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Some Convergence Properties of Exponential Series Expansions of Distributions*

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1. INTRODUCTION

Let f be a given real (Schwartz) distribution whose support is a compact subset of the positive real line, $0 < t < \infty$. For certain purposes in electrical network theory [1], it is desirable to approximate f in a generalized sense by a finite linear combination of terms of the form,

$$1_+(t)\, t^m e^{-At} \cos (Bt + C),$$

where $1_+(t)$ is the Heaviside unit step function, m is a nonnegative integer, and A, B, and C are real numbers with $A > 0$. If f is a continuous function on some open interval (a, b), then the approximations are also required to converge uniformly to f on every compact subset of (a, b). A method for accomplishing this was presented in [2]. The purpose of this paper is to prove the several convergence theorems that were merely stated in [2]. A more detailed discussion is given in [3] with a number of examples.

As is customary, $\mathscr{D}$ shall denote the space of infinitely differentiable functions $\phi(t)$ having compact supports. Its dual space $\mathscr{D}'$ consists of all Schwartz distributions. The number that an $f \in \mathscr{D}'$ assigns to a $\phi \in \mathscr{D}$ is denoted by $\langle f, \phi \rangle$. We say that a sequence $\{f_\nu\}_{\nu=1}^\infty$ converges in $\mathscr{D}'$ if every $f_\nu \in \mathscr{D}'$ and the sequence converges according to the weak topology of $\mathscr{D}'$ (i.e., for every $\phi \in \mathscr{D}$, $\{\langle f_\nu, \phi \rangle\}_{\nu=1}^\infty$ converges) [4]. The notation $f(t)$ for a distribution indicates only that the testing functions, on which f is defined, possess t as their independent variable.

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2. The Approximation Method

Let n, ν, α, and p be positive integers with $p > 1$ and $n = \nu^p$. Consider the following α-order Cesaro mean of a finite sum of cosines.

$$g_\nu^{[\alpha]}(t) = \frac{2}{\nu}\left\{\frac{1}{2} + \sum_{k=1}^{n} \frac{(n+1-k)(n+2-k)\cdots(n+\alpha-k)}{(n+1)(n+2)\cdots(n+\alpha)} \cos\frac{2\pi k t}{\nu}\right\}. \tag{1}$$

We shall show in a moment that $g_\nu^{[\alpha]}(t)$ converges in $\mathscr{D}'$ to the Dirac delta functional $\delta(t)$ as $\nu \to \infty$. With ρ being a positive number, we set

$$g_{\nu\rho}^{[\alpha]}(t) = e^{-\rho t}\, g_\nu^{[\alpha]}(t). \tag{2}$$

Then, the required approximation to f is obtained by multiplying the convolution,

$$f_{\nu\rho}^{[\alpha]} = g_{\nu\rho}^{[\alpha]} * f, \tag{3}$$

by $1_+(t)$, as we shall also prove.

3. Convergence in $\mathscr{D}'$

That (1) converges in $\mathscr{D}'$ to $\delta(t)$ as $\nu \to \infty$ is shown as follows. Let $\tilde{\phi}$ denote the Fourier transform of $\phi \in \mathscr{D}$.

$$\tilde{\phi}(\omega) = \int_{-\infty}^{\infty} \phi(t)\, e^{-i\omega t}\, dt.$$

Then,

$$\langle g_\nu^{[\alpha]}, \phi\rangle = \frac{1}{\nu}\sum_{k=-n}^{n} \frac{(n+1-|k|)\cdots(n+\alpha-|k|)}{(n+1)\cdots(n+\alpha)}\, \tilde{\phi}\left(\frac{2\pi k}{\nu}\right). \tag{4}$$

Restricting ν such that the support of ϕ is contained in $-\nu < t < \nu$ and noting that by Poisson's summation formula [5; *p.* 60]

$$\phi(0) = \frac{1}{\nu}\sum_{k=-\infty}^{\infty} \tilde{\phi}\left(\frac{2\pi k}{\nu}\right),$$

we may write

$$\langle \delta - g_\nu^{[\alpha]}, \phi\rangle = \frac{1}{\nu}\sum_{k=-n}^{n}\left[1 - \frac{(n+1-|k|)\cdots(n+\alpha-|k|)}{(n+1)\cdots(n+\alpha)}\right]\tilde{\phi}\left(\frac{2\pi k}{\nu}\right) + \frac{1}{\nu}\sum_{|k|>n}\tilde{\phi}\left(\frac{2\pi k}{\nu}\right). \tag{5}$$

