On some tractable classes in deduction and abduction

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Abstract

We address the identification of propositional theories for which entailment is tractable, so that every query about logical consequences of the theory can be answered in polynomial time. We map tractable satisfiability classes to tractable entailment classes, including hierarchies of tractable problems; and show that some initially promising conditions for tractability of entailment, proposed by Esghi (1993) and del Val (1994), surprisingly only identify a subset of renamable Horn.

We then consider a potential application of tractable entailment, through a reduction due to Esghi (1993) of certain abduction problems to a sequence of entailment problems. Besides clarifying the range of applicability of Esghi’s results from the semantic point of view, we show that the reduction can almost trivially fail to be in any of the basic tractable classes discussed in the first part of the paper.

We leave open the question of how to more broadly identify tractable entailment classes, as our examples suggest that there is room for progress in this area. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper discusses two related problems. First, we consider the identification of propositional theories $\Sigma$ for which the entailment problem is tractable, so that every query whether a clause $C$ is a logical consequence of $\Sigma$ can be answered in polynomial time. Typical knowledge bases consist of domain theories which are known to be consistent; the main issue in this context is not the consistency of the knowledge base.
but the ability to efficiently query it over and over. Work on knowledge compilation (see [2] for a review) has explored this problem in some depth, with the goal of making theories tractable for entailment. This paper deals with the complementary goal of identifying theories with tractable entailment. Our approach is based on the notion of “polynomial refutation completeness”, a straightforward generalization of the unit refutation completeness criterion defined in [10,13].

We provide, first, a general connection between tractable entailment and tractable satisfiability, which yields hierarchies of classes of problems with tractable entailment. The map is quite direct: any class with tractable satisfiability can under very mild conditions (closure under addition of unit clauses) be transformed into a class with tractable entailment. We then focus on other attempts to obtain tractable entailment. Esghi [13] provided a sufficient condition for tractability of entailment which was generalized by del Val in [10] to what we will call “orderly merge free” (OMF) theories. While Esghi’s condition can be tested in polynomial time, del Val conjectured that membership in OMF could not be decided in polynomial time in the general case. We show that OMF theories are always renamable Horn. Since entailment for renamable Horn theories is known to be tractable, these conditions do not add to the known repertoire of tractable classes. This result, while negative, allows us to see renamable Horn in a new light.

The second problem we address is abduction, i.e., the problem of finding a set of assumptions which explain a given proposition. Esghi [13] reduced abduction on an acyclic Horn theory $\Sigma$ to a sequence of propositional entailment problems with a “pseudo-completed” theory $C(\Sigma)$. If the entailment relation for the latter is tractable, then the corresponding abduction problem is also tractable. We show that, unfortunately, the propositional theories obtained by Esghi are never, except in trivial cases, renamable Horn, nor do they fall in any other of the (low-degree) polynomial entailment classes identified in this paper. In fact, as the examples will show, the syntactic reduction proposed by Esghi, namely pseudo-completion, often turns problems with tractable abduction (namely, definite and binary theories) into problems which are in no polynomially recognizable tractable entailment class, according to current knowledge.

These results on abduction depend only on the syntactic transformation defined by Esghi. But is it really abduction? Since our interest lies in classical logic-based abduction, we examine some counterintuitive quirks in Esghi’s non-standard definition of abduction. We show in particular that his results make sense only for acyclic Horn theories in which all abducibles occur only negatively. In this slightly more restricted but still fairly expressive subset of acyclic Horn, Esghi’s reduction could be useful, if it had tractable entailment. But for this we need to identify further tractable entailment classes than currently known. Indeed, some of the examples strongly suggest that this should be possible. We therefore leave open the question of how to identify broader tractable entailment classes.

The structure of this paper is as follows: Section 2 introduces polynomial refutation completeness as a criterion for tractable entailment, and maps under very weak conditions tractable satisfiability classes to tractable entailment classes. Section 3 shows the surprising result that the conditions of [10,13] for tractable entailment only identify renamable Horn

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2 A theory is Horn if every clause contains at most one positive literal, and definite if it is Horn and contains no negative clauses; a theory is binary if every clause contains at most two literals.
theories. Finally, Section 4 discusses the problem of abduction, first from the point of view of its reduction to supposedly tractable entailment problems, then to clarify semantic issues in the definition of abduction underlying such propositional reduction.

2. Tractable entailment: Refutation completeness

2.1. Preliminaries

In what follows, we assume familiarity with the standard terminology of propositional reasoning: atoms or variables, literals, clauses, resolution deductions, and unit resolution deductions. We will consider only theories \( \Sigma \) in clausal (CNF) form over a vocabulary \( \mathcal{P} \). \( \square \) denotes the empty clause. For a set of literals \( L \), define \( \neg L = \{ \neg l \mid l \in L \} \), and define \( \Sigma_L = \{ C \setminus \neg L \mid C \in \Sigma, C \cap L = \emptyset \} \). Intuitively, \( \Sigma_L \) simplifies \( \Sigma \) by unit resolving all the literals of \( L \), treated as unit clauses, and eliminating clauses subsumed by them. If \( L \) is a clause \( C \), then \( \neg C \) will also be conveniently treated as a set of unit clauses, the negation of \( C \) in clausal form. Given a theory \( \Sigma \), \( n \) denotes the number of distinct variables, and \( |\Sigma| \) the total length of \( \Sigma \), i.e., total number of literal occurrences.

We are interested in polynomial (hence incomplete) inference relations. An important special case is unit resolution: we write \( \Sigma \vdash_u C \) when clause \( C \) can be derived by unit resolution from the set of clauses \( \Sigma \). More generally, given any polynomial procedure \( P \) that can correctly detect the unsatisfiability of some class of theories \( \mathcal{C}_P \), we will consider the “inference relation” \( \vdash_P \) defined by \( \Sigma \vdash_P \square \) iff \( \Sigma \) is \( P \)-refutably, i.e., iff the procedure \( P \) determines \( \Sigma \) unsatisfiable. We could extend this definition to arbitrary clauses by defining \( \vdash_P C \) iff \( \Sigma \cup \neg C \vdash_P \square \); but this is not needed in this paper.

A renaming \( R \) is a function that maps every variable \( p \in \mathcal{P} \) to itself or its negation, i.e., \( R(p) \in \{ p, \neg p \} \). The renaming can be applied to a theory \( \Sigma \), also denoted \( R(\Sigma) \), by replacing every occurrence of \( p \) in \( \Sigma \) by \( R(p) \), and eliminating double negation. Given a class of theories \( \mathcal{C} \), the class renamable \( \mathcal{C} \) is defined as the class of theories which have a renaming in \( \mathcal{C} \).

