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Note

On Invex Sets and Preinvex Functions

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In this paper we consider the class of the invex function introduced by Hanson. We show that under certain condition an invex function defined on an invex set A is preinvex on A. Similarly, a quasiinvex function defined on an invex set A is prequasiinvex.

1. INTRODUCTION

In 1981 Hanson [3] introduced a class of functions with convex like property. This class is defined as follows.

DEFINITION 1.1. We say that a differentiable function $f : R^n \to R$ belongs to Hanson's class (or satisfies Hanson's condition) if there exists a function $\eta : R^n \times R^n \to R^n$ such that for any $x, y \in R^n$

$$f(x) - f(y) \geq (\eta(x, y))^t \nabla f(y), \tag{1.1}$$

where $\nabla f(y)$ is the gradient vector of f at y and for any column vector a, a^t denotes its row transpose.

The importance of Hanson's class of functions in mathematical programming is due to the following theorem observed by Hanson [3].

THEOREM 1.1. *Minimize $f(x)$ subject to*

$$g_i(x) \leq 0, \qquad 1 \leq i \leq m, \tag{1.2}$$

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where $f, g_i : R^n \to R$, $1 \le i \le m$, are once differentiable functions. Let $\bar{x} \in S = \{x \mid g_i(x) \le 0,\ 1 \le i \le m\}$ and let $\bar{x}$ satisfy the Karush Kuhn Tucker conditions [5, 6] *of optimality. Then $\bar{x}$ is a minimizer of f over S.*

The name invex with respect to η (a short form for invariant convex) has been given to a function satisfying the Hanson property with the function η by Craven [2]. This is because, if $\phi : R^n \to R^n$ is a differentiable and invertible transformation f satisfies the Hanson property with η iff $f \circ \phi$ satisfies the Hanson property with $\bar{\eta}(x, y) = J_{\phi}^{-1}\, \eta(\phi(x), \phi(y))$, where J_{ϕ}^{-1} denotes the Jacobian of ϕ^{-1}.

DEFINITION 1.2. Let $f : R^n \to R$. We say that f is pseudoinvex with respect to $\eta : R^n \times R^n \to R^n$ if

$$(\eta(x, y))'\nabla f(y) \ge 0 \Rightarrow f(x) \ge f(y).$$

DEFINITION 1.3. Let $f : R^n \to R$ be differentiable. We say that f is quasiinvex with respect to η, where $\eta : R^n \times R^n \to R^n$, if

$$f(x) \le f(y) \Rightarrow (\eta(x, y))'\nabla f(y) \le 0.$$

In this note we study invex sets. Although there are examples of such sets in the literature, they are mostly in R. See Weir and Mond [8]. We show that how to build such sets into R^n using invex sets in a lower dimensional space.

The main results proved in this note relate a differentiable function satisfying Hanson condition to a condition called preinvexity by Jeyakumar [4]. The notion of a preinvex function is defined in the next section, which also presents the main results.

2. INVEX SETS

DEFINITION 2.1. We call a set $A \subseteq R^n$ invex with respect to a given η: $R^n \times R^n \to R^n$ if

$$x, y \in A,\ 0 \le \lambda \le 1 \Rightarrow y + \lambda\eta(x, y) \in A.$$

Remark 2.1. It is to be noted that any set in R^n is invex with respect to $\eta(x, y) \equiv 0\ \forall\, x \in R^n, y \in R^n$. However, the only function $f: R^n \to R$ invex with respect to η is the trivial function $f(x) \equiv c\ \forall\, x \in R^n$, where c is a real number.

The definition essentially says that there is a path starting from y which is contained in A. We do not require that x should be one of the end points of the path. However, if we demand that x should be an end point of the path for every pair x, y then $\eta(x, y) = x - y$, reducing to convexity.

