the key component in providing mobility to both freight and passenger transport. Unfortunately, roads within the transport network will sometimes be closed for various kinds of reasons such as accidents, adverse weather conditions, natural disasters, or even terrorist attacks. Thus, in order to reduce the social and economic impacts from these natural or malevolent events, vulnerability analysis, which aims at identifying the weak spots in the transport network and the corresponding impacts upon failure, is vital in strategic planning to identify the critical areas/roads for network improvements.

In the literature, studies of vulnerability analysis could be divided into two main categories: devising definitions/measures for network vulnerability and developing efficient algorithm for evaluating network vulnerability. For the studies in the first category, majority of the previous works have considered the change of total travel time/distance after link failure as a measure of network vulnerability [1, 2, 3, 4]. Taylor et al. [1] have considered three different indexes: generalized travel cost, inverse of travel distance and ARIA index, which is a remoteness index adopted in Australia, for evaluating the accessibility between different zones within the Australian network. By comparing the changes of these indexes upon network degradation at different locations, the vulnerable links/spots in the Australian network are identified. Apart from the studies that solely consider...
travel time/distance, Chen et al. [5] have introduced the dimension of travel behavioral responses (destination choice, mode choice, etc) in vulnerability analysis. With such consideration, the formulation proposed by Chen et al. [5] could more precisely model the change of travel demands under network degradations and gives a more realistic estimation of network vulnerability. In the above studies, although vulnerability analyses are developed based on network performances and/or travel behaviors, the impact of network topology has not been considered explicitly. In the literature, the impact of network topology in vulnerability analysis is usually realized by considering the change of network/OD connectivity under infrastructure degradation(s) [6, 7, 8]. Kurauchi et al. [8] have incorporated the idea of network topology into critical link identification by introducing a link criticality index that is defined by the number of connected paths between OD pairs. In their study, the link that causes the largest decrease in the number of connected paths as it degrades will be identified as the most critical link. Murray et al. [6] and Matisziw et al. [7] have applied the connectivity-based vulnerability analysis to the communication network. For the second category, several studies have investigated the vulnerability analysis in large-scale networks (see, for example, [1, 9, 10]). Traditionally, the identification of critical link(s) within network is completed by network scan [1] that each network link is partially/completely degraded in turn for reassignment. The network performances (e.g. travel time) are then compared with the non-degraded case for reassignment.

In the literature, the majority of vulnerability analyses are developed under the network-modeling framework in which the transportation system is represented by nodes and links while the demands are defined at the zone centroids. Thus, results of the vulnerability analyses will largely depend on the details of network coding and the spatial representation (in terms of demand distribution and the road network coverage) of the model. Considering the characteristics of the network-modeling based framework, three critical issues may arise and cause biases on the result of vulnerability analysis. These three issues include:

(i) The result of vulnerability analysis depends on the details of network representation. In most cases, major routes, or main highway corridors, will be fully modeled but some sub-networks of minor roads may not be included in the network coding. Thus, the analysis of critical link will be biased by the network coding and may overestimate the importance of some major links due to the lack of alternative routes.

(ii) The arbitrary definition of demand locations in the network (i.e. zone centroids and centroid connectors) can potentially cause unrealistic result of the vulnerability analysis. Due to the discrete representation of demand distribution, a certain link in the network may be attached to a particular demand location/group, and, hence, the failure of this link will cause a major impact to these travelers.

(iii) The catastrophic disruptions in transport networks in recent years have involved different types of wide-area natural disasters (e.g. flooding, earthquake). By nature, this type of disruptions will affect an area instead of a particular link. In this emergency case, the evaluation of area accessibility requires a better geographic representation of the terrain as well as transport network (e.g. to identify the differences between bridge and typical surface road failures). The current network modeling framework cannot allow such representation.

Recently, attempts have been made to provide a more realistic representation of the network/area degradations and the corresponding impacts in the network-based vulnerability analysis. Jenelius and Mattsson [12] have introduced a grid-based approach for modeling the area-covering network disruptions. Their study shows that, due to the reduced influence from network redundancy, such area-covering disruption will give a markedly different result in vulnerability analysis as compared to the studies of single link failure. Taylor and Susilawati [13] have introduced a location-based accessibility metrics, which takes into account the continuously varied travel distances over the study area, for a more realistic representation of impacts from network degradation in vulnerability analysis. Despite the continuum descriptions in network degradations and impact considerations, the use of traditional network-based traffic assignment model in these studies [12, 13] restraints them from addressing the above critical issues effectively and comprehensively. Thus, the motivation of this paper is to effectively address the above three critical issues in regional vulnerability analysis. In this paper, the continuum modeling framework [14, 15, 16, 17, 18] is utilized to represent the regional or urban space including both geographic (e.g. river, mountain) and transport network representations. In the continuum model, demand will be continuously distributed in accordance to the actual population distribution (hence reducing the biases of the zonal representation discussed in ii). For the supply side of the transportation system, the continuum model
approximates the road network as a continuum and users are free to choose their routes in a two-dimensional space [16]. Thus, the continuum model framework proposed herein also allow for the analysis of wide-area disruption and represent all possible alternative spaces for vehicle and people movements under such drastic situations (hence removing the alternative route constraint discussed in i and considering the area-wide disruption discussed in iii). In this study, the vulnerability analysis for the continuum transportation system is formulated as a bi-level model. The upper-level model is a vulnerability analysis model that aims to determine the most vulnerable location(s) of the study region. The upper-level model will be formulated as an optimization problem with accessibility index of the entire study region as the objective function. The location of network degradation, which is determined by the upper-level model, will be input to the lower-level model, which is a continuum traffic equilibrium model [15, 16], for determining the path choices, travel costs and the corresponding sensitivity information. This information will be used in the upper-level model to search for the most vulnerable location(s).