2.2. Polynomial refutation completeness

Our criterion of tractability for entailment is \( P \)-refutation completeness, a simple generalization of the unit refutation completeness defined in [10,13]:

**Definition 1.** Let \( \vdash_p \) be as described above. A set of clauses \( \Sigma \) is \( P \)-refutation complete iff for any clause \( C : \Sigma \models C \) iff \( \Sigma \cup \neg C \vdash_p \square \). In particular, \( \Sigma \) is unit refutation complete when \( \vdash_p \) is \( \vdash_u \).

\( P \)-refutation completeness (P-RC) may informally be read “polynomial refutation completeness” when \( \vdash_p \) is not specified. Since by assumption \( \Sigma \cup \neg C \vdash_p \square \) can be determined in polynomial time, P-RC ensures tractable query answering, for arbitrary queries, for theories in the class decided by procedure \( P \). (The restriction to clausal queries is without loss of generality.) An important special case is when \( \vdash_p \) is \( \vdash_u \), which conforms
the class URC. While there are obvious decision procedures to determine if a theory is URC or P-RC, for any $\vdash_p$, no polynomial decision procedure is known.

$P$-refutation completeness is a much stronger criterion of tractability than usual, as it requires the polynomial solvability of the (countable) set of all entailment problems for a given theory, rather than requiring only the polynomial solvability of a single satisfiability problem (i.e., the special case of entailment of $\Box$). In fact, entailment is often interesting only when satisfiability is no longer an issue. From the point of view of querying a theory $\Sigma$, the usual one-shot satisfiability test merely tells us that $\Sigma$ is ready to use, that it is a consistent “domain theory” which we can now query. If $\Sigma$ is detected unsatisfiable, no other query is relevant but $\Box$; there is no problem of entailment.

Note that a theory can be very easy for satisfiability and extremely hard for entailment. To see this, pick your favorite “hard” instance of SAT, and add to every clause in the theory a new literal $l$. Detecting the satisfiability of the new theory is trivial: any interpretation satisfying $l$ will do. On the other hand, detecting whether the new theory entails $l$ is as hard as the SAT problem for the original theory.

Conversely, a theory may be easy for entailment (in fact, we can always make it easy, see, e.g., [2,10]), even if it is not in any of the known tractability classes for satisfiability. While every theory can be equivalently expressed in unit refutation complete form (e.g., by putting it in prime implicate form), most theories do not have an equivalent (clausal) theory which belongs to a tractable class for satisfiability, for example an equivalent renamable Horn theory.

2.3. From tractable satisfiability to tractable entailment

In this section, we address the question of recognizing URC theories, and more generally other classes of theories with tractable entailment problems, from classes of theories with tractable satisfiability problems. For reasons of space, we refer the reader to the literature on the classes of tractable satisfiability problems we consider: renamable Horn [14], extended Horn [3], QHorn [1], balanced formulas [5], split-Horn [18], bounded induced width [9].

The following proposition establishes a general relationship between tractable entailment and tractable satisfiability.

**Proposition 2.** Let $C$ be any class of theories satisfying:

(i) $\Sigma \in C$ is unsatisfiable iff $\Sigma \vdash_p \Box$; and

(ii) if $\Sigma \in C$ then $\Sigma \cup \{l\} \in C$, for any unit clause $l$.

Then every $\Sigma \in C$ is $P$-refutation complete.

In particular, the following classes of theories are unit refutation complete:

(a) renamable Horn;

(b) renamable extended Horn;

(c) renamable balanced formulas.

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3 For example, we can generate the set $P(I(\Sigma))$ of prime implicates of $\Sigma$. $\Sigma$ is in URC iff $\Sigma \cup \neg C \vdash_p \Box$ for any $\Sigma \in P(I(\Sigma))$. Similarly for $P$-refutation completeness.
Proof. If $\Sigma \in \mathcal{C}$ then $\Sigma \cup \neg C \in \mathcal{C}$ for any clause $C$, by (ii), and the conclusion follows from (i). Fixing $\vdash_P$ to be $\vdash_u$, renamable Horn theories clearly satisfy (i) and (ii). So do renamable extended Horn formulas and renamable balanced formulas, by a simple extension of results from [3] and [5], respectively, to the renamable case. □

Sometimes the definition of a class $\mathcal{C}$ requires the simplifying effect of assigning truth values to variables, as opposed to simply adding a unit clause. (For example, assigning a literal $l$ true may remove all non-Horn clauses from a theory $\Sigma$ by subsumption, whereas $\Sigma \cup \{l\}$ may still be non-Horn.) To account for this case we have:

**Proposition 3.** Let $\mathcal{C}$ be any class of theories satisfying:

(i) $\Sigma \in \mathcal{C}$ is unsatisfiable iff $\vdash_P \Box$; and

(ii) if $\Sigma \in \mathcal{C}$ then $\Sigma[\{l\}] \in \mathcal{C}$, for any literal $l$.

Then for every $\Sigma \in \mathcal{C}$ and non-tautologous clause $C$, we have: $\Sigma \models C$ iff $\vdash_P \neg_C \vdash_P \Box$.

Assume procedure $P$ performs a limited form of unit simplification, namely it replaces $\Sigma \cup \neg C$ by a theory not more complex than $(\Sigma \cup \neg C)_{\neg C} = \Sigma_{\neg C} \cup (\neg C)_{\neg C}$. (This is a pretty harmless assumption as most interesting $\vdash_P$ relations can use full unit resolution without increase in complexity.) It easily follows from Proposition 3 that $\Sigma \models C$ iff $\Sigma \cup \neg C \vdash_P \Box$, given (ii'). Hence the proposition is essentially asserting the $P$-refutation completeness of any $\Sigma \in \mathcal{C}$.

We thus have a very general relationship between tractable satisfiability and tractable entailment: for any class with tractable satisfiability, only conditions (ii) or (ii') are required to ensure tractable entailment. As pointed out in [18], almost all known tractable satisfiability classes satisfy both conditions; the only known exception seems to be the pair of nested hierarchies $\mathcal{F}_i$ and $\mathcal{F}_j$ defined in [8].

Proposition 2 lists some classes of theories which are URC. Using $\vdash_P$ instead of $\vdash_u$ buys us some interesting tractable classes which are not in URC. These include the class of QHorn formulas [1], which combines renamable Horn with binary and has satisfiability solvable in linear time; the class split-Horn [18], solvable in quadratic time; and theories with bounded induced width [9]. All these classes satisfy (ii) and (ii'), hence they guarantee tractable query answering using the satisfiability algorithm for the given class on $\Sigma \cup \neg C$.