Since ϕ is in $\mathscr{D}$, $\tilde{\phi}(\omega)$ is of rapid descent as $|\omega| \to \infty$, which means that for any α and a sufficiently large constant M (that depends on α)

$$\left| \tilde{\phi}\left(\frac{2\pi k}{\nu}\right) \right| \leqslant \frac{M}{1 + | k/\nu |^{\alpha+2}} .$$

Consequently, the second sum on the right-hand side of (5) converges to zero as $\nu \to \infty$ because

$$\left| \frac{1}{\nu} \sum_{|k|>\nu^p} \tilde{\phi}\left(\frac{2\pi k}{\nu}\right) \right| \leqslant \frac{1}{\nu} \sum_{|k|>\nu^p} M\left(\frac{\nu}{k}\right)^2 \leqslant \int_{|x|>\nu^{p-1}} \frac{M}{x^2}\, dx \to 0.$$

Furthermore, for $| k | \leqslant n$, we have the inequality,

$$1 - \frac{(n + 1 - | k |) \cdots (n + \alpha - | k |)}{(n + 1) \cdots (n + \alpha)} \leqslant 1 - \left(\frac{n + 1 - | k |}{n + 1}\right)^{\alpha}.$$

Hence, setting $C = (n + 1)/\nu$, we may dominate the first sum on the right-hand side of (5) by

$$\frac{1}{\nu} \sum_{k=-n}^{n} \left[1 - \left(\frac{n + 1 - | k |}{n + 1}\right)^{\alpha}\right] \left| \tilde{\phi}\left(\frac{2\pi k}{\nu}\right) \right|$$

$$\leqslant \frac{1}{\nu C^{\alpha}} \sum_{k=-n}^{n} \left[C^{\alpha} - \left(C - \left| \frac{k}{\nu} \right|\right)^{\alpha}\right] \frac{M}{1 + | k/\nu |^{\alpha+2}}$$

$$\leqslant \frac{1}{\nu} \sum_{\mu=1}^{\alpha} \binom{\alpha}{\mu} C^{-\mu} \sum_{k=-n}^{n} \left| \frac{k}{\nu} \right|^{\mu} \frac{M}{1 + | k/\nu |^{\alpha+2}} . \tag{6}$$

Given any $\epsilon > 0$, we can choose an N such that, for all $\nu > N$ and for $\mu = 1, 2, \cdots, \alpha$, the difference between

$$\frac{1}{\nu} \sum_{k=-n}^{n} \left| \frac{k}{\nu} \right|^{\mu} \frac{M}{1 + | k/\nu |^{\alpha+2}}$$

and

$$\int_{-\infty}^{\infty} \frac{M| x |^{\mu}}{1 + | x |^{\alpha+2}}\, dx \tag{7}$$

is bounded by ϵ. Note that (7) is a finite quantity. Moreover, as $\nu \to \infty$, $C \to \infty$. Consequently, the right-hand side of (6) converges to zero as $\nu \to \infty$. This proves our original assertion.

It now follows that the function (2) also converges in $\mathscr{D}'$ to $\delta(t)$. Since f has a bounded support, the distributional convolution (3) exists and, by the continuity of this process, (3) converges in $\mathscr{D}'$ to $\delta * f = f$ as $\nu \to \infty$ [6; chap. 5]. Moreover, the convolution (3) can be computed as follows [6; Sec. 5.5].

$$
\begin{aligned}
f_{\nu\rho}^{[\alpha]}(t) &= \langle f(\tau), g_{\nu\rho}^{[\alpha]}(t-\tau)\rangle \\
&= \frac{1}{\nu} \sum_{k=-n}^{n} \frac{(n+1-|k|)\cdots(n+\alpha-|k|)}{(n+1)\cdots(n+\alpha)} F\left(-\rho + \frac{i2\pi k}{\nu}\right) \\
&\qquad \times \exp\left(-\rho t + \frac{i2\pi k t}{\nu}\right). \qquad (8)
\end{aligned}
$$