The remainder of this paper is organized as follows. Section 2 defines the notations to be used and the specific equations of the problem. Section 3 discusses the formulations and solution algorithms for the upper- and lower-level model of the continuum vulnerability analysis. Numerical examples that demonstrate the effectiveness and usefulness of the model are given in Section 4 and the paper concludes in Section 5.

2. Notations and definitions

Consider an arbitrary-shaped region with multiple central business districts (CBDs) as shown in Figure 1, in which the road network is approximated as a continuum [14]. It is assumed that these CBDs are sufficiently compact as compared with the whole region. Different classes of users, who are continuously distributed over the region, will travel from their demand location to the CBDs along the least costly route within the two-dimensional space. Due to the differences of traveling environment in the surface road and expressway system, the expressways will be separately defined in the study region (Figure 1). In the definition of study region, the continuum modeling approach has two advantages: arbitrary and scale-free. As the study region for the continuum modeling approach is solely defined by the spatial locations of the outer and CBD boundaries (i.e. no specific shape is required), modelers could easily define the shape of the study region based on the scope of study (e.g. a province, a river basin, etc). As the finite element method is adopted for solving the continuum model (see Section 3.1.2 for details), problem size is only depended on the number of finite elements used, but not the size of the study area. Thus, it is possible to have a larger study area without substantially increase the problem size. Such scale-free nature of the continuum modeling approach and the corresponding solution algorithm enables its flexibility in modeling problems with different scales.

Let the study region be \( \Omega \), and the outer boundary be \( \Gamma \). Let the locations of CBDs be \( L_d, d \in D \), where \( D \) is the set of CBDs within the study region. To avoid singularity at the CBDs, it is assumed that each of the CBDs is of finite size and enclosed by a boundary, \( \Gamma_{cd} \). The travel cost per unit distance of travel (e.g. HKD/km) at
location \((x, y) \in \Omega\) for class \(m\) users is denoted by \(c_m(x, y)\), which is location dependent, and has the following functional relationship with traffic flows at that location:

\[
c_m(x, y) = a_m(x, y) + \frac{b_m(x, y)}{v(x, y)K(x, y)} \sum_i \sum_j |g_i(x, y)|, \quad \forall (x, y) \in \Omega, \quad i, m \in M, \quad j \in D,
\]

where \(M\) is the set of user classes considered in this study, \(a_m(x, y)\) and \(b_m(x, y)\), which are strictly positive scalar functions of the cost-flow relationship that reflects the local characteristics of the streets at location \((x, y) \in \Omega\), are respectively the free-flow and congestion-related parameter of class \(m\) users, \(K(x, y)\) is the road density, which is defined by the length of road per unit area (e.g. \(\text{km/km}^2\)), of location \((x, y)\) of the non-degraded system, \(v(x, y)\) is the percentage of road density remains at location \((x, y)\) after network degradation(s), \(g_i(x, y) = \left( f_{xmd}(x, y), f_{ymd}(x, y) \right)\) is the flow vector of class \(m\) users who are heading to CBD \(d\) and, \(f_{xmd}(x, y)\) and \(f_{ymd}(x, y)\) are the corresponding flow fluxes in the \(x\) and \(y\) directions, respectively. The flow vector \(g_i(x, y)\) indicates the movement direction of class \(m\) users at location \((x, y) \in \Omega\) when heading to CBD \(d\) in the two-dimensional plane, and

\[
|g_i(x, y)| = \sqrt{f_{xmd}^2(x, y) + f_{ymd}^2(x, y)}, \quad \forall (x, y) \in \Omega
\]

is the corresponding flow intensity that measures in a unit time the number of class \(m\) users who cross a small segment of unit width perpendicular to the flow direction (i.e. expressed in \(\text{veh/km/hr}\)). The percentage of road density remains at location \((x, y)\) after network degradation(s) is determined by,

\[
v(x, y) = \prod_{j \in J} v_j(x, y), \quad \forall (x, y) \in \Omega
\]

and

\[
v_j(x, y) = \begin{cases} \frac{1}{r_j} (v_j^* - v_j') \sqrt{(x_{j^c} - x)^2 + (y_{j^c} - y)^2} + v_j', & \forall (x, y) \in \Omega_j \\ 1, & \text{otherwise} \end{cases}
\]

where \(v_j(x, y)\) is the percentage of road density remains at location \((x, y)\) under network degradation \(j\); \(\Omega_j\) is the circular impact area of network degradation \(j\); \(x_{j^c}\) and \(y_{j^c}\) are respectively the \(x\)- and \(y\)-coordinate for the center of network degradation \(j\); \(r_j\) is the radius of the impact area for network degradation \(j\); \(v_j^*\) and \(v_j'\) are respectively the percentage of capacity remains at the boundary and center of the impact area \(\Omega_j\); \(J\) is the number of network degradations that are simultaneously considered. In this study, it is assumed that the degree of network degradation is highest, or the remaining road density is lowest, at the center of degradation (i.e. \(v_j' \leq v_j^*\)). Equation (1) represents an isotopic cost function because it depends only on the flow intensities but not on the flow directions. The congestion component includes the flow intensities of all classes of users because all users share the same road space. Each user creates an external cost to all other users through the causing of congestion. Basically, the calibration of the parameters \(a_m\) and \(b_m\) will be similar to the finding of area speed-flow relationship [19, 20, 21, 22, 23], or calibration of the macroscopic fundamental diagram [24, 25], for the study region. For the calibration of these parameters, traffic counts from different screen-lines within the study area are collected for estimating the average flow intensities within the area. Apart from the screen-line traffic counts, travel time surveys should also be carried out for estimating the travel costs across the study area. Lastly, with the collected flow intensities, estimated travel costs and road densities of the study region, the unit transportation cost function could be calibrated. As \(a_m\) and \(b_m\) are depended on the network characteristics (e.g. number of signalized junctions), the calibration will be repeated for areas with different network characteristics. In general, a function that describes the relationship between these parameters (i.e. \(a_m\) and \(b_m\)) and the network characteristics should be derived. For each combination of user class and CBD, the following flow conservation equation should be satisfied,