EXAMPLE 2.1. The following is an example of a bounded invex set in R, which is invex with respect to a nontrivial $\eta : R \times R \to R$. Let us consider the bounded set $[-7, -2] \cup [2, 10]$. This set is a bounded invex set with respect to η given as

$$\eta(x, y) = x - y, x \geq 0, y \geq 0$$
$$\eta(x, y) = x - y, x \leq 0, y \leq 0$$
$$\eta(x, y) = -7 - y, x \geq 0, y \leq 0$$
$$\eta(x, y) = 2 - y, x \leq 0, y \geq 0.$$

Examples of an invex set and an invex function in R have been given in Weir and Mond [8]. The following proposition enables us to construct invex sets in R^n, starting from an invex set in R.

PROPOSITION 2.1. *Suppose that* $S_1 \subseteq R$, $S_2 \subseteq R$ *such that* S_1 *is invex with respect to* $\eta_1 : R \times R \to R$ *and* S_2 *is invex with respect to* $\eta_2 : R \times R \to R$. *Then* $S_1 \times S_2 \subseteq R^2$ *is invex with respect to* $\eta : R^2 \times R^2 \to R^2$ *defined by*

$$\eta \begin{pmatrix} x_1, y_1 \\ x_2, y_2 \end{pmatrix} = \begin{pmatrix} \eta_1(x_1, y_1) \\ \eta_2(x_2, y_2) \end{pmatrix}.$$

Proof. This is easy to verify.

EXAMPLE 2.2. The above proposition shows that the following set in R^2 is invex with respect to η:

Let us consider the invex sets $S_1 = [-5, -2] \cup [2, 7]$, $S_2 = [-7, -2] \cup [2, 10]$ which are invex with respect to η_1, η_2, respectively, where η_1 and η_2 are given as

$$\eta_1(x, y) = x - y, x \geq 0, y \geq 0 \qquad \eta_2(x, y) = x - y, x \geq 0, y \geq 0$$
$$\eta_1(x, y) = x - y, x \leq 0, y \leq 0 \qquad \eta_2(x, y) = x - y, x \leq 0, y \leq 0$$
$$\eta_1(x, y) = -5 - y, x \geq 0, y \leq 0 \qquad \eta_2(x, y) = -7 - y, x \geq 0, y \geq 0$$
$$\eta_1(x, y) = 2 - y, x \leq 0, y \geq 0 \qquad \eta_2(x, y) = 2 - y, x \leq 0, y \geq 0.$$

Clearly, $S_1 \times S_2$ is invex with respect to $\eta = \binom{\eta_1}{\eta_2}$.

The following is an example of an invex set in R^2 which is not a cartesian product of two invex sets in R.

EXAMPLE 2.3. Let us consider the set $\{(u, v) \mid u \geq 0, v \geq 0, u + v \leq 3\} \cup \{(u, v) \mid u \geq 0, v \geq 0, 3u - 2v \geq 9, u \leq 5\}$. Also let $\eta = \binom{\eta_1}{\eta_2}$. This set is invex with respect to the function η specified as

$$\begin{aligned}
\eta_1 &= x_1 - y_1, && 0 \leq x_1 \leq 3,\ 0 \leq y_1 \leq 3 \\
\eta_2 &= x_2 - y_2, && 0 \leq x_2 \leq (3 - x_1),\ 0 \leq y_2 \leq (3 - x_1) \\
\eta_1 &= x_1 - y_1, && 3 \leq x_1 \leq (3 + \tfrac{2}{3}x_2),\ 3 \leq y_1 \leq (3 + \tfrac{2}{3}y_2) \\
\eta_2 &= x_2 - y_2, && 0 \leq x_2 \leq 3,\ 0 \leq y_2 \leq 3 \\
\eta_1 &= -y_1, && 3 \leq x_1 \leq (3 + \tfrac{2}{3}x_2),\ 0 \leq y_1 \leq 3 \\
\eta_2 &= -y_2, && 0 \leq x_2 \leq 3,\ 0 \leq y_2 \leq (3 - y_1) \\
\eta_1 &= 3 - y_1, && 0 \leq x_1 \leq 3,\ 3 \leq y_1 \leq (3 + \tfrac{2}{3}y_2) \\
\eta_2 &= -y_2, && 0 \leq x_2 \leq (3 - x_1),\ 0 \leq y_2 \leq 3.
\end{aligned}$$

The general problem of identifying classes of invex sets in R^n that are useful in the theory of optimization remains open. In what follows we consider nondifferentiable functions which have a convex like property over an invex set. Such functions have been called preinvex by Jeyakumar [4].