Here, F denotes the Laplace transform $\mathfrak{L}f$ of f.

$$
F\left(-\rho + \frac{i2\pi k}{\nu}\right) = \left\langle f(\tau), \exp\left(\rho\tau - \frac{i2\pi k\tau}{\nu}\right)\right\rangle. \qquad (9)
$$

Note that, since f is a real distribution, $F(\bar{s}) = \overline{F(s)}$. Hence, (8) is a real-valued function. Because f is the zero distribution on a neighborhood of $t = 0$, we may multiply both sides of (8) by $1_+(t)$ without upsetting the convergence in $\mathscr{D}'$. This will yield the required form for our approximation. Thus, we have established

THEOREM 1. *If f is a real distribution whose support is a compact subset of the open interval* $(0, \infty)$, *then $f_{\nu\rho}^{[\alpha]}(t)\, 1_+(t)$ converges in $\mathscr{D}'$ to f.*

This theorem remains valid even when f has a support that is unbounded on the right so long as the abscissa of convergence σ_1 of $\mathfrak{L}f$ satisfies $\sigma_1 < -\rho < 0$. This will insure that the convolution (3) has a sense and converges in $\mathscr{D}'$ to f as $\nu \to \infty$. See [3].

4. Uniform Convergence

We shall now show that the convergence is, in addition, pointwise and uniform on any compact subset of an open interval, on which f is a continuous function, so long as the order α of the Cesaro mean (1) is chosen large enough. We shall make use of the order of a distribution.

Let $\mathscr{D}_I$ denote the space of those elements of $\mathscr{D}$ whose supports are contained in a fixed finite closed interval I. Every distribution f possesses the following property [6; Sec. 3.3].

For every I there exists a constant C and an integer $q \geqslant 0$ such that for all ϕ in $\mathscr{D}_I$

$$|\langle f, \phi\rangle| \leqslant C \sup \left| \frac{d^q \phi}{dt^q} \right|,$$

where C and q depend in general upon I. If q can be chosen independently from I, the distribution f is said to be of finite order (on the real line). In this case the smallest nonnegative integer r that q can be for all choices of I is called the order of f. It is a fact that every distribution of bounded support is of finite order [6; corollary 3.4-2a].

THEOREM 2. *If the distribution $f(t)$ is of bounded support and of order r, if it is a continuous function on the finite interval $a < t < b$, and if $\alpha > r$, then $f_{\nu\rho}^{[\alpha]}(t)$ converges uniformly to $f(t)$ on $a_1 \leqslant t \leqslant b_1$, where $a < a_1 < b_1 < b$.*

In order to establish this theorem, we shall need two lemmas.

LEMMA 1. *For every integer q such that $0 \leqslant q \leqslant \alpha - 1$,*

$$\frac{d^q}{dt^q} g_\nu^{[\alpha]}(t)$$

converges to zero as $\nu \to \infty$ uniformly on $a \leqslant t \leqslant \nu/2$, where $a > 0$.

PROOF. Let K_n^α be defined by

$$g_\nu^{[\alpha]}(t) = \frac{2}{\nu} K_n^\alpha \left(\frac{2\pi t}{\nu}\right). \tag{10}$$

Therefore,

$$\left| \frac{d^q}{dt^q} g_\nu^{[\alpha]}(t) \right| = \left| \frac{2}{\nu} \frac{d^q}{dt^q} K_n^\alpha \left(\frac{2\pi t}{\nu}\right) \right|$$

and, by a known result [7; Vol. II, p. 60, Eq. (1.10)], the right-hand side is dominated by

$$\frac{M}{t^{\alpha+1}} \nu^{(q-\alpha)(p-1)} \tag{11}$$

on the interval, $\nu^{(1-p)} \leqslant t \leqslant \nu/2$. Here, M is some constant. Lemma 1 now follows immediately.

