DETERMINANTS OF LAPLACIANS ON GRAPHS

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In recent years, it has been observed that the determinant of a Laplacian on a manifold can often be expressed in terms of the closed orbits of a flow on that (or a related) manifold. Examples include the work of many authors (see, for example [2], [3], [6], [14], [16], [19], [20]) which uses the Selberg trace formula to express the determinant of a Laplacian acting on sections of a vector bundle over a Riemann surface in terms of the closed geodesics on the surface. Other examples can be found in the work of D. Fried relating Reidemeister Torsion (a combinatorial invariant of finite C-W complexes which can be expressed in terms of determinants of combinatorial Laplacians) to the closed orbits of flows of various types ([5], [6], [7]). In this paper we show that this relationship is, in fact, fundamental, and can be seen on the level of graphs.

Let $G$ be a finite graph (all terms will be defined precisely in Section 1) with a weight attached to each vertex and edge. These weights induce an inner product on the spaces $V^*$ (the complex functions on the set of vertices of $G$) and $E^*$ (the complex functions on the set of edges). If $\delta$ is the usual coboundary operator from $V^*$ to $E^*$ we define a (combinatorial) Laplacian $\Delta$ by

$$\Delta = \delta^* \delta$$

where $\delta^*$ is the adjoint of $\delta$ with respect to the inner products.

More generally, let $\rho : \pi_1(G) \to S^1 = \{z \in \mathbb{C} \text{s.t.} |z| = 1\}$ be a representation, and let $V^*_\rho$ denote the complex functions on the vertices of $\tilde{G}$, the universal cover of $G$, which transform via $\rho$ under the action of $\pi_1(G)$. (Note that $\dim V^*_\rho = \dim V^* = \text{the number of vertices in } G$).

There is a natural Laplacian

$$\Delta_\rho : V^*_\rho \to V^*_\rho.$$

We derive a formula for the characteristic polynomial of $\Delta_\rho$ in terms of the closed orbits of flows on $G$. In particular, we prove

**Theorem 1.** Write $\text{Det}(\Delta_\rho + \lambda) = \sum_k C_k \lambda^k$ then

$$C_k = \sum_{\text{vector fields } X \text{ on } G \text{ with exactly } k \text{ zeroes}} \frac{W(X)}{\prod_{\gamma \text{ of } X} (1 - \rho(\gamma))}$$

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where $W(X)$ is a weight attached to $X$ and depends in a simple way on the weights attached to the vertices and edges. If the weights of the vertices and edges are all equal to 1, then $W(X) = 1$ for every $X$.

Rather than define precisely the terms on the right hand side (this will be done later) we present a very simply example.

Consider the following graph $G$:

![Graph G](image)

Fig. 1.

Note that $\pi_1(G)$ is a free group on one generator, which we have labelled $\theta$. Suppose all vertices and edges are given a weight equal to 1. Given a representation

$$\rho : \pi_1(G) \to S^1$$

we have a Laplacian $\Delta_\rho : V^*_\rho \to V^*_\rho$. From Theorem 1 we learn

$$\text{Det} \Delta_\rho = \sum_{\text{nowhere zero prime closed orbits } y \text{ of } X} \sum_{\text{vector fields } X \text{ on } G} (1 - \rho(y)).$$  \hspace{1cm} (1)

A vector field on $G$ is an assignment, to every vertex $v$, of an edge leaving $v$. There are 4 nowhere zero vector fields on $G$, represented schematically by the following 4 figures.

![Vector fields](image)

Fig. 2.

Each vector field induces a map on the vertices. Vector field (i), for example, maps $v_1$ to $v_2$ via edge $e_1$, and $v_2$ to $v_1$ via edge $(-e_1)$. This vector field has one prime closed orbit $v_1 \to v_2 \to v_1$ (an orbit is prime if it is not a multiple repetition of a smaller orbit), which traces out the path $y_1 = e_1 + (-e_1)$. The path $y_1$ represents the element 1 in $\pi_1(G)$, so $\rho(y_1) = 1$, $(1 - \rho(y_1)) = 0$ and thus the vector field (i) does not contribute to the sum (1).

The same is true for the vector field (ii).

The vector field (iii) has a closed orbit $y_3 : v_1$ goes to $v_2$ via $e_1$, and $v_2$ goes to $v_1$ via $e_2$. This closed orbit traces out the path $e_1 + e_2 = \theta$. In this case

$$\prod_{\text{prime closed orbits } \gamma} (1 - \rho(y)) = 1 - \rho(\theta).$$
The closed orbit of the vector field (iv) traces out the path $\theta^{-1}$ so

$$\prod_{\text{prime closed orbits } \gamma \text{ of (iv)}} (1 - \rho(\gamma)) = 1 - \rho(\theta^{-1}).$$

Summing the contributions from these 4 vector fields yields

$$\text{Det } \Delta_{\rho} = (1 - \rho(\theta)) + (1 - \rho(\theta^{-1})) = 2 - \rho(\theta) - \rho(\theta^{-1}).$$

Continuing further, here $\dim V^*_P = 2$, so

$$\text{Trace } (\Delta_{\rho}) = \text{Coefficient of } \lambda \text{ in } \text{Det}(\Delta_{\rho} + \lambda) = \sum_{\text{vector fields } X} \sum_{\text{non-stationary prime closed orbits } \gamma} (1 - \rho(\gamma)).$$

There are 4 vector fields which assign the zero vector to exactly one vertex.