\[
\nabla \cdot g_{adm}(x, y) + q_{adm}(x, y) = 0, \quad \forall (x, y) \in \Omega, \quad d \in D, \quad m \in M.
\]
where \( q_{md}(x, y) \) is the demand (in veh/hr) of class \( m \) users at location \((x, y) \in \Omega\) who choose CBD \( d \). Assuming no flow across the boundary of the study area, we have the following boundary condition,

\[
f_{md}(x, y) = 0, \quad \forall (x, y) \in \Gamma, \ d \in D, \ m \in M.
\]

However, it is not difficult to extend the model to represent a given demand pattern that enters or leaves the study region at the boundary. In such case, we only need to replace the above constraint with \( f_{md}(x, y) \cdot n(x, y) = g_{md}(x, y) \), where \( g_{md}(x, y) \) is the demand function (in veh/hr) and \( n(x, y) \) is the unit normal vector pointing away from the study region at the boundary point \((x, y) \in \Gamma\). During road degradations, the corresponding travel costs through the degraded location(s) will increase due to the induced congestion and detours. Thus, the vulnerability of a location is measured by the change of total travel cost, or other equivalent measures, upon its degradation. In this study, the continuum form of Hansen integral accessibility index [26], which depends on the total travel costs of the users, is adopted for measuring the overall impact of network degradation. The accessibility index for class \( m \) users who are heading to CBD \( d \) (\( AI_{md} \)) is defined as follows:

\[
AI_{md} = \frac{\iint_{\Omega} q_{md}(x, y) d\Omega}{\iint_{\Omega} u_{md}(x, y) d\Omega}
\]

where \( u_{md}(x, y) \) is the total travel cost (in HKD) of class \( m \) users traveling to CBD \( d \) from location \((x, y) \). For each location \((x, y) \), the accessibility of class \( m \) users heading to CBD \( d \) is defined by reciprocal of the corresponding total travel cost (i.e. \( u_{md} \)). A location with large total travel costs will provide a low accessibility, and vice versa. The accessibility index of class \( m \) users heading to CBD \( d \) (\( AI_{md} \)) for the entire study region (\( \Omega \)) is defined as the weighted average, which the demand \( q_{md} \) is taken as the weight, of the corresponding accessibilities at each location \((x, y) \) within the study region. Due to the continuum nature of the problem, a double integration is adopted for evaluating the weighted average. By aggregating the accessibility indexes for all combinations of user classes and CBDs, the accessibility index for the entire study region is defined as follows:

\[
AI = \sum_{m \in M} \sum_{d \in D} \left[ AI_{md} \int_{\Omega} q_{md}(x, y) d\Omega \right] = \sum_{m \in M} \sum_{d \in D} \left[ \int_{\Omega} q_{md}(x, y) d\Omega \right] = \sum_{m \in M} \sum_{d \in D} \left[ \frac{\int_{\Omega} q_{md}(x, y) d\Omega}{\int_{\Omega} u_{md}(x, y) d\Omega} \right]
\]

From Equation (8), it could be seen that the accessibility index \( AI \) will decrease as the total travel costs (\( u_{md} \)) increase. Thus, the most vulnerable location, which has the largest increase in total travel cost, is the location with minimum accessibility index as it degrades. For the sake of clarity, the location indexes (\( x \) and \( y \)) for the aforementioned parameters/variables will be omitted in the following model formulations and solution algorithms.

3. Regional vulnerability analysis

In this study, the regional vulnerability analysis is formulated as a bi-level model as shown in Figure 2. The upper-level model is a vulnerability analysis model that aims to determine the most vulnerable locations, \((x_{jr}, y_{jr})\), \( \forall j \in J \), of the study region. For each step in the upper-level model, the locations of network degradations are input into the lower-level model for equilibrium assignment. The optimal total travel costs (\( u_{md} \)) and related sensitivity information from the lower-level model is feedback to the upper-level model for determining the decent direction.
In this section, the formulation and solution algorithm for the lower-level model are introduced in Section 3.1. Section 3.2 introduces the upper-level model and the corresponding solution algorithm.