LEMMA 2. *If η is some small positive constant, then as $\nu \to \infty$ the following limits hold.*

$$\int_\eta^{\nu/2} | g_\nu^{[\alpha]}(x) | \, dx \to 0 \tag{12}$$

$$\int_{-\nu/2}^{-\eta} | g_\nu^{[\alpha]}(x) | \, dx \to 0 \tag{13}$$

$$\int_{-\eta}^{\eta} g_\nu^{[\alpha]}(x) \, dx \to 1. \tag{14}$$

PROOF. Using the same estimate on K_n^α as in the previous proof, we have

$$\int_{\eta}^{\nu/2} | g_\nu^{[\alpha]}(x) | \, dx = \frac{2}{\nu} \int_{\eta}^{\nu/2} \left| K_n^\alpha \left(\frac{2\pi x}{\nu} \right) \right| dx$$

$$\leqslant (2\pi)^\alpha \int_{2\pi\eta/\nu}^{\pi} \frac{M}{\nu^{p\alpha} y^{\alpha+1}} \, dy < \frac{M}{\alpha \eta^\alpha \nu^{\alpha(p-1)}} \to 0$$

as $\nu \to \infty$. The limit (13) can be shown in the same way.

Finally, note that

$$\int_{-\nu/2}^{\nu/2} g_\nu^{[\alpha]}(x) \, dx = 1,$$

as is clear from (1). The limit (14) now follows from this result and from (12) and (13). Q.E.D.

Another fact that we shall use subsequently is that $g_\nu^{[\alpha]}(t) \geqslant 0$ for every positive α [7; Vol. I, p. 88].

PROOF OF THEOREM 2. For the sake of clarity, we sketch $f(\tau)$ in the figure, where it is understood that $f(\tau)$ is a continuous function on $a < \tau < b$ but may be a singular distribution on $x < \tau < a$ and $b < \tau < y$. We also illus-

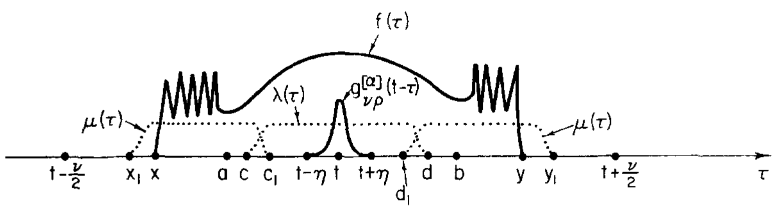

FIG. 1.

trate there $g_{\nu p}^{[\alpha]}(t - \tau)$ as well as other values of τ that we shall use. It is assumed throughout that

$$t - \frac{\nu}{2} < x_1 < x < a < c < c_1 < t - \eta < t < t + \eta$$
$$< d_1 < d < b < y < y_1 < t + \frac{\nu}{2}$$

and that the support of $f(\tau)$ is contained in $x \leqslant \tau \leqslant y$.

Let $\mu(\tau)$ and $\lambda(\tau)$ be infinitely differentiable functions that satisfy the following conditions: (We use brackets to denote closed intervals.) $\mu(\tau) = 1$ on neighborhoods of $[x, a]$ and $[b, y]$ and the support of $\mu(\tau)$ is contained in $[x_1, c_1]$ and $[d_1, y_1]$. $\lambda(\tau) = 1$ on a neighborhood of $[c_1, d_1]$ and the support of $\lambda(t)$ is contained in $[c, d]$. Moreover, $\mu(\tau) + \lambda(\tau) = 1$ on a neighborhood of $[x, y]$.

Now,

$$\begin{aligned} f_{\nu\rho}^{[\alpha]}(t) &= f(t) * g_{\nu\rho}^{[\alpha]}(t) \\ &= \langle f(\tau), \mu(\tau)\, e^{-\rho(t-\tau)}\, g_\nu^{[\alpha]}(t-\tau)\rangle + \langle f(\tau), \lambda(\tau)\, e^{-\rho(t-\tau)}\, g_\nu^{[\alpha]}(t-\tau)\rangle. \end{aligned} \tag{15}$$