We next use Proposition 3 to map results on hierarchies of polynomially solvable satisfiability problems to hierarchies of theories with polynomially solvable entailment problems, following the very general approach of Pretolani [18] (other hierarchies will be mentioned later).

**Definition 4** (Pretolani [18]). Let $\Pi_0$ be any tractable “base class”, such that the $\Pi_0$ satisfiability problem can be decided in some polynomial time $O(\tau)$, and such that membership in $\Pi_0$ can also be recognized in polynomial time.

Define the hierarchy of nested classes $\{\Pi_i\}$ by:

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4 Technically, the class $\Delta_0$, which is contained in every other class in the hierarchies, does not satisfy (ii) nor (ii'). We conjecture that this can be fixed by restricting $\Delta_0$ to contain only Horn theories.
• \( \Sigma \in \Pi_i \) iff \( \Sigma \in \Pi_{i-1} \) or there exists a (candidate) literal \( l \) such that \( \Sigma \setminus \{l\} \in \Pi_{i-1} \) and \( \Sigma \setminus \{\neg l\} \in \Pi_i \).

In particular, define the following hierarchies: the Horn hierarchy \( \{H_i\} \) is based on \( \Pi_0 = H_0 = \text{Horn} \); \( \{B_i\} \) comes from \( \Pi_0 = B_0 = \text{binary} \); the renamable Horn hierarchy \( \{R_i\} \) is induced by \( R_0 = \text{renamable Horn} \); and \( \{Q_i\} \) by \( \Pi_0 = Q_0 = \text{QHorn} \).

This definition implies a potential decomposition of any \( \Sigma \in \Pi_i \) into \( O(n^i) \) theories, which can be solved by repeated calls to the (polynomial) satisfiability algorithm for \( \Pi_0 \).

The decomposition tree has polynomially (with degree \( i \)) bounded size, and is a proper backtrack tree, with left branches labeled by candidate literals, and right branches by the negation of the corresponding left branch. Each of the at most \( O(n^i) \) leaves of the tree can be reduced to a \( \Pi_0 \)-theory using \( \Sigma \) and the set of literal labels along the path from root to leaf. The requirement of polynomial recognizability of \( \Pi_0 \) guarantees that the tree can be built in polynomial time [18].

**Definition 5.** Let \( P\Pi_i \) be the satisfiability algorithm for a given class \( \Pi_i \), and let \( \vdash_{\Pi_i} \) be the associated inference relation.

A single generic algorithm, PSAT, introduced in [18], implements all the \( P\Pi_i \) procedures, but we treat them formally as different for each \( \Pi \) and \( i \).

**Proposition 6.** Assume \( \Pi_0 \) satisfies condition (ii') from Proposition 3. Then for every \( \Sigma \in \Pi_i \) and non-tautologous clause \( C \), \( \Sigma \models C \) iff \( \Sigma \setminus C \vdash_{\Pi_i} \).

**Proof.** Pretolani [18] shows that if the base class \( \Pi_0 \) satisfies (ii') then so does any \( \Pi_i \).

The result then follows from Proposition 3. \( \square \)

This is to say, essentially, that \( \Pi_i \) is \( P\Pi_i \)-refutation complete, i.e., \( \Pi_i \subseteq P\Pi_i - \text{RC} \), provided \( \Pi_0 \) satisfies (ii'), as it most often is the case (see discussion above). The complexity of query answering in \( P\Pi_i - \text{RC} \) is identical to the complexity of satisfiability in \( \Pi_i \). This yields, for example, query time of \( O(n^i |\Sigma|) \) for \( H_i \) and \( B_i \); and of \( O(n^{i+1} |\Sigma|) \) for \( R_i \) or \( Q_i \).

These divergences in complexity between, say, classes \( R_i \) and \( H_i \) are exclusively due to the differing costs of recognizing membership in the respective classes. This motivates a second, better mapping from tractable satisfiability to entailment. There are two major costs in deciding \( \Pi_i \)-satisfiability. One is solving the \( O(n^i) \) \( \Pi_0 \)-problems into which \( \Sigma \) is decomposed, which takes time \( O(n^{i+1}) \), which is at best, but also commonly, \( O(n^i |\Sigma|) \). But the decomposition of \( \Sigma \) must be found first. This is the recognition problem for each class \( \Pi_i \), which alone accounts for the above divergences in complexity.

The key observation in order to improve query time is that this recognition problem needs to be solved only once for entailment; the same decomposition can be used for all queries, which need only solve the \( \Pi_0 \) problems. And as we will see, any recognition overhead that arises in the context of satisfiability can be amortized in the context of entailment with just a few queries.
A decomposition of any $\Sigma \in \Pi_i$ has the form of an implicit binary tree, with branches labeled with literals, and nodes labeled with theories. In this tree, a node has level $h \leq i$ if it is labeled with a theory $\Gamma \in \Pi_h$; its left branch is labeled with a candidate literal $l$ for $\Gamma$, and its left child with the theory $\Gamma[l] \in \Pi_{h-1}$; and its right branch is labeled with $\neg l$ and its right child with $\Gamma[\neg l] \in \Pi_h$. The root node is labeled with $\Sigma$ and has level $i$. The size of the tree and the number of leaves is $O(n^i)$. Let $j$ be the label of a leaf (which must have level 0), and let $L_j$ be the set of branch labels (literals) along the path from the root node to $j$. Then, first, $L_j \supseteq L_j \supseteq C$ for every $1 \leq j \leq k$. Second, if the set of leaves is $A_1, \ldots, A_k$, with their paths labeled with sets $L_1, \ldots, L_k$, respectively, then $\Sigma \equiv \bigvee_{1 \leq j \leq k}(L_j \land A_j) \equiv \bigvee_{1 \leq j \leq k}(L_j \land \Sigma_{L_j})$. Note incidentally that $L_j$ and $\Sigma_{L_j}$ have no variables in common.

**Definition 7.** Let $\Sigma \in \Pi_i$ have a decomposition $T$, with its fixed associated level $i$. Define the inference relation $\vdash^T_{\Pi}$ as:

$$\Sigma \cup \neg C \vdash^T_{\Pi} \square \quad \text{iff} \quad \Sigma_{L_j} \cup L_j \cup \neg C \vdash_{\Pi_0} \square \quad \text{for every } 1 \leq j \leq k.$$ 

The associated satisfiability procedure, call it $P^T_{\Pi}$, simply checks the $\Pi_0$-refutability of the $O(n^i)$ subproblems. If every subproblem is unsatisfiable, then it returns $\Sigma \models C$; otherwise it returns $\Sigma \not\models C$. Note, as a slight optimization, that subproblems where $L_j$ is inconsistent need to be recognized as such just once (at cost $O(n^i)$ per $j$); there is no need to reconsider them in every query. This may substantially reduce the number of $\Pi_0$ tests per query.