3.1. Lower-level model

3.1.1. Model formulation

In this study, the lower-level model is a continuum traffic equilibrium model \([15, 27]\) with updated transportation cost function based on network degradations from the upper-level model (Equation 1). The continuum traffic equilibrium model adopted in this study is defined by the following set of governing equations:

\[
\begin{align*}
\left(a_m + \frac{b_m}{vK} \sum_{i=1}^{M} \sum_{j=1}^{D} |f_{ij}| \right) f_{md} + \nabla u_{md} &= 0 \quad \forall (x,y) \in \Omega, \quad \forall i, m \in M, \quad j, d \in D \\
\nabla f_{md} + q_{md} &= 0 \quad \forall (x,y) \in \Omega, \quad m \in M, \quad d \in D \\
\end{align*}
\]

(9a) (9b)

\[
\begin{align*}
\nabla f_{md} &= 0 \quad \forall (x,y) \in \Gamma, \quad m \in M, \quad d \in D \\
\end{align*}
\]

(9c)

\[
\begin{align*}
u_{md} &= 0 \quad \forall (x,y) \in \Gamma_{cd}, \quad m \in M, \quad d \in D \\
\end{align*}
\]

(9d)

In Equation (9d), as users on \(\Gamma_{cd}\) are already at the boundary of CBD, transportation cost should not be incurred (i.e. \(u_{md} = 0\)). Equation (9b) and (9c) are respectively the continuity equation and boundary condition of the traffic flow \(f_{md}\). From Equation (9a), it could be seen that //\(md\)\(\nabla\)\(f_{md}\), i.e. the flow vector is directly opposite to the gradient of the corresponding total travel cost \(u_{md}\) in \(\Omega\). For any used path \(p\) from a class \(m\) user’s demand location \((O)\) to CBD \(d\) \((\Gamma_{cd})\), the total travel cost can be obtained as

\[
C_{sup} = \int_{p} c_m d\xi = \int_{p} c_m \frac{f_{md}}{|f_{md}|} \cdot d\xi = -\int_{p} \nabla u_{md} \cdot d\xi = -\left[ u_{md} (\Gamma_{cd}) - u_{md} (O) \right] = u_{md} (O),
\]

(10)

using Equation (9a), (9d) and the fact that \(f_{md}/|f_{md}|\) is a unit vector that parallel to \(d\xi\). In contrast, for any unused path \(\tilde{p}\) from a user’s demand location \((O)\) to CBD \(d\) \((\Gamma_{cd})\), the total travel cost can be obtained as

\[
C_{sup} = \int_{\tilde{p}} c_m d\xi \geq \int_{\tilde{p}} c_m \frac{f_{md}}{|f_{md}|} \cdot d\xi = -\int_{\tilde{p}} \nabla u_{md} \cdot d\xi = -\left[ u_{md} (\Gamma_{cd}) - u_{md} (O) \right] = u_{md} (O).
\]

(11)

The inequality in Equation (11) is due to the fact that for some segments along route \(\tilde{p}\), the normal vectors \(f_{md}/|f_{md}|\) and \(d\xi\) are not parallel and hence \(d\xi \geq (f_{md}/|f_{md}|) \cdot d\xi\) for these segments. Therefore, for any unused path, the total travel cost is greater than or equal to that of the used routes. In this way, the model will guarantee that the users will choose his/her route in the study region in a user-optimal manner with respect to unit transportation cost function.
3.1.2. Solution algorithm

The finite element method (FEM) is used to approximate the continuous variables in the study region [28]. As there is no explicit objective function for the lower-level model, Galerkin formulation of the weighted residual technique is adopted [29]. In the Galerkin formulation, the governing equations (Equation 9a and 9b) are defined at each finite element node and a nodal residual vector, \( r_s \), is used to represent the extent to which the governing equations are locally satisfied around node \( s \) [27]. For the finite element node \( s \), the nodal residual vector of class \( m \) users who are heading to CBD \( d \) \( (r_smd) \) is defined as follows:

\[
\begin{align*}
\mathbf{r}_{smd}(\Psi) &= \\
&= \left\{ \begin{array}{l}
\sum_{e \in E_s} \int_{\Omega_s} \left( a_m + \frac{b_m}{vK} \sum_{i \in M} \sum_{j \in D} f_{ij} \right) f_{md} \left( \frac{\partial u_{md}}{\partial x} \right) N_s d\Omega \\
+ \sum_{e \in E_s} \int_{\Omega_s} \left( a_m + \frac{b_m}{vK} \sum_{i \in M} \sum_{j \in D} f_{ij} \right) f_{mdy} \left( \frac{\partial u_{md}}{\partial y} \right) N_s d\Omega \\
+ \sum_{e \in E_s} \int_{\Omega_s} (\nabla f_{md} + q_{md}) N_s d\Omega
\end{array} \right.
\end{align*}
\]  

(12)

where \( E_s \) is the set of finite elements that contain the finite element node \( s \), \( \Omega_s \) denotes the area of finite element \( e \), and \( N_s \) denotes the local interpolation function of finite element node \( s \). In this formulation, the boundary conditions (Equation 9c and 9d) are enforced by taking a zero weight function [29]. For the global satisfaction of the governing equations, we require that

\[
\mathbf{R}(\Psi) = \text{Col}(\mathbf{r}_{smd}(\Psi)) = 0,
\]  

(13)

where \( \Psi \) is the solution vector of the lower-level model (user equilibrium assignment model) and Equation (13) defines a system of governing equations for each node within the study region. We apply the Newton-Raphson algorithm with a line search to solve the problem, in which the iterative equation is derived as

\[
\Psi_{k+1} = \Psi_k - \lambda J(\Psi_k)^{-1} \mathbf{R}(\Psi_k),
\]  

(14)

where \( J(\Psi_k) \) is the Jacobian matrix of vector \( \mathbf{R}(\Psi_k) \) in iteration \( k \) of the solution algorithm, and \( \lambda \) is the step size, which is determined by the golden section method [30]. The solution procedure is summarized as follows.

Solution Procedure A

Step A1: Find an initial solution \( \Psi_0 \). Set \( k = 0 \).

Step A2: Evaluate \( \mathbf{R}(\Psi_k) \) and \( J(\Psi_k) \).