![Fig. 3.](image)

In these diagrams, the vertex without an indicated vector is assigned the zero vector, and is stationary under the induced map. In each case, there are no non-stationary closed orbits, so for each of these vector fields we have, vacuously,

$$\Pi(1 - \rho(\gamma)) = 1.$$

Summing over the vector field yields

$$\text{Trace } \Delta_{\rho} = 4.$$

The coefficient of $\lambda^2$ is a sum over the vector fields with exactly 2 zeroes. There is only one such vector field (which is zero at both $v_1$ and $v_2$) and it has no non-stationary closed orbits, so it contributes 1 to the sum. Therefore, (as is a priori clear), the coefficient of $\lambda^2$ is 1.

Summarizing, we have learned

$$\text{Det}(\Delta_{\rho} + \lambda) = (2 - \rho(\theta) - \rho(\theta^{-1})) + 4\lambda + \lambda^2.$$

We note that the expression

$$\prod_{\text{prime closed orbits } \gamma} (1 - \rho(\gamma))$$

also appears (modulo exponents of $\pm 1$) in [5] and [7], and 'regularized' in [6], as the formula for the Reidemeister Torsion in terms of closed orbits of a flow.

Theorem 1 is related to a remarkable formula which first appeared in the work of Kirchoff [15] in the context of electrical circuits. As a special case, suppose $G$ is a connected graph, and we consider $\Delta$, the usual combinatorial Laplacian (i.e. take $\rho = 1$). This operator has a 1-dimensional kernel (corresponding to the constant function). Let $\text{Det}'\Delta$ denote the product of the non-zero eigenvalues of $\Delta$. Then

**Theorem 2.** ([15]). $\text{Det}'\Delta = (\# \text{ of vertices of } G) \times (\# \text{ of maximal trees of } G)$
This formula has been rediscovered many times (see, for example [1], [4]) and in Section 3 we derive this result from the main theorem of this paper.

It is definitely worth noting that a different, more direct, graph-theoretic analogue of Selberg’s trace formula appears in the work of Hashimoto ([9], [10], [11]), and earlier in the work of Ihara ([12], [13], see also [18]). There, motivated by questions concerning the structure of discrete cocompact subgroups of algebraic groups over \( p \)-adic fields, the authors are lead to study a zeta function associated to a finite graph (see the discussion in [17]). They are able to relate this zeta function to the characteristic polynomial of the Laplacian when the graph \( G \) is regular (i.e. all vertices bound the same number of edges) [12], and, more generally, when \( G \) is a semi-regular bipartite graph (see [9] [10] for definitions and a precise statement of results).

1. PRELIMINARIES

In this section we quickly review the necessary definitions and notation

1.1 Graphs. In what follows, \( G \) will denote a finite oriented graph. This is, \( G \) consists of a finite set \( V \) of vertices, and a finite set \( E \) of edges. An element \( e \in E \) is an ordered pair \((v_1, v_2)\) of vertices. We write \( v_1 = o(e), v_2 = t(e) \) the origin of \( e \), and \( v_2 = t(e), v_1 \) is the terminus of \( e \). Together, \( v_1 \) and \( v_2 \) are the extremities of \( e \) and we write \( \{v_1, v_2\} = \text{ext}(e) \).

Note that we allow loops (edges \( e \) with \( o(e) = t(e) \)) and multiple edges (edges \( e_1 \) and \( e_2 \) with \( \text{ext}(e_1) = \text{ext}(e_2) \)).

A circuit in \( G \) is a sequence \( v_0, e_0, v_1, e_1, v_2, e_2, \ldots, v_{k-1}, e_{k-1}, v_k = v_0 \) where \( v_0, \ldots, v_{k-1} \) are distinct vertices \( e_0, \ldots, e_{k-1} \) are distinct edges, and for each \( i = 0, \ldots, k - 1 \), \( \{v_i, v_{i+1}\} = \text{ext}(e_i) \). A tree is a graph with no circuits.

If \( G' \) is another oriented graph, with vertices \( V' \) and edges \( E' \), a map \( \sigma \) from \( G \) to \( G' \) is a pair of maps

\[
\sigma_V : V \to V', \quad \sigma_E : E \to E'
\]

such that for all \( e \in E \)

\[
o(\sigma_E(e)) = \sigma_V(o(e)), \quad t(\sigma_E(e)) = \sigma_V(t(e)).
\]

An isomorphism is a map such that \( \sigma_V \) and \( \sigma_E \) are bijections.