**Proposition 8.** Assume that $\Pi_0$ satisfies (ii'), and that $P\Pi_0$ performs unit resolution, i.e., $\vdash_{\Pi_0}$ is at least as strong as $\vdash_u$. Then

1. Any $\Pi_i$ is $P^T_{\Pi}$-refutation complete, i.e., a subset of $P^T_{\Pi} \cap RC$, for some $T$.
2. $\vdash^T_{\Pi}$ can be decided in time $O(n^i)$, independently of recognition time for $\Pi_i$, using space $O(n^i)$.

**Proof.** (1) Let $\Sigma \in \Pi_i$. To prove (1), we need to show that $\Sigma \models C$ iff $\Sigma \cup \neg C \vdash^T_{\Pi} \square$, i.e., iff $\Sigma_{L_j} \cup L_j \cup \neg C \vdash_{\Pi_0} \square$ for every $1 \leq j \leq k$.

Since $\Sigma \equiv \bigvee_{1 \leq j \leq k}(L_j \land \Sigma_{L_j})$, we have that

$$\Sigma \cup \neg C \equiv \bigvee_{1 \leq j \leq k} (L_j \land \Sigma_{L_j} \land \neg C).$$

Hence $\Sigma \models C$ iff $\Sigma \cup \neg C \models \square$ iff $\Sigma_{L_j} \cup L_j \cup \neg C \models \square$ for every $1 \leq j \leq k$.

Thus it only remains to show

(i) $\Sigma_{L_j} \cup L_j \cup \neg C \models \square \quad \text{iff} \quad \Sigma_{L_j} \cup L_j \cup \neg C \vdash_{\Pi_0} \square$.

We note first that, since $\Sigma_{L_j}$ and $L_j$ contain no variables in common, $\Sigma_{L_j} \cup L_j \cup \neg C \models \square$ iff (a) $L_j \cup \neg C$ is inconsistent or (b) $\Sigma_{L_j} \cup \neg C$ is inconsistent.

(a) If $L_j \cup \neg C$ is inconsistent, it contains complementary literals, and this is detected by unit resolution (this includes the case in which $C$ is a tautology). Hence in this case

(ii) $L_j \cup \neg C \models \square \quad \text{iff} \quad L_j \cup \neg C \vdash_{\Pi_0} \square$. 

(b) So suppose $L_j \cup \neg C$ is consistent. We need to show that

$$\Sigma_{L_j} \cup \neg C \models \square \iff \Sigma_{L_j} \cup \neg C \vdash_{\Pi_0} \square.$$  

Claims (ii) and (iii) together prove (i). We prove (iii).

Since $PT$ uses unit resolution, and $L_j \cup \neg C$ is consistent, $\Sigma_{L_j} \cup \neg C \vdash_{\Pi_0} \square \iff \Sigma_{L_j} \cup \neg C \vdash_{\Pi_0} \square$. By condition (ii'), $\Sigma_{L_j} \cup \neg C \in \Pi_0$, since $\Sigma_{L_j} \in \Pi_0$. Hence $\Sigma_{L_j} \cup \neg C \vdash_{\Pi_0} \square \iff \Sigma_{L_j} \cup \neg C \vdash_{\Pi_0} \square$ (because of membership in $\Pi_0$) $\Sigma_{L_j} \cup \neg C \models \square$ iff $\Sigma_{L_j} \cup \neg C \models \square$. This establishes (iii).

(2) For time, $\vdash_T$ involves solving at most $O(n^i)$ $\Pi_0$-problems, nothing else. As for space, the decomposition must be stored to be reused, but this requires only branch labels; leaf labels, the $\Pi_0$-theories obtained from the decomposition, can be reconstructed with only linear overhead per query from the tree and $\Sigma$, as a byproduct of unit resolution. $\square$

In particular, for the hierarchy $R_i$, the relation $\vdash_T$ can be decided in $O(n^i | \Sigma |)$ for each individual query on a $\Sigma \in R_i$. Furthermore, after only $O(n)$ queries, the amortized cost per query (the ratio of total query time plus initial recognition time to number of queries made) is also $O(n^i | \Sigma |)$. We say that recognition time is “linearly amortizable” for a class $R_i$. Both properties are particularly useful, as, e.g., $R_i$ is considerably broader than $H_i$. These properties should be contrasted with the $O(n^i+1 | \Sigma |)$ complexity for testing satisfiability of theories in $R_i$, and with the query complexity determined by the weaker $PR$-refutation completeness.

Note incidentally that $\vdash_T$ is defined in terms of $\vdash_{R_0}$, the satisfiability algorithm for renamable Horn, which is simply $\vdash_u$. Thus $\vdash_T$ can be seen as a polynomially bounded size backtrack decomposition of $\Sigma \in R_i$ into $O(n')$ URC theories.

Similar remarks about amortizability apply to other classes as well, provided membership in $\Pi_0$ is polynomial, from which we can derive polynomial recognition for $\Pi_i$ using the techniques of [18]. A polynomial number of queries suffices to amortize the cost of recognition.

The main point is therefore that the recognition problem needs to be solved only once for entailment, and any recognition overhead can be amortized over many queries. This is important, and clearly differentiates the entailment and “one-shot satisfiability” contexts, yielding a somewhat more favorable complexity picture for entailment than satisfiability.

2.4. Discussion

The relationship between tractable satisfiability and entailment classes established above is quite general, as it covers almost all known tractable satisfiability classes, as discussed above. It also has broad coverage of theories, as the hierarchies have the property that for any $\Sigma$ there is some $i$ such that $\Sigma \in \Pi_i$.

Propositions 2 and 3 do have some limitations. There is, first, the recognition problem. No polynomial time algorithm is known for recognizing extended Horn or balanced theories, nor for the hierarchies of tractable inference relations defined in [7]; and recognition of the hierarchy renamable generalized Horn is NP-hard [11]. Thus even if $\Sigma$ or $\Sigma \cup \neg C$ is, say, extended Horn, we cannot infer the satisfiability of $\Sigma \cup \neg C$ from the failure to unit refute it; for this we need to know (recognize) that $\Sigma \cup \neg C$ is extended Horn. One of the
main goals of tractable entailment is to provide guarantees on query time once a knowledge base is “deployed” for use. Without polynomial recognizability, such guarantees are much harder to provide.

Second, there are many classes of theories which trivially satisfy (i), for any \( P \), but not (ii) or (ii'), and for which tractable entailment is a relevant issue, but satisfiability is not. For example, the class \( C_{\text{SAT}} \) consisting of all satisfiable theories trivially satisfies (i), and yet many theories in \( C_{\text{SAT}} \) are not, say, unit refutation complete.\(^5\) As pointed out above, it is for satisfiable theories that tractability of entailment is an issue.