Let P be a bound on $e^{-\rho t}$ for $a < t < b$ and let Ω denote the set, $x_1 \leqslant \tau \leqslant c_1$ and $d_1 \leqslant \tau \leqslant y_1$. Since f is of order r, the first term on the right-hand side of (15) is dominated by

$$\begin{aligned} C \sup_{\tau\in\Omega} &\left| \frac{\partial^r}{\partial\tau^r} [\mu(\tau)\, e^{-\rho(t-\tau)}\, g_\nu^{[\alpha]}(t-\tau)] \right| \\ &\leqslant CP \sup_{\tau\in\Omega} \left| \sum_{q=0}^{r} \binom{r}{q} \frac{\partial^{r-q}}{\partial\tau^{r-q}} [\mu(\tau)\, e^{\rho\tau}]\, \frac{\partial^q}{\partial\tau^q} g_\nu^{[\alpha]}(t-\tau) \right|. \end{aligned}$$

By Lemma 1 and the fact that $g_\nu^{[\alpha]}$ is an even function, we see that the right-hand side of the last inequality converges to zero as $\nu \to \infty$ uniformly for $c_1 + \eta < t < d_1 - \eta$. Since we are free to choose c and c_1 arbitrarily close to a, as well as d and d_1 arbitrarily close to b, and η arbitrarily small, the same comment holds for any closed interval $[a_1, b_1]$ that is interior to the open interval (a, b).

Next consider

$$\langle f(\tau), \lambda(\tau)\, e^{-\rho(t-\tau)}\, g_\nu^{[\alpha]}(t-\tau)\rangle = \int_c^d f(\tau)\, \lambda(\tau)\, e^{-\rho(t-\tau)}\, g_\nu^{[\alpha]}(t-\tau)\, d\tau.$$

We shall estimate this integral by considering in turn the intervals, $[c, t-\eta]$, $[t+\eta, d]$ and $[t-\eta, t+\eta]$. The integral on $[c, t-\eta]$ is dominated by

$$\begin{aligned} &P[\sup_{c<\tau<d} | f(\tau)\, \lambda(\tau)\, e^{\rho\tau} |] \int_{t-\nu/2}^{t-\eta} | g_\nu^{[\alpha]}(t-\tau) |\, d\tau \\ &= P[\sup_{c<\tau<d} | f(\tau)\, \lambda(\tau)\, e^{\rho\tau} |] \int_\eta^{\nu/2} | g_\nu^{[\alpha]}(x) |\, dx. \end{aligned}$$

The last expression, which is independent of t, converges to zero as $\nu \to \infty$ according to Lemma 2. As before, this conclusion holds for all t in any closed interval $[a_1, b_1]$ that lies inside the open interval (a, b).

A similar argument can be constructed for the interval $[t + \eta, d]$.

Finally, noting that $\lambda(\tau) = 1$ on $[t - \eta, t + \eta]$, we may write

$$\int_{t-\eta}^{t+\eta} f(\tau)\, \lambda(\tau)\, e^{-\rho(t-\tau)}\, g_\nu^{[\alpha]}(t - \tau)\, d\tau$$

$$= \int_{t-\eta}^{t+\eta} [f(t) + \epsilon_1(t, \tau)]\, [1 + \epsilon_2(t - \tau)]\, g_\nu^{[\alpha]}(t - \tau)\, d\tau. \tag{16}$$

Here, $\epsilon_1(t, \tau)$ is the error term for $f(\tau)$ and $\epsilon_2(t - \tau)$ is the error term for $e^{-\rho(t-\tau)}$. Moreover, $\epsilon_1(t, \tau)$ is uniformly continuous for $c < t < d$ and $t - \eta < \tau < t + \eta$ so that $\epsilon_1(t, \tau) \to 0$ as $\eta \to 0$ uniformly for $c < t < d$. Also, with $| t - \tau | < \eta$, $\epsilon_2(t - \tau) \to 0$ as $\eta \to 0$ uniformly on $c < t < d$ because of the continuity of the exponential function. In view of (14) and the fact that $g_\nu^{[\alpha]}(x) \geqslant 0$ for $\alpha = 1, 2, 3, \cdots$, we can first choose η so small and subsequently ν so large that (16) is close to $f(t)$ to within any desired degree of accuracy uniformly on $c < t < d$. This completes the proof.