DEFINITION 2.2. Let $A \subseteq R^n$ be an invex set, with respect to $\eta : R^n \times R^n \to R^n$. Let $f : A \to R$. We say that f is preinvex if $f(x^2 + \lambda\eta(x^1, x^2)) \leq \lambda f(x^1) + (1 - \lambda)f(x^2)$, $\forall\ x_1, x_2 \in A$, $0 \leq \lambda \leq 1$.

DEFINITION 2.3. *Let* $A \subseteq R^n$ be an invex set, with respect to $\eta : R^n \times R^n \to R^n$. We say that f is prequasiinvex with respect to η if $x^1, x^2 \in A$, $0 \leq \lambda \leq 1$, implies that $f(x^2 + \lambda\eta(x^1, x^2)) \leq \max(f(x^1), f(x^2))$ for all x_1, $x_2 \in A$, $0 \leq \lambda \leq 1$.

Also we say that f is prepseudoinvex with respect to η if $f(x^1) < f(x^2) \Rightarrow f(x^2 + \lambda\eta(x^1, x^2)) \leq \lambda f(x^1) + \lambda(1 - \lambda)b(x^1, x^2)$ for all $0 \leq \lambda \leq 1$, where $b : R^n \times R^n \to R^1$ is a positive function. These definitions are due to Pini [7].

Pini [7] shows that if f is defined on an invex set A and if it is preinvex and differentiable then f is also invex with respect to η. The converse is not true in general and Pini [7] gives an example.

In what follows, we shall show that with the following condition imposed on η, a differentiable function which is invex on A, with respect to η, is also preinvex.

Condition C. Let $\eta : R^n \times R^n \to R^n$; we say that the function η satisfies the condition C if for any x^1, x^2,

$$\eta(x^2, x^2 + \lambda\eta(x^1, x^2)) = -\lambda\eta(x^1, x^2),$$

$$\eta(x^1, x^2 + \lambda\eta(x^1, x^2)) = (1 - \lambda)\eta(x^1, x^2)$$

for all $0 \leq \lambda \leq 1$.

THEOREM 2.1. *Suppose that A is a preinvex set with respect to η and suppose that $f : X \to R$ is differentiable where X is open and $X \supseteq A$. Further suppose that f is invex with respect to η on A and that η satisfies condition C. Then f is preinvex with respect to η on A.*

Proof. Suppose that $x^1, x^2 \in A$. Let $0 < \lambda < 1$ be given and look at $\bar{x} = x^2 + \lambda\eta(x^1, x^2)$. Note that $\bar{x} \in A$. By, the invexity of f we have

$$f(x^1) - f(\bar{x}) \geq \eta(x^1, \bar{x})'\nabla f(\bar{x}). \tag{2.1}$$

Similarly, the invexity condition applied to the pair x^2, $\bar{x}$ yields

$$f(x^2) - f(\bar{x}) \geq \eta(x^2, \bar{x})'\nabla f(\bar{x}). \tag{2.2}$$

Now, multiplying (2.1) by λ and (2.2) by $(1 - \lambda)$ and adding, we note that $\lambda f(x^1) + (1 - \lambda)f(x^2) - f(\bar{x}) \geq (\lambda\eta(x^1, \bar{x})' + (1 - \lambda)\eta(x^2, \bar{x})'\nabla f(\bar{x})$. However, by condition C, $\lambda\eta(x^1, \bar{x})' + (1 - \lambda)\eta(x^2, \bar{x})' = 0$. Hence, the conclusion of the theorem follows.