In fact, carving interesting subclasses of \( C_{\text{SAT}} \) for polynomial entailment looks like a promising approach. For example, the class \( B_{\text{SAT}} \) of satisfiable binary theories is easily shown URC. Interestingly, this class does not satisfy (ii), as adding a literal may destroy satisfiability and thus membership in \( B_{\text{SAT}} \); further, general binary theories (i.e., \( B_0 \)) do not even satisfy (i), for \( \vdash P \) being \( \vdash \mu \). Monotone theories (no negative literals) and renamable monotone are also URC, simply because they are satisfiable theories in prime implicate form; in fact, in this case entailment reduces to table lookup. There are more complex subsets of \( C_{\text{SAT}} \) for which it would be interesting to identify tractable entailment subclasses. An example is provided in our discussion of abduction in Section 4. We will there see how entailment can be a problem for whole classes of theories which are known to be satisfiable.

3. Identifying URC theories: The OMF class

In this section, we discuss two sufficient conditions for unit refutation completeness that have appeared in the literature. We prove the surprising result that both conditions identify only a subset of renamable Horn. Both are based on Esghi’s [13] concept of a “tied chain”.

**Definition 9.** A tied chain in a set of clauses \( \Sigma \) is a sequence of triples \( (x_1, C_1, y_1), \ldots, (x_n, C_n, y_n) \) such that:

- for \( 1 \leq i \leq n \): \( C_i \in \Sigma \), \( x_i, y_i \in C_i \), and \( x_i \neq y_i \);
- for \( 1 \leq i < n \): \( y_i \) and \( x_{i+1} \) (the link literals of the chain) are complementary literals;
- \( x_1 = y_n \), called the tied literal of the chain.

For example, \( \Sigma = \{ [p, q, r], [\neg r, s], [\neg s, p] \} \) contains a tied chain with \( p \) as tied literal and \( r \) and \( s \) as link symbols. Tied chains are closely related to merge resolvents, resolvents obtained from parent clauses which contain literals in common (which must be “merged” in the resolvent). Resolving the first two elements of the chain above we obtain \( \{ p, q, s \} \), which resolved with the third clause \( \{ \neg s, p \} \), gives us the merge resolvent \( \{ p, q \} \), with \( p \) as merge literal. In fact, every resolution deduction of a merge resolvent can be traced back to a sequence of resolution steps on an underlying tied chain [10].

Tied chains can easily be found in polynomial time, as follows:

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\(^5\) Of course, membership in \( C_{\text{SAT}} \) is NP-complete, but that’s a different issue.
Definition 10. The tied chain graph $G_T(\Sigma)$ of a set of clauses $\Sigma$ is the directed graph whose nodes are all literals whose symbol occurs in $\Sigma$, and such that there is an edge from $l$ to $l'$ iff $\{\neg l, l'\} \subseteq C$ for some clause $C \in \Sigma$.

It can be shown that $\Sigma$ contains a tied chain with tied literal $l$ iff there is a path in $G_T(\Sigma)$ from $\neg l$ to $l$ [13], in which case the link literals can be directly read off the path. $G_T$ has an interesting symmetry property which we will later use:

Lemma 11. Every path in $G_T(\Sigma)$ from a literal $l_0$ to a literal $l_m$ has a symmetric path from $\neg l_m$ to $\neg l_0$ with the sign of every intermediate node inverted.

Proof. $G_T(\Sigma)$ is defined so that there is an edge $(l_i, l_j)$ iff there is an edge $(\neg l_j, \neg l_i)$. Thus from the path $l_0 \rightarrow l_1 \rightarrow \cdots \rightarrow l_{m-1} \rightarrow l_m$ we can obtain a mirror path $\neg l_m \rightarrow \neg l_{m-1} \rightarrow \cdots \rightarrow \neg l_1 \rightarrow \neg l_0$. $\square$

Esghi [13] showed that the absence of tied chains is a sufficient condition for unit refutation completeness. This condition was generalized in [10] by introducing a total ordering $<$ of the symbols of $\Sigma$, extended in the obvious way to literals (e.g., $p < \neg q$ iff $p < q$).

Definition 12. A tied chain $T$ in $\Sigma$ with tied literal $l$ is free with respect to $<$ iff $l < l'$ for some link literal $l'$ of $T$.

Definition 13. A literal $l$ is a free literal with respect to $<$ iff every tied chain with $l$ as tied literal is free with respect to $<$. A symbol $p \in P$ is a free symbol with respect to $<$ iff both $p$ and $\neg p$ are free literals with respect to $<$.

Theorem 14 (del Val [10]). Suppose there exists an ordering $<$ of the symbols in a set of clauses $\Sigma$ such that every tied chain (and thus every symbol) in $\Sigma$ is free with respect to $<$. Then $\Sigma$ is unit refutation complete.

The theorem is proven by showing that any consequence of $\Sigma$ can be derived without resolving upon descendants of merge literals, a condition shown in [10] to be sufficient for unit refutation completeness. This motivates the following definition:

Definition 15. A theory is orderly merge free (OMF) iff there exists a total ordering $<$ of its symbols such that every symbol is free with respect to $<$.

Thus the theorem tells us that OMF theories are unit refutation complete and therefore tractable, not just for satisfiability but also for entailment. But deciding membership in OMF was conjectured intractable in [10], a conjecture that we will refute in a rather surprising way. For this purpose, we constructively characterize the class OMF as follows.
3.1. A constructive characterization of OMF

Definition 16. A set of clauses $\Sigma$ is in the class OMF1 iff it contains only empty clauses, or there exists a symbol $p$ such that:

1. $\Sigma_p$ is in OMF1, where $\Sigma_p$ is the result of removing every occurrence of $p$ or $\neg p$ from the clauses of $\Sigma$; and
2. $p$ is a non-tied symbol in $\Sigma$, i.e., neither $p$ nor $\neg p$ is the tied literal of any tied chain of $\Sigma$.

Theorem 17. The classes OMF and OMF1 are identical.

Proof. (OMF1 $\subseteq$ OMF) Suppose $\Sigma \in$ OMF1. We show by induction on the number of symbols in $\Sigma$ that there exists a total ordering $<$ of these symbols such that every one of them is free with respect to $<$. If $\Sigma$ has only one symbol then every symbol is free trivially in the only possible ordering. Inductively, suppose $\Sigma$ has $n > 1$ symbols. By definition of OMF1, there exists a symbol $p$ which is not tied in $\Sigma$, and such that $\Sigma_p$ is in OMF1. Since $\Sigma_p$ has fewer symbols, by inductive hypothesis there exists an ordering $<'$ of the symbols of $\Sigma_p$ such that every tied chain of $\Sigma_p$ is free with respect to $<'$.