The universal cover of \( G \), denoted by \( \tilde{G} \) (and whose vertices and edges we will denote by \( \tilde{V} \) and \( \tilde{E} \)), is a tree equipped with a group of automorphisms \( \pi_1(G) \), and a map \( \pi : \tilde{G} \to G \), such that

1. \( \pi_1(G) \) acts freely on \( \tilde{G} \) (i.e. for all \( \gamma \in \pi_1(G) \), if \( \gamma \neq 1 \) then \( \gamma(\tilde{e}) \neq \tilde{e} \) for all \( \tilde{e} \in \tilde{E} \) and \( \gamma(\tilde{v}) \neq \tilde{v} \) for all \( \tilde{e} \in \tilde{E} \))

2. For any vertices \( \tilde{v}_1, \tilde{v}_2 \in \tilde{V} \), \( \pi(\tilde{v}_1) = \pi(\tilde{v}_2) \) if and only if there is a \( \gamma \in \pi_1(G) \) with \( \gamma \tilde{v}_1 = \tilde{v}_2 \). The same property must hold for edges.

It is a classical fact that \( \pi_1(G) \) is a free group. ([17] Theorem 4)

1.2. Laplacians on graphs. Let \( V^* \) denote the vector space of complex functions \( V \mapsto \mathbb{C} \) and \( E^* \) the vector space of complex functions \( E \mapsto \mathbb{C} \). There is a canonical “coboundary” map

\[
\delta : V^* \to E^*
\]

defined by

\[
(\delta f)(e) = f(t(e)) - f(o(e)).
\]
If $V^*$ and $E^*$ are equipped with inner products, we can form the operator $\delta^*$, the adjoint of $\delta$, and thus a Laplacian

$$\Delta = \delta^* \delta : V^* \to V^*.$$  

We will restrict ourselves to inner products of a special type: Let 

$$W : V \to \mathbb{R}^{>0}$$

be a weight function. Then for $f_1, f_2 \in V^*$ we define the $L^2$-inner product by 

$$\langle f_1, f_2 \rangle = \sum_{v \in V} W(v) f_1(v) f_2(v).$$

We also choose a weight function for the edges, which we also denote by $W$, and define the analogous $L^2$-inner product on $E^*$.

Remark. If one is modelling a smooth manifold $M$ by the graph $G$, with the vertices representing disjoint regions of $M$, one usually takes $W(v) =$ volume of region $v$. If $G$ represents a simple electrical circuit, with vertices joined by resistors, one usually takes $W(v) = 1$ for all $v$ and $W(e) =$ (resistance of edge $e)^{-1}$.

Let $\tilde{V}^*$ denote the complex functions on the vertices of $\tilde{G}$. Then $\pi_1(G)$ acts on $\tilde{V}^*$ by

$$(\gamma f)(\tilde{v}) = f(\gamma \tilde{v})$$

for $f \in \tilde{V}^*$, $\tilde{v} \in \tilde{V}$ and $\gamma \in \pi_1(G)$. The space $V^*$ is naturally identified with the elements of $\tilde{V}^*$ invariant under this action. Let

$$\rho : \pi_1(G) \to S^1 = \{ z \in \mathbb{C} \text{ with } |z| = 1 \}$$

be a homomorphism. We can now twist the space $V^*$ by $\rho$. Define 

$$V^*_{\rho} = \{ f \in \tilde{V}^* | \gamma f = \rho(\gamma) f \text{ for all } \gamma \in \pi_1(G) \}.$$

Define $E^*_{\rho}$ in the analogous fashion. The coboundary operator $\delta$ on $\tilde{G}$ maps $V^*_{\rho}$ to $E^*_{\rho}$.

The weight functions on $V$ and $W$ induce inner products on $V^*_{\rho}$ and $E^*_{\rho}$ as follows: For every $v \in V$ choose a lift $\tilde{v} \in \tilde{V}$, that is a vertex $\tilde{v}$ of $\tilde{G}$ such that $\pi \tilde{v} = v$. Similarly, choose a lift $\tilde{e} \in \tilde{E}$ of every $e \in E$. Now for $f_1, f_2 \in V^*_{\rho}$ define 

$$\langle f_1, f_2 \rangle = \sum_{v \in \tilde{V}} W(v) f_1(\tilde{v}) f_2(\tilde{v}).$$

Similarly, for $g_1, g_2 \in E^*_{\rho}$ define

$$\langle g_1, g_2 \rangle = \sum_{e \in \tilde{E}} W(e) g_1(\tilde{e}) g_2(\tilde{e}).$$

It is easy to see that these inner products are independent of the $\tilde{v}$'s and $\tilde{e}$'s that we've chosen.