Remark 2.2. The above proof is similar to the proof in the convex case.

Similarly, we can prove the following.

THEOREM 2.2. *Let $A \subseteq R^n$ be invex with respect to η and let $f: X \to R$ be differentiable on X, where X is an open set containing A. Suppose that f is quasiinvex with respect to η on A and that η satisfies condition C. Then f is prequasiinvex on A.*

Proof. Suppose that $x^1, x^2 \in A$ and let $f(x^1) \leq f(x^2)$. Consider the set

$$\Omega = \{x \mid x = x^2 + \lambda\eta(x^1, x^2), f(x) > f(x^2), 0 \leq \lambda \leq 1\}.$$

In order to show that f is prequasiinvex, we have to show that $\Omega = \phi$. Note that if $\Omega \neq \phi$ then by, continuity of f, the set

$$\Omega' = \{x \mid x = x^2 + \lambda\eta(x^1, x^2), f(x) > f(x^2), 0 < \lambda < 1\}$$

is also nonempty. Hence, it is sufficient to show that $\Omega' = \phi$, to complete the proof.

Suppose now that $\bar{x} \in \Omega'$. We then have $\bar{x} = x^2 + \bar{\lambda}\eta(x^1, x^2)$, for some $0 < \bar{\lambda} < 1$ and $f(\bar{x}) > f(x^2) \geq f(x^1)$. By the definition of quasiinvexity it follows, considering the pair $\bar{x}$ and x^1, that

$$(\eta(x^1, \bar{x}))'\nabla f(\bar{x}) \leq 0. \tag{2.3}$$

Similarly, considering the pair x^2 and $\bar{x}$, it follows that

$$(\eta(x^2, \bar{x}))'\nabla f(\bar{x}) \leq 0. \tag{2.4}$$

Hence by condition C, we have

$$-\bar{\lambda}\eta(x^1, x^2)'\nabla f(\bar{x}) \leq 0 \tag{2.5}$$

and

$$(1 - \bar{\lambda})\eta(x^1, x^2)'\nabla f(\bar{x}) \leq 0. \tag{2.6}$$

Now (2.5) and (2.6), together with the fact that $0 < \bar{\lambda} < 1$, imply that

$$\eta(x^2, x^1)'\nabla f(\bar{x}) = 0. \tag{2.7}$$

Note that (2.7) holds for any $\bar{x} \in \Omega'$. Now suppose that $\Omega' \neq \phi$. Let $\bar{x} \in \Omega'$ and let

$$\bar{x} = x^2 + \bar{\lambda}\eta(x^1, x^2).$$

By the continuity of f we can find $\lambda^* < \bar{\lambda} < \hat{\lambda} < 1$ such that for all $\lambda \in (\lambda^*, \hat{\lambda})$, we have

$$f(x^2 + \lambda\eta(x^1, x^2)) > f(x^2),$$
$$f(x^2 + \lambda^*\eta(x^1, x^2)) = f(x^2)$$

(It is possible that $\lambda^* = 0$.) Let $h(\lambda) = f(x^2 + \lambda\eta(x^1, x^2))$; we have $h(\lambda^*) = f(x^2)$.

Now, by the mean value theorem applied to the function $h : [\lambda^*, \hat{\lambda}]$ we have

$$h(\bar{\lambda}) - h(\lambda^*) = \left.\frac{dh}{d\lambda}\right|_{\lambda=\tilde{\lambda}},$$

where $\tilde{\lambda} \in (\lambda^*, \hat{\lambda})$ or

$$f(x^2 + \bar{\lambda}\eta(x^1, x^2)) - f(x^2) = \eta(x^1, x^2)'\nabla f(x^2 + \tilde{\lambda}\eta(x^1, x^2)).$$

The right-hand side is positive by our hypothesis, but the left-hand size is zero by (2.7), as $x^2 + \tilde{\lambda}\eta(x^1, x^2) \in \Omega'$, by construction; hence, we have a contradiction. The proof follows in this case. The proof is similar in case $f(x^2) \leq f(x^1)$.