Step A3: If the relative error, \( \left| \mathbf{R}(\Psi_k) \right| / \left| \Psi_k \right| \), is less than an acceptable error \( \varepsilon \), then terminate, and \( \Psi_k \) is the solution.

Step A4: Otherwise, apply the golden section method to determine the step size \( \lambda^* \), which minimizes the norm of the residual vector \( \left| \mathbf{R}(\Psi_k - \lambda J(\Psi_k)^{-1} \mathbf{R}(\Psi_k)) \right| \). Then, set \( \Psi_{k+1} = \Psi_k - \lambda^* J(\Psi_k)^{-1} \mathbf{R}(\Psi_k) \).

Step A5: Replace \( \Psi_k \) with \( \Psi_{k+1} \). Set \( k = k + 1 \) and go to Step A2.

3.2. Upper-level model

3.2.1. Model formulation

This section introduces the regional vulnerability analysis model (upper-level model) that aims to find the most vulnerable location(s) by minimizing the accessibility index of the study region. A constrained optimization problem of the degradation location is defined as follows.
Minimize \[ AI(\Phi) = \frac{\sum \sum \int_{\Omega} q_{md}(x,y) \, d\Omega}{\sum \sum \int_{\Omega} u_{md}^*(x,y) \, d\Omega} \] (15a)

Subject to \((x_j, y_j) \in \Omega, \forall j \in J\), (15b)

where \( AI(\Phi) \) is the accessibility index of the study region \((\Omega)\), which depends on the demands \((q_{md})\) and equilibrium travel costs \((u_{md}^*)\) from the lower-level model; \( \Phi = \text{Col}\left(x_j, y_j, j = 1...J\right) \) is the vector of the coordinates for the center of degradations. Constraint (15b) is to ensure that all the centers of degradation \((x_j, y_j)\) has to be within the study region \(\Omega\).

3.2.2. Solution algorithm

Similar to the study by Ho and Wong [27], the convex combination method [30] is adopted for solving the minimization problem (15). To apply the convex combination method, the continuous variable in the minimization problem (15) is discretized through the application of the FEM [29]. Based on the convex combination method, a linear program for determining the descent direction of the minimization problem (15) could be set up as follows:

Minimize \[ \nabla AI(\Phi) \cdot v \] (16a)

Subject to \[ \frac{1}{2} \left( x_i^1 y_i^1 + x_i^2 y_i^2 + x_i^{aux} y_i^{aux} - x_i^1 y_i^1 - x_i^{aux} y_i^{aux} - x_i^2 y_i^2 \right) \geq 0, \forall j \in J, l \in \tilde{L}, \] (16b)

where \( v = \text{Col}(x_i^{aux}, y_i^{aux}, j = 1...J) \) is the auxiliary solution vector for the linear program (16), \((x_i^1, y_i^1)\) and \((x_i^2, y_i^2)\) are respectively the start and end node of a boundary segment \(l\), \(\tilde{L}\) is the set of boundary segments (after finite element discretization) for both of the outer boundary \((\Gamma)\) and CBD boundaries \((\Gamma_{cd})\). For each of the segment \(l\) in the outer boundary (CBD boundaries), the start and end nodes are taken in an anti-clockwise (clockwise) direction. The left-hand-side of constraint (16b) defines the area of a triangle enclosed by the nodes \((x_i^1, y_i^1)\), \((x_i^2, y_i^2)\) and \((x_i^{aux}, y_i^{aux})\). The non-negativity of these areas is to ensure the centers of network degradation \((x_i^{aux}, y_i^{aux})\) has to be located between the outer and CBD boundaries (i.e. within the study region \(\Omega\)).

For finding the decent direction in the convex combination method, derivatives (i.e. \( \partial u_{md}^*/\partial x_j \) and \( \partial u_{md}^*/\partial y_j \)) must be calculated and could be derived from a sensitivity analysis of the lower-level variables \((u_{md}^*)\) with respect to the upper-level variables \((x_j, y_j)\). In this study, the sensitivity analysis will be completed by the finite difference method. With the sensitivity information, the maximum decent direction of the minimization problem (15) could be found by solving the linear program (16). The following solution procedure is adopted to solve the regional vulnerability analysis model (the upper-level model).

Solution Procedure B

Step B1: Set \(k = 0\) and find a initial solution for the upper-level model, \(\Phi_0\).

Step B2: With \(\Phi_0\), solve the lower-level model base on solution procedure A to find the solution for the lower-level model, \(\psi_0\).

Step B3: Using \(\psi_0\), evaluate the sensitivity matrix based on the finite difference approach.

Step B4: With the sensitivity matrix from the lower-level model, the auxiliary vector \(\tilde{\Phi}_0\) is found by solving the linear program (16).

Step B5: Apply the golden section method (with the smallest search interval of \(\delta\)) to determine the step size \(\lambda^* \in [0,1]\), which minimizes the objection function, \(AI(\tilde{\Phi}_0 + \lambda^* (\tilde{\Phi}_0 - \Phi_0))\), in Equation (15a). Then, set \(\tilde{\Phi}_1 = \tilde{\Phi}_0 + \lambda^* (\tilde{\Phi}_0 - \Phi_0)\).
Step B6: If \( AI(\Phi_{k}) < AI(\Phi_{k+1}) \), then set \( \Phi_{k+1} = \Phi_{k+1}, k = k + 1 \) and go to Step B2; otherwise stop and \( \Phi_{k} \) is the solution to the upper-level model, and \( \Psi_{k} \) is the corresponding solution to the lower-level model.