Using these inner products, we can define an adjoint $\delta^*$

$$\delta^* : E^*_{\rho} \to V^*_{\rho}$$

and thus a Laplacian

$$\Delta_{\rho} = \delta^* \delta : V^*_{\rho} \to V^*_{\rho}.$$  

We now work more concretely. Using the above chosen lifts, we construct a convenient
basis for $V^*_{p}$: Given $v \in V$ define an element $v^* \in V^*_{p}$ by setting (for $\tilde{w} \in \tilde{V}$)

$$v^*_p(\tilde{w}) = \begin{cases} 
\rho(\gamma) & \text{if } \tilde{w} = \gamma \tilde{v} \\
0 & \text{if } \pi(\tilde{w}) \neq v
\end{cases}$$

These $v^*_p$ define a basis of $V^*_{p}$. We can define the analogous basis of $E^*_p$. With respect to these bases, the operator $\delta$ is represented by the "relationship matrix" $R_p$, whose rows are indexed by $E$ and columns by $V$, where

$$\begin{cases} 
0 & \text{if } v \notin \text{xt}(e) \\
\rho(\gamma_1) - \rho(\gamma_2) & \text{if } t(\tilde{e}) = \gamma_1 \tilde{v} \text{ and } o(\tilde{e}) = \gamma_2 \tilde{v} \\
\rho(\gamma) & \text{if } t(\tilde{e}) = \gamma \tilde{v} \text{ and } o(\tilde{e}) \neq v \\
- \rho(\gamma) & \text{if } o(\tilde{e}) = \gamma \tilde{v} \text{ and } t(\tilde{e}) \neq v
\end{cases}$$

Now let $W_v$ denote the diagonal square matrix, with rows indexed by $V$, whose $v$th diagonal entry is $W(v)$. Similarly, define $W_e$ to be the diagonal matrix, with rows indexed by $E$, whose $e$th diagonal entry is $W(e)$. With respect to the inner products induced by the weight function $W$, the adjoint of $\delta$ is represented by the matrix

$$W^{-1}_v R^t_p W_e$$

(where $R^t_p$ is the conjugate transpose of the matrix $R_p$). Therefore with respect to our chosen basis, the operator $\Delta_p$ is represented by the matrix

$$W^{-1}_v R^t_p W_e R_p.$$  \hspace{1cm}(2)

### 1.3. Vector fields on graphs

A nowhere zero vector field $X$ on the graph $G$ is a map

$$X : V \to \{1, -1\} \times E$$

which satisfies $v = o(X(v))$ for all $v \in V$ (where we define $o(-e) = t(e)$ for $e \in E$). That is, for each $v \in V$ we choose an edge $X(v)$ leaving $v$. A vector field $X$ induces a (discrete time) flow $\phi_X$ on the vertices, where

$$\phi_X : V \to V$$

maps each vertex $v$ to the vertex in the direction $X(v)$. More precisely, for all $v \in V$,

$$\phi_X(v) = t(X(v)).$$

We can allow our vector fields to have zeroes. A vector field $X$ with zeroes on $G$ is a map

$$X : V \to \{1, -1\} \times E \cup \{0\}$$

where we require that for $v \in V$ either $X(v) = 0$ or $v = o(X(v))$. Then $X$ induces a flow $\phi_X$ as before, with the additional stipulation that $\phi_X(v) = v$ if $X(v) = 0$.

A periodic orbit of $X$ is sequence of distinct vectors, $\gamma = v_0, v_1, \ldots, v_k$ with $\phi_X(v_0) = v_1, \phi_X(v_1) = v_2, \ldots, \phi_X(v_k) = v_0$.

There is a natural way to lift this flow to $\tilde{G}$. In $\tilde{G}$ we have

$$\phi^{k+1}_X(\tilde{v}_0) = \tilde{y} \tilde{v}_0$$

for some $\tilde{y} \in \pi_1(G)$. We will frequently identify the periodic orbit $\gamma$ with the element $\tilde{y}$. In particular, we will write $\rho(\gamma)$ for $\rho(\tilde{y})$.

If $X(v) = 0$, then $\phi_X(v) = v$, so $v$ itself forms a periodic orbit. In this case we call $v$ a stationary point. A non-stationary periodic orbit is any periodic orbit not of this type. Note
that if \( X(u) \neq 0 \) and \( o(X(u)) = t(X(u)) = v \) then \( \phi_v(v) = v \) and again \( v \) itself forms a periodic orbit, but in this case it is a non-stationary periodic orbit.

2. THE MAIN THEOREM

**Theorem 1.** Write \( \text{Det}(\Delta \rho + \lambda) = \sum_{k=0}^{\lvert V \rvert} C_k \lambda^k \) then

\[
C_k = \sum_{\text{vector fields } X \text{ with exactly } k \text{ zeroes}} W(X) \prod_{\gamma \text{ non-stationary prime periodic orbits of } X} (1 - \rho(\gamma))
\]

where \( W(X) = \prod_{X(v) \neq 0} \frac{W(X(v))}{W(v)} \).

**Proof of the Theorem.** In Section 1.2 we defined a basis \( \{\tilde{v}_\rho\} \) of \( V^*_\rho \) whose elements are indexed by elements of \( V \). Expressing \( \Delta \rho \) as a matrix with respect to this basis we have

\[
C_k = \sum_{U \subseteq V} \sum_{\sigma \in \text{Perm}(U)} (-1)^{\lvert \sigma \rvert} \prod_{e \in U} (\Delta \rho)_{\pi_e, \sigma(e)}
\]

where \( \text{Perm}(U) \) denotes the permutation group of \( U \), and \( \lvert \sigma \rvert \) is the parity of \( \sigma \). For \( U \subseteq V \) let

\[
C_U = \sum_{\sigma \in \text{Perm}(U)} (-1)^{\lvert \sigma \rvert} \prod_{e \in U} (\Delta \rho)_{\pi_e, \sigma(e)}.
\]

The theorem follows from the identity

\[
C_U = \sum_{\text{vector fields } X \text{ such that } \text{zero}(X) = Y \subseteq U} W(X) \prod_{\gamma \text{ non-stationary prime periodic orbits of } X} (1 - \rho(\gamma)) \tag{3}
\]

which we now prove.