Consider the ordering $<$ obtained from $<'$ by putting $p$ at the end of $<'$ (i.e., for every symbol $q$ in $\Sigma_p$, $q < p$ holds, and for every two symbols $q$ and $r$ in $\Sigma_p$, $q < r$ holds iff $q <' r$). Let $T$ be any tied chain of $\Sigma$, with tied literal $l$. By hypothesis of the theorem, $p$ and $\neg p$ are not tied, and are therefore different from $l$; thus $l$ occurs in $\Sigma_p$. If $p$ or $\neg p$ is a link literal of $T$ then by construction $l < p$, hence $T$ is free with respect to $<$. Finally, if $p$ or $\neg p$ occur in the chain in other position than as link or tied literals (or if they do not occur at all), then there is a tied chain $T_p$ in $\Sigma_p$ which differs from $T$ only in that the clauses in $T_p$ are obtained from clauses in $T$ by deleting $p$ or $\neg p$. $T_p$ must be free with respect to $<'$ by inductive hypothesis, and thus $T$ must be free with respect to $<$, since they both have the same tied and link literals. Thus every tied chain of $\Sigma$ is free with respect to $<$, as desired.

(OMF $\subseteq$ OMF1) Suppose $\Sigma \in$ OMF. The proof is by induction on the number of symbols in $\Sigma$. The base case is trivial. For the induction, let $<$ be a total ordering which makes every symbol free, and let $p$ be the last symbol in this ordering. Since $G_T(\Sigma_p)$ is a subgraph of $G_T(\Sigma)$ (removing $p$ and $\neg p$ from $\Sigma$ can only remove links from $G_T(\Sigma)$), clearly $\Sigma_p \in$ OMF, and by inductive hypothesis, $\Sigma_p \in$ OMF1. Furthermore, $p$ is not a tied symbol in $\Sigma$, as otherwise $p$ would not be free with respect to $<$. Hence $\Sigma \in$ OMF1.

Membership in OMF1, and thus in OMF, is polynomial, but this becomes moot in view of the next section.

3.2. OMF and renamable Horn

Recall that a theory is renamable Horn [14] iff there exists a uniform renaming of its symbols that makes the theory Horn. Lewis [16] shows that $\Sigma = [C_1, \ldots, C_m]$, where each $C_i = [l_{i1}, \ldots, l_{in_i}]$ is renamable Horn iff the following set of binary clauses is satisfiable:
\[ \Sigma_B = \bigcup_{i=1}^m \bigcup_{1 \leq j < k \leq n_i} \{l_{ij}, l_{lk}\}. \]

Note that \( G_T(\Sigma_B) = G_T(\Sigma) \). Furthermore, \( \Sigma_B \) is unsatisfiable (and hence \( \Sigma \) is not renamable Horn) if there is some symbol \( p \) such that both \( p \) and \( \lnot p \) are tied literals of some chain, in other words, iff there is a path from \( p \) to \( \lnot p \) and vice versa in \( G_T(\Sigma_B) \), see, e.g., [17, Theorem 9.1]. These two facts allow us to prove the main result of this section.

**Theorem 18.** \( \text{OMF} \subseteq \text{renamable Horn} \). \(^6\)

**Proof.** Suppose \( \Sigma \) is not renamable Horn, so that for some symbol \( p \) there is in \( \Sigma_B \) a tied chain \( T_B^p \) with tied literal \( p \) and a tied chain \( T_B^{\lnot p} \) with tied literal \( \lnot p \). Let \( P_p \) and \( P_{\lnot p} \) be the corresponding paths in \( G_T(\Sigma_B) = G_T(\Sigma) \) from, respectively, \( p \) to \( \lnot p \) and \( \lnot p \) to \( p \); let \( T_p \) and \( T_{\lnot p} \) be the tied chains corresponding to these two paths in \( \Sigma \); and let \( \Sigma^* \) be the set of clauses of \( \Sigma \) involved in these two latter chains.

Consider any intermediate node \( l \) of \( P_p \). There are paths in \( G_T(\Sigma) \) from \( l \) to \( \lnot p \) (a subpath of \( P_p \)), from \( \lnot p \) to \( p \) (the path \( P_{\lnot p} \)), and from \( p \) to \( \lnot l \) (the mirror path of Lemma 11 corresponding to the subpath of \( P_p \) from \( l \) to \( \lnot p \)). Thus there is a path from \( l \) to \( \lnot l \) (and similarly from \( \lnot l \) to \( l \)) for any link literal \( l \) of \( T_p \); and any link or tied literal in any of these paths is a link or tied literal of \( T_p \). A similar assertion holds for \( T_{\lnot p} \). Thus every link symbol of \( T_p \) or \( T_{\lnot p} \) is tied in \( \Sigma \) and in \( \Sigma^* \). Let \( p_1, \ldots, p_k \) be the symbols of \( \Sigma^* \) which are not link or tied symbols of \( T_p \) or \( T_{\lnot p} \). Then by definition \( \Gamma = (\ldots(\Sigma^*_{p_1})\ldots)_{p_k} \) is not in OMF1, since all symbols in \( \Gamma \) are tied. Since \( G_T(\Gamma) \) is a subgraph of \( G_T(\Sigma) \), \( \Sigma \not\in \text{OMF1} \), hence by Theorem 17, \( \Sigma \not\in \text{OMF} \). \( \square \)

Thus the class OMF does not identify any new tractable theories, not even under the stronger criterion of unit refutation completeness, as renamable Horn theories are also in URC by Proposition 2.

The proof of inclusion in renamable Horn makes essential use of tied chains which are embedded within other tied chains, e.g., the chain \( T_{\lnot p} \) is embedded in the tied chains for the link literals of \( T_{\lnot p} \). Thus one might wonder whether it would possible to ignore such chains with embedded tied chains, requiring symbols to be free only with respect to tied chains without embedded subchains. The following example shows that having all symbols free in this sense fails to guarantee even unit refutability, and thus of course unit refutation completeness. Let \( \Sigma = \{p_0 \lnot p_1, \lnot p_0 \lnot p_1, p_2 p_1, \lnot p_2 p_1\} \). Clearly \( \Sigma \) is unsatisfiable, but not unit refutable. The only tied chains without embedded tied chains are \((\lnot p_1, p_0 \lnot p_1, p_0)\), \((\lnot p_0, \lnot p_0 \lnot p_1, \lnot p_1)\) and \((p_1, p_2 p_1, p_2), (\lnot p_2, \lnot p_2 p_1, p_1)\), so the ordering \( p_1 < p_0 < p_2 \) is such that every such tied chain is free, and thus such every symbol is free in this weak sense.