4. Numerical example

In this paper, two numerical examples will be presented for demonstrating the characteristics of the proposed model in regional vulnerability analysis and its capability to address the three critical issues of the network-based vulnerability analysis discussed in Section 1. In the first example, the vulnerability analyses of the continuum and discrete modeling framework will be compared. Figure 3a shows the continuum network and the finite element mesh adopted in the solution algorithm for this numerical example. As mentioned previously, the expressway and surface road network will be modeled separately in this continuum network. Based on the continuum network (Figure 3a), a discrete network, which only considered the expressways in the continuum network, is setup (Figure 3b). In this numerical example, only 2 OD pairs, \( (O_1, D) = 3000 \text{veh/hr} \) and \( (O_2, D) = 4000 \text{veh/hr} \), are considered for a clear demonstration of the model characteristics. In the continuum model, these demands are distributed in a small area that is next to the corresponding location in the expressway system (i.e. \( O_1 \) and \( O_2 \) in Figure 3a). In this numerical example, the acceptable relative error (\( \epsilon \)) for the lower-level model and the smallest search interval of (\( \delta \)) of the upper-level model are respectively taken as \( 1 \times 10^{-5} \) and 0.05.

Area I and Area II in Figure 3a are the areas of which the model setups (i.e. demand distribution and unit transportation cost function) of the continuum model will be updated for model testing. The details of these updates will be discussed in the later part of this section. In this numerical example, single class of users are considered and their unit transportation cost function (Equation 1) on the expressway is taken as

\[
c(x, y) = 0.2 + \frac{0.0002}{v(x, y)} f(x, y)
\]

where the ratio \( b_m(x, y)/K(x, y) \) in Equation (1) is taken as 0.0002 in the above equation.

For the surface roads in the continuum model, the unit transportation cost will be higher than that in the expressway system and the details of setting this cost will be discussed in the later part of this numerical example.
For the discrete network, the link cost function will be taken in a form of \( C_i = a_i + b_i V_i \) of which the adopted parameters (i.e. \( a_i \) and \( b_i \)) are shown in the Table in Figure 3b. In order to setup a discrete network that is comparable to the continuum model, the parameters \( a_i \) and \( b_i \) are taken such that the equilibrium travel costs of the 2 OD pairs in the non-degraded scenario will be similar in these two networks. Such comparison will be shown in the results of the base case in this numerical example. The accessibility index for the discrete network will be similar to the Hansen integral accessibility index \cite{26} and takes the following form:

\[
AI_{\text{discrete}} = \frac{Q_{o,d} + Q_{d,o}}{C_{o,d} + C_{d,o}} = \frac{3000 + 4000}{7000}
\]

where \( C_{o,d} \) and \( C_{d,o} \) are respectively the average travel cost (in HKD) of OD pair \( O_1D \) and \( O_2D \).

Before demonstrating the strength of the continuum modeling approach in regional vulnerability analysis, a base case, which the continuum model is setup to mimic discrete network, should be solved as a base for comparison. In the base case, the transportation cost function for the surface road network of the continuum model is assumed to be 100 times of that for the expressway network (Equation 17). This assumption is to ensure the traffic will only travel on the expressway network of the continuum model, which mimics the exclusion of surface roads in the discrete network. Apart from the non-degraded network, 4 different degradation scenarios are considered in this base case. The locations of expressway degradations in these 4 scenarios are marked by the “x” mark in Figure 3a. In the continuum model, the link degradation is implemented to the network by tremendously increase the unit transportation cost of a 0.75~1.0 km section on the corresponding expressway link. For the discrete network, as usual, the link degradation is fulfilled by removing the link from the network. After the equilibrium assignments of the discrete and continuum model (the lower-level model in Section 3.1) for the non-degraded network and the 4 different scenarios, the corresponding average travel cost, accessibility index and rank of vulnerable expressway links are shown in Table 1.

### Table 1 Comparison of the discrete and continuum model for the base case

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average travel cost (HKD)</th>
<th>Accessibility index</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Discrete model</td>
<td>Continuum model</td>
<td>Discrete model</td>
</tr>
<tr>
<td>No degradation</td>
<td>Origin 1 (O_1)</td>
<td>86.5</td>
<td>73.7</td>
</tr>
<tr>
<td>Scenario 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scenario 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scenario 3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scenario 4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As origin 1 (O_1) is disconnected from the network in scenario 1, the average travel cost is infinity for this origin in the discrete network. On the other hand, the corresponding travel cost is relatively high (HKD 164.5) for the continuum case as users in this case is diverted to the local network, which has a much high transportation cost. Similar situations also occur in scenario 3 and 4. Based on the accessibility index, scenarios (degraded expressway links) are ranked for their impact on network accessibility. In this base case, both the continuum and discrete model give the same ranking of the degraded expressway links. As expected, link 2 (Scenario 4) is the most vulnerable expressway link as it is the only link that connected to the destination. Thus, the degradation of this link will affect all the users in the system. Link 5 (Scenario 2) is the least vulnerable link as there is parallel route to take up the demand under link degradation. This is also true for links, 3, 4 and 6.