As in (2), with respect to our chosen basis, \( \Delta \rho \) is represented by the matrix

\[
W \cdot \tilde{R}_\rho \cdot W \cdot R_\rho.
\]

Thus, we have

\[
C_U = \sum_{\sigma \in \text{Perm}(U)} (-1)^{\lvert \sigma \rvert} \prod_{e \in U} \sum_{e \in E} W^{-1}_e (\tilde{R}^e)_{\pi_e, \sigma(e)} \cdot W_e R_e, \sigma(e)
\]

\[
= \sum_{\sigma} (-1)^{\lvert \sigma \rvert} \prod_{e \in U} \sum_{e \in E} W^{-1}_e \tilde{R}_e, \sigma(e)
\]

\[
= \sum_{\sigma} (-1)^{\lvert \sigma \rvert} \sum_{\text{Maps } Y: U \rightarrow E} \prod_{e \in U} W^{-1}_e \tilde{R}_{Y(e), \sigma(e)}
\]

\[
= \sum_{\text{Maps } Y: U \rightarrow E} \left( \prod_{e \in U} W^{-1}_e \tilde{R}_{Y(e)} \right) \sum_{\sigma \in \text{Perm}(U)} (-1)^{\lvert \sigma \rvert} \prod_{e \in U} \tilde{R}_{Y(e), \sigma(e)} \tag{4}
\]

Now define

\[
M(U) = \{ \text{Maps } Y: U \rightarrow E \text{ such that for all } v \in U, v \in \text{ext}(Y(v)) \}\]
and

\[ P(U, Y) = \{ \sigma \in \text{Perm}(U) \text{ s.t. for all } v \in U, \, \sigma(v) \in \text{ext}(y(v)) \}. \]

It is enough for the first sum in (4) to be taken over \( M(U) \), and the second sum to be taken over \( P(U, Y) \), as otherwise either \( R_{Y(v), v} = 0 \) or \( R_{Y(v), \sigma(v)} = 0 \).

For \( Y \in M(U) \) define

\[ W(Y) = \prod_{v \in U} W_v^{-1} W_{Y(v)}. \]  

Then we have

\[ C_U = \sum_{Y \in M(U)} W(Y) \prod_{v \in U} R_{Y(v), \sigma} R_{Y(v), v} \left( \prod_{v \in U} \sum_{\sigma \in \text{Perm}(U), \sigma} (-1)^{|\sigma|} \prod_{v \in U} R_{Y(v), \sigma} R_{Y(v), \sigma(v)}. \right) \]  

To simplify further, we define the set \( U_0(Y) \subset U \) by

\[ U_0(Y) = \{ v \in U \text{ s.t. } o(Y(v)) = t(Y(o)) = v \}. \]

That is, \( U_0 \) is the set of vertices such that the edge \( Y(v) \) goes from \( v \) to itself. Let \( U_1(Y) = U - U_0(Y) \). Then every \( \sigma \in P(U, Y) \) must fix every \( v \in U_0(Y) \), and thus \( \sigma \) is the extension to \( U \) of a permutation in \( P(U_1(Y), Y) \). Now we can rewrite (6) as

\[ C_U = \sum_{Y \in M(U)} W(Y) \prod_{v \in U_0} \tilde{R}_{Y(v), \sigma} \tilde{R}_{Y(v), v} \left( \prod_{v \in U_1} \sum_{\sigma \in \text{Perm}(U_1), \sigma} (-1)^{|\sigma|} \prod_{v \in U_1} \tilde{R}_{Y(v), \sigma} \tilde{R}_{Y(v), \sigma(v)}. \right) \]

We will evaluate separately the expressions in the two pairs of brackets.

(i) Evaluation of \( \prod_{v \in U_0} \tilde{R}_{Y(v), \sigma} \tilde{R}_{Y(v), v} \).

If \( v \in U_0(Y) \), then the oriented edge \( Y(v) \) describes a loop \( \gamma_v \in \pi_1(G) \). Thus the lift \( Y(v) \) satisfies

\[ \tilde{Y}(v) = (\gamma_v, \gamma_v \tilde{v}) \]

for some \( \gamma \in \pi_1(G) \). This implies

\[ \tilde{R}_{Y(v), v} = \rho(\gamma_v) \rho(\gamma) - \rho(\gamma) = (\rho(\gamma_v) - 1)^2 \]

and

\[ \tilde{R}_{Y(v), \sigma} \tilde{R}_{Y(v), v} = (\rho(\gamma_v^{-1}) - 1)(\rho(\gamma_v) - 1) = (1 - \rho(\gamma_v)) + (1 - \rho(\gamma_v^{-1})). \]