Remark 2.3. It is easy to show that a differentiable function which is pre-quasiinvex with respect to η, on a set A which is invex with respect to η, is also quasiinvex. Our theorem 2.2 is a converse of this under condition C.

COROLLARY 2.1. *Suppose that $g : R^n \to R$ is quasiinvex with respect to η. Further, suppose that condition C is satisfies by η then $S = \{x \mid g(x) \leq 0\}$ is also invex with respect to η.*

Proof. This is clear from pre-quasiinvexity of g under the condition of the corollary.

The above corollary is useful in constrained minimization. We can easily show the following result.

THEOREM 2.3. *Let $f: R^n \to R$ be a differentiable and pre-pseudoinvex function on an invex set A with respect to a function $\eta : R^n \times R^n \to R^n$. Then f is quasiinvex with respect to η.*

THEOREM 2.4. *If $f: R^n \to R$ is pre-pseudoinvex then f is pre-quasiinvex with respect to the same η.*

THEOREM 2.5. *Let $f : R^n \to R$ be a pre-pseudoinvex function with respect to η. Also assume that $\phi : R \to R$ is a nondecreasing function. Then, the composite function $\phi_0 f$ is pre-pseudoinvex with respect to η.*

THEOREM 2.6. *Let $f: R^n \to R$ be a pre-pseudoinvex function on an invex set $A \subseteq R^n$ with respect to $\eta : R^n \times R^n \to R^n$. Assume that $\eta(x, y) \neq 0$ whenever $x \neq y$. Then every strict local minimizer of the function f is also a strict global minimizer. The set of points which are strict global minimizers is invex with respect to η.*

Remark 2.4. Like pre-pseudoiinvexity the above two theorems also hold in the case of pre-quasiinvexity (see Pini [7]).

In the following example we can verify that condition C holds. This shows that condition C may hold for a large class of functions η, rather than just for the trivial case $\eta(x, y) = x - y$.

EXAMPLE 2.4. Consider Example 2.1 for the bounded set $[-7, -2] \cup [2, 10]$. In this set condition C holds with respect to η given as

$$\begin{aligned}
\eta(x, y) &= x - y, & x \geq 0, y \geq 0 \\
\eta(x, y) &= x - y, & x \leq 0, y \leq 0 \\
\eta(x, y) &= -7 - y, & x \geq 0, y \leq 0 \\
\eta(x, y) &= 2 - y, & x \leq 0, y \geq 0.
\end{aligned}$$

References

1. A. Ben-Israel and B. Mond, What is Invexity? *Bull. Austral. Math. Soc. Ser. B* **28** (1986), 1–9.
2. B. D. Craven, Invex functions and constrained local minima, *Bull. Austral. Math. Soc. Ser. B* **24** (1981), 357–366.
3. M. Hanson, On sufficiency of the Kuhn Tucker conditions, *J. Math. Anal. Appl.* **80** (1981), 545–550.
4. V. Jeyakumar, Strong and weak invexity in mathematical programming, *Methods Oper. Res.* **55** (1985), 109–125.
5. W. Karush, "Minima of Functions of Several Variables with Inequalities as Side Conditions," M.S. thesis, Department of Mathematics, University of Chicago, 1939.
6. H. W. Kuhn and A. W. Tucker, "Nonlinear Programming; Proceedings, 2nd Berkley Symposium on Mathematical Statistics and Probability" (J. Neyman, Ed.), Univ. of California Press, Berkeley, CA, 1951.
7. R. Pini, Invexity and generalized convexity, *Optimization* **22** (1991), 513–525.
8. T. Weir and B. Mond, Preinvex functions in multiple objective optimization, *J. Math. Anal. Appl.* **136** (1988), 29–38.