From the results of the base case, it could be seen that, with the proper setting of transportation cost function (i.e. the tremendously large travel cost in the surface street network), the continuum model is capable to replicate the result of vulnerability analysis in the discrete network model. Based on the continuum model developed in the base case, the following three tests will in turn relax the following constraints in the discrete modeling framework: consideration of surface road network (Test 1), demand definition (Test 2) and extent of network degradation (Test 3). The aim of these 3 tests is to demonstrate how the realistic representation of degradation and transportation system in the continuum modeling approach could affect the results of vulnerability analysis.
Impact of surface road network (Test 1)

In the setup of discrete network, due to the scale of problem and availability of information, most of the minor road links are omitted. For example, apart from the 6 links in the discrete network (Figure 3b), a large proportion of surface road is not considered in the vulnerability analysis of this discrete network. In this test, the impact of surface road network on vulnerability analysis is studied by incorporating an area (Area I in Figure 3a) with comparable transportation cost into the continuum model in the base case. The unit transportation cost in Area I is taken as twice of that of the expressway system (Equation 17). Based on the above modification and the setups in the base case, the continuum models for the 5 scenarios (1 non-degraded and 4 scenarios of degradation) in the base case are solved and the results are shown in Table 2 below. Comparing the average travel costs (1st and 2nd column of Table 2) of the two origins in Scenario 4 with that in the base case (3rd and 4th column of Table 1), it could be seen that the average travel cost is reduced by 50%~55%. This could be explained by the presence of surface road network (Area I) for providing certain level of network connectivity, which at a comparable cost with the expressway, under the degradation on expressway link 2. Due to the decrease in the average travel costs, the accessibility index for scenario 4 (degradation of expressway link 2) is increased from 0.00520 in the base case to 0.01138 in this test. Thus, the expressway link 2 (Scenario 4) is becoming less vulnerable (i.e. the rank is changed from 1 to 3). This test demonstrates that the consideration of surface road network in the continuum model, which allows the possibility of having alternative routes during network degradation, will affect the results of regional vulnerability analysis.

Table 2 Average travel costs, accessibility indexes and rankings for Test 1 and Test 2

<table>
<thead>
<tr>
<th>Average travel cost (HKD)</th>
<th>Accessibility index</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1</td>
<td>Test 2</td>
<td></td>
</tr>
<tr>
<td>Origin (O₁)</td>
<td>Origin (O₂)</td>
<td></td>
</tr>
<tr>
<td>86.4</td>
<td>73.7</td>
<td>68.5</td>
</tr>
<tr>
<td>Scenario 1</td>
<td>Origin (O₁)</td>
<td>Origin (O₂)</td>
</tr>
<tr>
<td>162.5</td>
<td>71.9</td>
<td>68.8</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>84.7</td>
<td>94.7</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>85.3</td>
<td>161.3</td>
</tr>
<tr>
<td>Scenario 4</td>
<td>98.0</td>
<td>86.3</td>
</tr>
</tbody>
</table>

Impact of demand distribution (Test 2)

In the setup of discrete model in the base case, demands are assumed to be concentrated at the zone centroids and entered into the network at O₁ and O₂. As discussed, definition of centroid connectors will seriously and unrealistically affects the result of vulnerability analysis as it assumes the demand of an area is loaded to the network at 1 to 2 points. In order to remove this unrealistic setup, this test has assumed the demand of O₁ (3000 veh/hr) is evenly distributed on Area II, which represents the actual allocation of demand, in Figure 3a. Also, the transportation cost in Area II is assumed to be twice of that of the expressway (Equation 17). Based on the above modifications and original setups in the base case, the continuum models for the 5 scenarios in the base case are solved and the results are shown in Table 2 above. Comparing the average travel costs of origin 1 in Scenario 1 (degradation of expressway link 1) with that of no degradation, it could be seen that there is only a very minor increase (from HKD 68.5 to HKD 68.8), which different from the doubling of average cost in the base case (3rd column of Table 1). Such different could be explained by the fact that not all the demand from Area II will enter into the expressway system at location O₁ and affected by the degradation (the mark “x” on link 1 in Figure 3a). Instead, the demand in this area will continuously enter the expressway system along the section of link 1 that is connected to Area II. As a result, the impact of the degradation of link 1 is diminished as compared to base case. Due to the decrease in average travel costs, the accessibility index for scenario 1 (degradation of expressway link 1) is increased from 0.01052 in the base case to 0.01410 in this test. Thus, the expressway link 1 (Scenario 1) is becoming less vulnerable (i.e. the rank is changed from 3 to 4). This test demonstrates that the consideration of continuous demand distribution in the continuum model, which leads to a more realistic modeling of path choice within the study area, will also affect the results of regional vulnerability analysis.

Impact of the extent of degradation (Test 3)

In the vulnerability analysis of discrete modeling approach, a single link could only be degraded (spatially) as a whole. This assumption only reasonable if all the links within the study region are considered. But for the case of regional study, which only the strategic links are included, the impact of degrading one link, especially the one with a long distance, will be overestimated. It is because traffic will usually divert back to the degraded link
from the local network after the degraded section. Also, in the discrete network, the impact of extent of
degradation (e.g. the impacts of 10m or 20m degradation on a 100m link) could not be precisely modeled. In
order to test for the impacts for different extents, or lengths, of degradation, a continuum model with surface
road network (i.e. continuum network used in Test 1) is adopted.