Therefore

\[ \prod_{v \in U_0(Y)} \tilde{R}_{Y(v), \sigma} \tilde{R}_{Y(v), v} = \prod_{v \in U_0} (1 - \rho(\gamma_v)) + (1 - \rho(\gamma_v^{-1})). \]

(ii) Evaluation of

\[ \sum_{\sigma \in \text{Perm}(U_1), \sigma} (-1)^{|\sigma|} \prod_{v \in U_1} \tilde{R}_{Y(v), \sigma} \tilde{R}_{Y(v), \sigma(v)}. \]  

The map \( Y: U \rightarrow E \) defines a flow

\[ \phi_Y: V \rightarrow V \]

by defining, for \( v \in U \), \( \phi_Y(v) = \tilde{v} \) where \( \{v, \tilde{v}\} = \text{ext}(Y(v)) \), and for \( v \notin U \), \( \phi_Y(v) = v \).

We can restate the definitions of \( U_1(Y) \) and \( P(U_1, Y) \) conveniently in terms of \( \phi_Y \) as follows:

\[ U_1(Y) = \{ v \in V \text{ s.t. } \phi_Y(v) \neq v \} \]

\[ P(U_1, Y) = \{ \sigma \in \text{Perm}(U_1) \text{ s.t. for all } v \in U_1, \sigma(v) = v \text{ or } \sigma(v) = \phi_Y(v) \}. \]
We now investigate how these concepts relate to the periodic orbits of $\phi_Y$. Suppose $\gamma = \{v_0, v_1, \ldots, v_k\}$, $k \geq 1$, is a prime periodic orbit of $Y$ (i.e. the $v_i$'s are distinct, $\phi_Y(v_i) = v_{i+1}$ for $i = 0, 1, \ldots, k-1$, and $\phi_Y(v_k) = v_0$). Since $\phi_Y(v) = v$ for every $v \notin U_1$, we must have $\gamma \subset U_1$.

It follows from the above definition that for $\sigma \in P(U_1, Y)$ $\sigma$ must map $\gamma$ to itself. Furthermore, restricted to $\gamma$ either $\sigma = \phi_Y$ or $\sigma = \text{identity}$.

Let $\{\gamma_1, \ldots, \gamma_t\}$ be the set of prime periodic orbits of $Y$ which are contained in $U_1$. Then the $\gamma_i$'s are disjoint, and $\sigma$ maps each $\gamma_i$ to itself. If $v \in U_1$ is not an element of a periodic orbit then for every $\sigma \in P(U_1, Y)$ $\sigma(v) = v$. This can be seen as follows:

Consider the sequence

$v, \phi_Y(v), \phi_Y^2(v), \ldots$

Since $V$ is finite, the sequence must repeat, so that $\phi_Y^i(v) = \phi_Y^j(v)$ for some $i < j$. Thus $\phi_Y^i(v)$ belongs to a periodic orbit of $\phi_Y$.

Suppose $\sigma(v) \neq v$. Then we must have

$\sigma(v) = \phi_Y(v)$, $\sigma(\phi_Y(v)) = \phi_Y^2(v)$, \ldots, etc.

Thus, since $\sigma$ maps $U_1$ to itself, we must have $\phi_Y^k(v) \in U_1$ for all $k$. This implies that $\phi_Y^i(v)$ is an element of some $\gamma_m$. But then we have, for some $n$,

$\phi_Y^n(v) \notin \gamma_m$, $\sigma(\phi_Y^n(v)) = \sigma(\phi_Y^{n+1}(v)) \in \gamma_m$

which is a contradiction.

Summarizing, we have proven that for $\sigma \in P(U_1, Y)$, $\sigma = \text{identity}$ on the complement of the periodic orbits of $\phi_Y$ in $U_1$. Restricted to each periodic orbit $\gamma$ in $U_1$, $\sigma = \phi_Y$ or $\sigma = \text{identity}$. Therefore, each $\sigma \in P(U_1, Y)$ can be identified with the set of closed orbits of $\phi_Y$ on which $\sigma = \phi_Y$.

If $\sigma = \phi_Y$ on the closed orbits $\{\gamma_1, \ldots, \gamma_t\}$ and $\sigma = \text{identity}$ otherwise, then it is easy to see that

$$|\sigma| = \sum_{i=1}^t |\gamma_i| - 1 \quad (9)$$

In addition,

$$\prod_{v \in U_1} R_{Y(v), e} R_{Y(v), \sigma(v)} = \prod_{i=1}^t \left( \prod_{v \in \gamma_i} R_{Y(v), e} R_{Y(v), \phi_Y(v)} \right)$$