In addition, the convergence is pointwise and uniform on the semiinfinite interval $z \leqslant t < \infty$ so long as $\rho > 0$ and z is larger than any support point of f. More precisely, we have

THEOREM 3. *If the distribution $f(t)$ has a bounded support contained in the finite interval, $x \leqslant t \leqslant y$, if the order of f is r, if $\rho > 0$, and if $\alpha > r$, then $f_{\nu\rho}^{[\alpha]}(t)$ converges to zero uniformly on $z \leqslant t < \infty$, where $z > y$.*

PROOF. Let

$$x_1 < x < y < y_1 < z \leqslant t$$

and let $\theta(t)$ be an infinitely differentiable function that equals one on a neighborhood of $[x, y]$ and is zero outside $[x_1, y_1]$. We wish to show that

$$| f_{\nu\rho}^{[\alpha]}(t) | = | \langle f(\tau), \theta(\tau)\, e^{-\rho(t-\tau)}\, g_\nu^{[\alpha]}(t - \tau) \rangle |$$

can be made less than any preassigned $\epsilon > 0$ uniformly on $z \leqslant t < \infty$ by choosing ν large enough. We may again write

$$| f_{\nu\rho}^{[\alpha]}(t) | \leqslant C \sup_{x_1 \leqslant \tau \leqslant y_1} \left| \frac{\partial^r}{\partial \tau^r} [\theta(\tau)\, e^{-\rho(t-\tau)}\, g_\nu^{[\alpha]}(t - \tau)] \right|$$

$$= C \sup_{x_1 \leqslant \tau \leqslant y_1} \sum_{q=0}^{r} \binom{r}{q} \left| \frac{\partial^{r-q}}{\partial \tau^{r-q}} [\theta(\tau)\, e^{-\rho(t-\tau)}] \frac{\partial^q}{\partial \tau^q} g_\nu^{[\alpha]}(t - \tau) \right|. \tag{17}$$

First, consider the case where $t - \nu/2 \leqslant \tau \leqslant y_1$. On this interval we have that

$$\left| \frac{\partial^{r-q}}{\partial \tau^{r-q}} [\theta(\tau)\, e^{-\rho(t-\tau)}] \right| \leqslant B \qquad (q = 0, 1, \cdots, r)$$

for all $t \geqslant z$; here, B is a sufficiently large constant. Also, by lemma 1, for a given $\epsilon_1 > 0$ there is an $N > 0$ such that for all $\nu > N$

$$\left| \frac{\partial^q}{\partial \tau^q} g_\nu^{[\alpha]}(t - \tau) \right| < \epsilon_1 \qquad (q = 0, 1, \cdots, \alpha - 1)$$

uniformly on $t - \nu/2 \leqslant \tau \leqslant t - z + y_1$. Therefore, each term within the summation on the right-hand side of (17) is bounded by $\binom{r}{q} B\epsilon_1$ whenever $t - \nu/2 \leqslant \tau \leqslant y_1$. The only other pertinent case occurs when $-\infty < \tau \leqslant t - \nu/2$. By another known result [7; Vol. II, p. 60, Eq. (1.9)],

$$\left| \frac{\partial^q}{\partial \tau^q} g_\nu^{[\alpha]}(t - \tau) \right| \leqslant B\nu^{(q+1)(p-1)}. \tag{18}$$

Moreover, every term inside the summation sign of (17) has an exponential damping factor. Consequently, in view of (18) and the fact that $t - \tau \geqslant \nu/2$, we see that each such term converges to zero as $\nu \to \infty$ uniformly for all $t \geqslant z$. These results prove the theorem.

Theorem 1 can be extended to the case where the support of f extends indefinitely toward $+\infty$ so long as $\sigma_1 < -\rho < 0$, where σ_1 is the abscissa of convergence of the Laplace transform of f. That is, we can state

THEOREM 4. *If the distribution $f(t)$ has its support bounded on the left and is of order r, if its Laplace transform has a region of convergence,* Re $s > \sigma_1$, *if $\sigma_1 < -\rho < 0$, if $f(t)$ is a continuous function on the finite interval $a < t < b$, and if $\alpha > r$, then $f_{\nu\rho}^{[\alpha]}$ converges uniformly to $f(t)$ on $a_1 \leqslant t \leqslant b_1$, where $a < a_1 < b_1 < b$.*

The proof of this theorem is quite similar to the proofs of Theorem 2 and 3 and can be found in [3].

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