Though Theorem 18 is a negative result, it also allows us to see renamable Horn in a new light; for example, as prohibiting tied chains, and therefore merges, on both signs of a variable. As a data point, Horn theories can only have negative merge literals.

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\(^6\) The inclusion is in fact proper; see example in Section 4.1.
4. Abduction

We now turn to a potential application of tractable entailment, namely abduction, the problem of finding explanations for facts. In particular, we will be interested in the identification by Esghi [13] of a class of acyclic Horn theories\(^7\) for which, it is claimed, the abduction problem is tractable.

The standard definition of abduction is as follows:

**Definition 19.** Consider a set of clauses \(\Sigma\) over a vocabulary \(\mathcal{P}\), a set of assumption (abducible) symbols \(A \subseteq \mathcal{P}\), and a goal \(g \in \mathcal{P}\). A set of symbols \(E\), treated as a set of unit clauses, is an (abductive) explanation for \(g\) iff:

1. \(E \subseteq A\);
2. \(\Sigma \cup E\) is consistent;
3. \(\Sigma \cup E \models g\).

An explanation \(E\) is minimal if no \(E' \subset E\) is an explanation.\(^8\)

Esghi [13] proposes a non-standard definition of abduction, by imposing the additional requirement that explanations be closed under entailment; that is, that for any \(p \in A\), if \(\Sigma \cup E \models p\) then \(p \in E\). Let us call these closed explanations C-explanations. For any \(E \subseteq A\), let \(Cl(E) = \{p \in A \mid \Sigma \cup E \models p\}\). Then the requirement for a C-explanation \(E\) is that \(E = Cl(E)\).

4.1. Abduction as tractable entailment?

Esghi [13] reduces C-abduction problems on an acyclic Horn theory \(\Sigma\) to entailment problems with respect to a pseudo-completed theory derived from \(\Sigma\), as follows. For any non-negative clause \(C_i\) of \(\Sigma\), let \(c_i\) be a new symbol, to be used as a name of \(C_i\). \(C_i\) can be written as \(Q \supset p\), where \(Q\) is a conjunction of atoms, and \(p\) is an atom; let \(body(c_i) = Q\) and \(head(c_i) = p\). For any \(p \in \mathcal{P}\), let \(cl(p) = \{c_i \mid head(c_i) = p\}\) be the set of (names of) clauses with \(p\) positive. For any set of symbols \(P \subseteq \mathcal{P}\), define the theory \(onlyif(\Sigma, P) = \bigcup_{p \in P} onlyif(\Sigma, \{p\})\), where \(onlyif(\Sigma, \{p\})\) is the conjunction of the following two formulas:

\[
p \supset \bigvee_{c_i \in cl(p)} (c_i \supset body(c_i)).
\]

Intuitively, the pseudo-completed theory \(C(\Sigma, P) = \Sigma \cup onlyif(\Sigma, P)\) corresponds to applying Clark’s completion [4] to all symbols in \(P\); by using the “clause names” \(c_i\), the

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\(^7\) The graph of a Horn theory consists of one node for each symbol, and an edge from a symbol \(p\) to a symbol \(q\) iff \(p\) is the antecedent of a clause with head \(q\), i.e., if \(p\) occurs negatively and \(q\) positively in a clause. The theory is acyclic if this graph is acyclic.

\(^8\) More general definitions are also common, e.g., allowing assumptions to be literals rather than symbols, and allowing the goal to be a literal or even a clause. This additional generality is not needed for this section. See also [12].
resulting theory can be trivially transformed to clausal form without incurring in the usual combinatorial explosion of transformation to clausal form.

Let \( G = \mathcal{P} \setminus A \) be the set of non-abducible symbols. Esghi shows that any minimal C-explanation can be found in polynomial time, for acyclic Horn theories, whenever \( C;G \) is unit refutation complete. In fact, \( P \)-refutation completeness is sufficient as well. Now the question is how to identify cases where \( C;G \) satisfies this condition. Let \( \Gamma = \{ p \cup q \supset r, s \supset r \} \), with \( A = \{ p, q, s \} \). Then \( C(\Sigma, \{ p \}) = \Sigma \cup \{ p \supset c_1 \vee c_2, c_1 \supset q, c_2 \supset r \} \). All positive literals are tied in \( C(\Sigma, \{ p \}) \) and no negative literal is tied; thus \( C(\Sigma, \{ p \}) \neq \emptyset \). It suffices, though, to add the clause \( \neg q \vee \neg r \) to \( \Sigma \) to make the completion non-renamable.

Consider also \( \Gamma = \{ p \land q \supset r, s \supset r \} \), with \( A = \{ p, q, s \} \). \( \Gamma \cup onlyif(\Gamma, \{ r \}) \) is not renamable Horn. In fact, \( C(\Gamma, \{ r \}) \) is not QHorn nor extended Horn, and it is trivial to find examples which are not split-Horn, to take a few of the classes we have considered above. Thus even the most trivial theories fail to satisfy any of the battery of tests for tractability assembled in previous sections.

Yet, as pointed out by a referee, both examples are in URC. It is easy tough to find completions with non-URC completions. Let \( \Omega \) be:

\[
\{ \neg a_1 \lor \neg b_1, \, \neg a_1 \lor b_1, \, \neg a_2 \lor \neg b_2, \, \neg a_2 \lor b_2, \, a_1 \supset g, \, a_2 \supset g, \, a_3 \supset g \},
\]

where \( onlyif(\Omega, \{ g \}) \) is:

\[
\{ g \supset (c_1 \lor c_2 \lor c_3), \, c_1 \supset a_1, \, c_2 \supset a_2, \, c_3 \supset a_3 \}.
\]

It is not difficult to see that \( C(\Omega, \{ g \}) \cup \{ g \} \cup \{ \neg a_3 \} \models \square \) yet \( C(\Omega, \{ g \}) \cup \{ g \} \cup \{ \neg a_3 \} \not\models \square \), hence \( C(\Omega, \{ g \}) \) is not URC.

The theory \( \Omega \) is relatively simple, though it definitely has some more structure than the previous examples. In particular, the first two pairs of clauses yield merge resolvents \( \neg a_1 \) and \( \neg a_2 \), equivalently they form tied chains; and we know from [10] that failure of URC can always be linked to merge resolvents (see also beginning of Section 3). So perhaps it still makes sense to look for merge-related conditions in order to broaden the polynomially recognizable subclasses of URC. At least, broader classes are needed to encompass simple examples such as the pseudocompleted \( \Gamma \) above.