because if $v \in U_1$ and $\sigma(v) = v$ then

$$R_{Y(v), e} R_{Y(v), \sigma(v)} = |R_{Y(v), e}|^2 = 1.$$

Fixing $i, 1 \leq i \leq r$, we will now evaluate

$$\prod_{v \in \gamma_i} R_{Y(v), e} R_{Y(v), \phi_Y(v)}.$$

We can simplify this product by noticing that it is independent of the orientation on the edges, the chosen lifts of the edges, and the chosen lifts of the vertices: Changing the orientation of $Y(v)$ multiplies both $R_{Y(v), e}$ and $R_{Y(v), \phi_Y(v)}$ by $-1$, leaving the product unchanged. Varying the lift of $Y(v)$, replacing $Y(v)$ by $\gamma Y(v)$, multiplies $R_{Y(v), e}$ and $R_{Y(v), \phi_Y(v)}$ by $\rho(\gamma)$, so $R_{Y(v), e} R_{Y(v), \phi_Y(v)} \phi_Y(v)$ is unchanged. Varying the lift of $v$, replacing $\tilde{v}$ by $\gamma \tilde{v}$ multiplies $R_{Y(v), e}$ by $\rho(\gamma^{-1})$. But $v = \phi_Y(v)$ for some $v' \in \gamma_i$ and $R_{Y(v'), \phi_Y(v')}$ is also multiplied by $\rho(\gamma^{-1})$ so again the product is unchanged.

Now if $\gamma_i = \{v_0, v_1, \ldots, v_k\}$, choose orientations and lifts so that $Y(v_0)$ goes from $\tilde{v}_0$ to $\tilde{v}_1$, $Y(v_1)$ goes from $\tilde{v}_1$ to $\tilde{v}_2$, etc. Continue in this fashion, eventually choosing $Y(v_k)$ so that it leaves from $\tilde{v}_k$. Then $t(Y(v_0)) = \gamma_i \tilde{v}_0$, where $\gamma_i \in \pi_1(G)$ is the image in $\pi_1(G)$ of the curve $\gamma_i$.
(so that \( \rho(\gamma_i) = \rho(\tilde{\gamma}_i) \)). Then

\[
R_Y(v_i), v_i = -1 \text{ for } i = 0, 1, \ldots, k
\]

\[
R_Y(v_i), \phi Y(v_i) = 1 \text{ for } i = 0, 1, \ldots, k - 1
\]

\[
R_{Y(v_k), \phi Y(v_k)} = \rho(\gamma_i).
\]

Thus

\[
\prod_{v \in \gamma_i} R_Y(v), v \cdot R_Y(v), \phi Y(v) = (-1)^{|\gamma_i|} \rho(\gamma_i)
\]  

Combining (9) and (10), if \( \sigma = \{\gamma_1, \ldots, \gamma_r\} \) we have

\[
(-1)^{|\sigma|} \prod_{j = 1}^r \prod_{v \in \gamma_j} R_Y(v), v \cdot R_Y(v), \sigma(v) = (-1)^r \prod_{i = 1}^r \rho(\gamma_i)
\]

Therefore, the sum (8) is equal to

\[
\sum_{\text{subsets } \{\gamma_1, \ldots, \gamma_r\} \text{ of the set of periodic orbits of } Y \text{ in } U} (-1)^r \prod_{i = 1}^r \rho(\gamma_i) = \prod_{\text{periodic orbits } \gamma \text{ of } Y \text{ in } U} (1 - \rho(\gamma)).
\]  

Combining (7) and (11) we learn

\[
\mathcal{E}_U = \sum_{Y \in \mathcal{M}(U)} W(Y) \prod_{v \in \text{ext}(Y)} ((1 - \rho(\gamma_v)) + (1 - \rho(\gamma_v^{-1}))) \prod_{\text{periodic orbits } \gamma \text{ of } Y \text{ in } U} (1 - \rho(\gamma)).
\]  

(iii) Interpretation of (12) in terms of vector fields.

Let \( X(U) \) denote the vector fields on \( G \) whose zero set is precisely \( V \setminus U \). Recall that \( X \in X(U) \) is a map \( X: U \to \{1, -1\} \times E \) satisfying \( u = 0(X(u)) \) for every \( u \in U \). Projecting onto the second factor (i.e. ignoring the \( \pm 1 \)) we get a map \( \tilde{X}: U \to E \) satisfying \( v = o(X(v)) \) for every \( v \in U \). For each \( X \in \tilde{X} \), \( X(v) = \pm Y(v) \) and satisfies \( v = o(Y(v)) \). For \( v \in U_1(Y) \), \( v \) is only one endpoint of \( Y(v) \), so the sign is uniquely determined. For \( v \in U_0(Y) \), \( v \) is both endpoints of \( Y(v) \) so the sign can be chosen arbitrarily.

For every such \( X \), the closed orbits of \( X \) will include the periodic orbits of \( Y \) in \( U_1 \), and, for each \( v \in U_0(Y), \gamma_v \) or \( \gamma_v^{-1} \), depending on the sign attached to \( Y(v) \). Thus

\[
\mathcal{E}_U = \sum_{\tilde{X} \in \mathcal{M}(U)} W(X) \prod_{v \in U_0(Y)} ((1 - \rho(\gamma_v)) + (1 - \rho(\gamma_v^{-1}))) \prod_{\text{periodic orbits } \gamma \text{ of } Y \text{ in } U_1} (1 - \rho(\gamma))
\]

\[
= \sum_{\tilde{X} \in \mathcal{M}(U)} \sum_{\text{vector fields } X \in \tilde{X}(U) \text{ at } Y = \tilde{Y}} W(X) \prod_{\text{non-stationary periodic orbits } \gamma \text{ of } X} (1 - \rho(\gamma))
\]

\[
= \sum_{\text{vector fields } X \in \tilde{X}(U) \text{ at } Y = \tilde{Y}} W(X) \prod_{\text{non-stationary periodic orbits } \gamma \text{ of } X} (1 - \rho(\gamma))
\]

as desired.