Table 3 Average travel costs and accessibility indexes for Test 3

<table>
<thead>
<tr>
<th></th>
<th>Average travel cost (HKD)</th>
<th>Accessibility index</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Origin 1 ($O_1$)</td>
<td>Origin 2 ($O_2$)</td>
</tr>
<tr>
<td>No degradation</td>
<td>86.3</td>
<td>73.6</td>
</tr>
<tr>
<td>0.8 km</td>
<td>90.3</td>
<td>82.3</td>
</tr>
<tr>
<td>1.6 km</td>
<td>97.3</td>
<td>91.1</td>
</tr>
<tr>
<td>2.4 km</td>
<td>103.8</td>
<td>97.0</td>
</tr>
<tr>
<td>3.2 km</td>
<td>107.8</td>
<td>102.0</td>
</tr>
</tbody>
</table>

In this test, 4 different lengths of degradation (0.8 km, 1.6 km, 2.4 km and 3.2 km) on link 2 are considered
and the corresponding results are shown in Table 3 above. Note that the degradations in this test will be started
from the destination ($D$) and extended along link 2. As expected, the longer the section of expressway link is
degraded, the higher the average travel costs. For example, the average travel cost of origin 1, is increase from
HKD 90.3 for the case of 0.8 km degradation to HKD 107.8 for the case of 3.2 km degradation. Such increase is
not difficult to understand as the longer the section is degraded, the longer the users have to travel in the surface
road network (Area I), which has a higher transportation cost. Comparing the accessibility indexes in this test
(last column of Table 3) with that for test 1 (5th column in Table 2), it could be seen that link 2 (Scenario 4) will
become more vulnerable than link 1 (Scenario 1 with 0.8 km degradation) if the degradation of link 2 is over 2.4
km. This test demonstrates the flexibility of the continuum modeling approach in considering different extent of
degradations that would lead to a different conclusion in regional vulnerability analysis.

After comparing of the vulnerability analyses of the continuum and discrete modeling framework in the first
numerical example, the second numerical example focuses on finding the most vulnerable location in the study
region based on the bi-level model introduced in Section 3. In this numerical example, a study region with 2
CBDs and an expressway network, which is shown in Figure 1, is adopted. Similar to the previous example,
single class of users is considered. The unit transportation cost for the expressway will be taken as the same as in
the first example (Equation 17), while the unit transportation cost for the surface road network will be taken as
twice of that for the expressway. Figure 4 shows the spatial distribution of demand, which has a highest demand
density at the origins ($O_1$ and $O_2$) and decreases as it moves further away. In this numerical example, the total
number of vehicles from the two demand locations (Figure 4) to CBD 1 and CBD 2 are 11,383 veh/hr and
15,917 veh/hr respectively. As discussed in Section 2, the degradation considered in this numerical example is
taken as a circular shape and with a radius of 500 m. $\nu^f_j$ and $\nu^o_j$ of the impact area are respectively taken as
0.0011 and 0.0010. Based on the above setups, the bi-level model introduced in Section 3 is solved for the case
of single degradation within the study region.

Figure 4 shows the center of degradation for each iteration of the upper-level model, while the corresponding
accessibility indexes are shown in Figure 5b. For this numerical example, the upper-level model converges in 5
iterations, which takes approximately 10.5 hours using a personal computer with Quad Core 2.83 GHz CPU and 4 GB RAM. With the scale-free nature of the continuum model (Section 2) and an effective definition of finite element mesh (e.g. finer mesh in congested area and coarser mesh in the rural area) in the solution algorithm, similar computational time (say 10 to 20 hours) may also be achieved for cases of large-scale networks and regions (e.g. the 7,762 km² Bangkok Metropolitan Area (BMA) considered in Luathep et al. [11]). Also, such computational time is comparable to that of the sensitivity-based approach of the network-modeling framework (20.7 hour) and much faster than the traditional network scanning approach (346.3 hour) on a simplified network in BMA [11]. In this numerical example, the most vulnerable location (with the accessibility index of 0.0869 as it is degraded) is found to be next to CBD 1, which is centered at location 6 in Figure 5a. Despite the higher demand at CBD 2, location 6 is still more vulnerable than the vicinity of CBD 2 (e.g. location A in Figure 5a). It is because there are two expressways leading into CBD 2 and users could easily detour to the non-degraded expressway in case of degradation in either one of them. For the case that the degraded area covers both of the expressway (e.g. at location A), due to the size of the degradation is fixed at 500m radius, the detour in the surface road network will be much less than that for the degradation at location 6.

5. Conclusions

In this paper, a continuum modeling approach for vulnerability analysis is proposed to address the three critical issues faced by the traditional vulnerability analyses in network-modeling framework: 1) Lack of alternative routes due to the missing of minor roads in model setup; 2) Unrealistic definition of demand locations by zone centroids and centroid connectors, and; 3) Unrealistic representation of wide-area disaster that leads to network degradation. In this paper, the search of the most vulnerable locations in an arbitrary study region is formulated into a bi-level model. The lower-level model is formulated as a set of differential equations for finding the equilibrium flow patterns, travel costs and sensitivity information based on the locations of network degradations defined in the upper-level model. This lower-level model is formulated based on the Galerkin method with the weighted residual technique and the FEM. A Newtonian algorithm is adopted to solve this model, and a golden section method is used to determine the step size. The most vulnerable locations of the study region are determined in the upper-level model such that the accessibility index of the study region is minimized. This upper-level model is formulated as a constrained optimization problem that based on the optimized flows and travel costs from the lower-level model. The upper-level model is solved by the convex combination method after the discretization of the problem by FEM. In this paper, two numerical examples are completed to demonstrate the characteristics of the proposed continuum framework for regional vulnerability analysis. From the results of the numerical examples, it could be concluded that: 1) the continuum modeling framework could effectively address the three critical issues faced by the traditional network-based vulnerability analysis, and; 2) the proposed bi-level formulation and the corresponding solution algorithm for continuum vulnerability analysis could effectively and efficiently identify the most vulnerable location(s) within the study area. The future direction of research will be focused on improving the efficiency of searching the most vulnerable locations in the upper-level model.
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References