Remarkably, all the examples discussed in this section have tractable abduction problems, since they are definite or binary theories, yet the reduction to a pseudo-completed theory cannot be recognized as tractable for entailment, given current recognition algorithms and known tractable classes. That is, as much as we can tell, so far the reduction actually appears to increase the complexity of abduction.

4.2. The scope of C-explanations

If new tractable, say URC, classes are discovered, as it is likely, we would like results relative to the complexity of pseudo-completed theories to be directly transferable to the standard notion of abduction. We show in this section that Esghi’s C-explanations do not provide an adequate definition of abduction for acyclic Horn theories, unless we

\[\text{\footnotesize 9 The binary case is not considered in either [19] or [12], but it is easy to see that it's tractable.}\]
restrict their scope to acyclic Horn theories in which abducibles are only allowed to occur negatively. This is still fairly expressive in the context of abduction, which has a bleaker complexity picture than entailment, but is nevertheless a restriction.

This is the positive part. If abducibles are not allowed to occur positively, as heads of Horn clauses, then C-explanations and explanations are identical:

**Proposition 20.** Suppose all symbols of $A$ occur only negatively in $\Sigma$. Then $E$ is a minimal explanation of $g$ iff it is a minimal C-explanation.

**Proof.** By the assumption, clearly $Cl(E) = E$ whenever $\Sigma \cup E$ is consistent. It follows that explanations and C-explanations are identical, and therefore the same holds for minimal ones. \[ \square \]

This case is interesting because it corresponds to a common restriction of abduction, e.g., [6,15], where we consider only acyclic Horn theories, and the abducibles are restricted to be the “roots” of the graph of the Horn theory, i.e., those nodes without incoming edges. If we see the Horn clauses as expressing causal relationships, the restriction amounts to requiring that all explanations are in terms of “primitive causes”, those which are not caused by anything else in the given theory.\[ ^{10} \]

Let us now turn to the general case, where abducibles are allowed as heads of Horn clauses. Clearly, every C-explanation is an explanation, and for every explanation $E$ there is a C-explanation, namely $Cl(E)$. The relationship breaks when we consider minimal explanations, due to the interaction between the closure and minimality requirements.

Consider any abducible $g \in A$ such that $\Sigma \cup \{g\}$ is consistent (otherwise $g$ would not have explanations of either kind). The unique minimal C-explanation of $g$ is the trivial explanation $E_g = Cl(\{g\}) = \{ p \in A \mid \Sigma \cup \{g\} \models p \}$, as any other C-explanation of $g$ must contain $Cl(\{g\})$. This is fine for “primitive causes”, abducibles which only occur negatively, as they can only be trivially explained; but it is counterintuitive for other abducibles. If all abducibles are “primitive” then the problem does not arise. Note also that C-explanations fail to distinguish between the empty explanation (which is the only minimal one when $\Sigma \models g$), and the trivial explanation $\{g\}$.

For $g \notin A$, the results are more intuitive: any minimal C-explanation $E$ can be transformed into a standard minimal explanation by (what we will call) \textit{s-minimizing} $E$, a procedure described in [19] which consists in successively removing symbols from $E$ until we obtain a set $E' \subseteq E$ such that it still holds that $\Sigma \cup E' \models g$ but such that for no $p \in E'$ it holds that $\Sigma \cup (E' \setminus \{p\}) \models p$. C-explanations can be s-minimized in polynomial time, since the test $\Sigma \cup E \models p$ is tractable for $\Sigma$ Horn. Note that the result of s-minimization depends on the order in which symbols are removed from $E$.

However, not all minimal explanations of $g \notin A$ can be obtained in this way from \textit{minimal} C-explanations. For example, let $\Sigma = \{ p \supset q, q \supset g \}$, where $A = \{ p, q \}$. The only minimal C-explanation of $g$ is $\{q\}$, from which we cannot obtain the minimal explanation $\{p\}$ by s-minimization.\[ ^{10} \]

\[ ^{10} \text{Another way to put this restriction is that the definite clauses of } \Sigma \text{ cannot have abducibles in the head, i.e., as positive literals. Konolige [15] does not allow negative clauses, while } [6] \text{ requires them to contain only abducible atoms. In contrast, Esghi allows for unrestricted negative clauses.} \]
In summary, one can find a (minimal, standard) explanation by the following procedure: first find a minimal C-explanation, and then s-minimize it. This procedure guarantees that at least one minimal standard explanation is found, if there is one. But it is incomplete in two senses: first, for abducibles it can only produce the trivial explanation (which may be s-minimized into the empty one); second, some standard minimal explanations cannot be found.

There are at least two ways to circumvent these problems. First, if we require that abducibles occur only negatively then both forms of incompleteness disappear. Second, we can instead s-minimize non-minimal C-explanations: if \( E \) is a minimal explanation, then \( E \) can be obtained from the (non-necessarily minimal) C-explanation \( Cl(E) \) by s-minimization. For \( g \in A \), we can find a nontrivial explanation if \( g \) is the first literal removed from the C-explanation, no matter the order chosen to remove other symbols. However, in order to find all minimal explanations we need to try all possible s-minimizations of every C-explanation.

The limitations of C-explanations are inherent to the (pseudo-)completion idea itself. Esghi carefully limits completion to non-abducible symbols, hence the completion also fails to produce nontrivial explanations for abducible symbols. But this is for good reason, as completing the non-primitive abducibles may introduce inconsistencies. In particular, suppose \( \Sigma = \{ p \supset r, q \supset s, r \land s \supset g, \neg p \lor \neg q \} \), and \( A = \{ p, q, r, s \} \). Then \( \Sigma \cup \{ g \} \) is inconsistent. Yet \( \Sigma \cup \{ g \} \) is consistent, and furthermore \( E = \{ r, s \} \) is one among other perfectly valid explanations of \( g \). Intuitively, by completing \( r \) and \( s \) we are insisting that they must have nontrivial explanations, which they only have if considered in isolation, but not jointly. As a result of the completion, the incompatibility of \( p \) with \( q \) becomes incompatibility of \( r \) with \( s \) (and of \( p \) with \( s \), and \( q \) with \( r \)), thus ruling out all acceptable explanations of \( g \).

5. Discussion

We address the identification of theories for which entailment is tractable. Using the concept of polynomial refutation completeness, we map tractable satisfiability classes to tractable entailment, up to and including polynomial hierarchies. We also show that some promising conditions for unit refutation completeness reduce to renamable Hornness.

We then apply some of these results to a reduction due to Esghi of propositional abduction problems to entailment problems. Besides clarifying the scope of Esghi’s results from the semantic point of view, we show that the reduction can almost trivially fail to be in any of the (“base”) tractable classes.

We leave open the question of how to more widely identify unit refutation complete or other tractable entailment classes.

References