This proves formula (3) and completes the proof of the theorem.
3. KIRCHHOFF’S THEOREM

Let $G$ be a connected graph. Suppose we set the weights $W(v)$ to be 1 and allow the $W(e)$'s to be arbitrary positive real numbers. One arrives at this situation when modelling electrical circuits, where $W(e) = (\text{resistance of } e)^{-1}$ ([1]). Furthermore, we take our representation to be trivial ($\rho(\gamma) = 1$ for all $\gamma \in \pi_1(G)$). We denote the resulting Laplacian by $\Delta$. This operator has a 1 dimensional kernel, corresponding to the constant function. Write $\text{Det}'\Delta$ for the product of the non-zero eigenvalues of $\Delta$. Then the following theorem appears implicitly in the work of Kirchoff ([15], see also [1] [4]).

\textbf{Theorem 2.}

\[ \text{Det}'\Delta = (\text{# of vertices in } G) \times \sum_{T \text{ maximal tree in } G} \prod_{\text{edges in } T} W(e) \]

where a maximal tree (also called a spanning tree) is a connected subgraph of $G$ which contains every vertex and has no circuits.

In this section we indicate the relationship between Kirchoff’s theorem and Theorem 1 of this paper.

It follows from Theorem 1 that

\[ \text{Det}'\Delta = \text{the coefficient of } \lambda \text{ in } \text{Det}(\Delta + \lambda) \]

\[ = \sum_{\text{vector fields } X \text{ with exactly 1 zero}} \prod_{\text{edges in } X} W(e) \prod_{\text{non-stationary prime periodic orbits } \gamma \text{ of } X} (1 - \rho(\gamma)). \]

Since $\rho(\gamma) = 1$ for every $\gamma$, the only vector fields which contribute to this sum are those with no periodic orbits. Therefore

\[ \text{Det}'\Delta = \sum_{\text{vector fields } X \text{ with 1 zero and no periodic orbits}} \sum_{\text{edges in } X} W(e). \]  

(13)

For $v \in V$, define $X(v)$ by

\[ X(v) = \{ \text{vector fields } X \text{ with } \text{zero}(X) = \{v\} \text{ such that } X \text{ has no non-stationary periodic orbits} \}. \]

Now (13) becomes

\[ \text{Det}'\Delta = \sum_{\text{vector fields } X \text{ with 1 zero and no periodic orbits}} \sum_{\text{edges in } X} W(e). \]  

(14)

Every $X \in X(v)$ is a map

\[ X : V \setminus \{v\} \to \pm E. \]

The edges in the image of $X$ form a tree (since $X$ has no periodic orbit). Furthermore, the image contains $|V| - 1$ edges and thus must be a maximal tree ([17] Proposition 12). In fact, we have the following lemma.

\textbf{Lemma 3.} For every $v \in V$, every maximal tree in $G$ appears as the image of exactly one $X \in X(v)$.

\textbf{Proof.} Fix the vertex $v \in V$ and let $T$ be a maximal tree in $G$. By ([8] Theorem 1.2.3) there are at least 2 vertices of $G$ which are extremities of exactly one edge in $T$. Thus, we can find a vertex $v' \neq v$ with this property. Say $v' \in \text{ext}(e')$. We begin our construction of $X \in X(v)$ by setting $X(v) = \pm e'$, where the sign is uniquely determined by the property $v' = o(X(v))$. Note that if $T$ is to be the image of $X$ we must define $X(v)$ in this fashion.
Consider the subtree $T' = T - \{v', e'\}$. There are at least 2 vertices of $C_1$ which are the extremities of exactly 1 edge in $T$ (note that $v'$ is not an extremity of any edge in $T'$), so we can repeat the above process, defining $X$ on another vertex. Continuing this process yields a vector field $X \in X(o)$ such that $T = \text{image}(X)$ and from the above construction it is clear that such an $X$ is unique. \[ \Box \]

In particular, for every $v \in V$

$$\sum_{X(v)} \prod_{e \in X(v)} W(e) = \sum_{\text{maximal trees } T} \prod_{e \in T} W(e). \quad (15)$$

Substituting (15) into (14) yields Kirchhoff's theorem.

Generalizing this formula, if we allow the weights $W(v)$ to be arbitrary, the above argument shows

$$\text{Det} \Delta = \left[ \sum_{v \in V} \left( \prod_{v' \neq v} W(v') \right)^{-1} \right] \times \left[ \sum_{\text{maximal trees } T} \left( \prod_{e \in T} W(e) \right) \right].$$

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