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On resolutions of generalized metric spaces

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Abstract

We study the relation between generalized metric spaces and their resolutions. Here, we consider the class of spaces with G_δ-diagonals, M_3-, M_1-spaces, M_3–μ-spaces and developable spaces.

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1. Introduction

All spaces are assumed to be regular T_1 and all mappings to be continuous. The letter $\mathbb{N}$ denotes all positive numbers. For a space X, $\tau(X)$ denotes the topology of X. The concept of resolutions of spaces was originally given by Fedorčuk [2] and Watson [8] brought it in the limelight. He showed how nice properties of topological spaces can be destroyed by taking resolutions. In this paper, we study what kind of generalized metric spaces can be preserved to the resolutions. Especially, we consider the classes of spaces with G_δ-diagonals, M_3-spaces, M_1-spaces, M_3–μ-spaces and developable (Moore) spaces. These spaces have the position indicated by the implication: M_3–μ-space $\to$ M_1-space $\to$ M_3-space $\to$ G_δ-diagonal $\leftarrow$ developable space. As the study with the same direction, we have the results of Richardson and Watson [7], where they consider the metrizability of

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resolutions. We give the definition of the resolution of a space. Let X and Y_x, $x \in X$, be spaces and for each x, $f_x : X \setminus \{x\} \to Y_x$ a mapping. We endow the set

$$Z = \bigcup \bigl\{ \{x\} \times Y_x \mid x \in X \bigr\}$$

with a topology whose base consists of

$$\bigl\{ U \otimes V \mid U \in \tau(X),\ V \in \tau(Y_x),\ x \in X \bigr\},$$

where

$$U \otimes V = \{x\} \times V \cup \bigcup \bigl\{ \{x'\} \times Y_{x'} \mid x' \in U \cap f_x^{-1}(V) \bigr\}, \quad U \in \tau(X),\ V \in \tau(Y_x).$$

We call Z the *resolution* of a space X (at each point $x \in X$ into Y_x by the mapping f_x). It is easy to see that the projection $\pi : Z \to X$ defined by $\pi((x, y)) = x$ for each $(x, y) \in Z$ is continuous. It is to be noted that the operation $A \otimes B$ is defined similarly for any subsets A, B of X, Y_x, $x \in X$, respectively. Finally, we point out that any resolution of regular spaces is also regular [1].

2. The resolutions of generalized metric spaces

We call a subset Λ of a space X *F_σ-discrete* in X if $\Lambda = \bigcup\{\Lambda_n \mid n \in \mathbb{N}\}$, where each Λ_n is discrete and closed in X. In general, let

$$\Lambda = \bigl\{ x \in X \mid |Y_x| > 1 \bigr\}.$$

We call that the projection $\pi : Z \to X$ is *closed at x* in X if for each open neighborhood O of $\pi^{-1}(x) = \{x\} \times Y_x$ in Z, there exists an open neighborhood U of x in X such that $\pi^{-1}(U) \subset O$. It is easy to see that π is a closed mapping if and only if π is closed at each point $x \in X$. Let

$$\Omega = \{x \in X \mid \pi \text{ is closed at each } x \text{ in } X\}.$$

Obviously, if Y_x is a singleton, then $x \in \Omega$, that is, $X \setminus \Lambda \subset \Omega$. More generally, it is easily checked that Y_x is compact, then $x \in \Omega$. For a subset A of a space X, let A^d be the derived set of A in X, that is, the set of all accumulation points of A in X.

Proposition 1. *If the resolution Z has a G_δ-diagonal, then Λ is represented as a countable union $\bigcup\{\Lambda(n) \mid n \in \mathbb{N}\}$, where for each n, $\Lambda(n)^d \cap \Omega = \emptyset$.*

Proof. Suppose that Z has a G_δ-diagonal sequence $\{\mathcal{U}(n) \mid n \in \mathbb{N}\}$. For each $x \in \Lambda$, take distinct points $p(x), q(x) \in Y_x$. There exists $n(x) \in \mathbb{N}$ such that

$$\bigl(x, q(x)\bigr) \notin S\bigl(\bigl(x, p(x)\bigr), \mathcal{U}\bigl(n(x)\bigr)\bigr).$$

Let

$$\Lambda(n) = \bigl\{ x \in X \mid n(x) = n \bigr\}, \quad n \in \mathbb{N}.$$

Then $\Lambda = \bigcup\{\Lambda(n) \mid n \in \mathbb{N}\}$. Assume that for some $n \in \mathbb{N}$, $\Lambda(n)^d \cap \Omega \neq \emptyset$. Take $x \in \Lambda(n)^d \cap \Omega$. Since π is closed at x and $x \in \Lambda(n)^d$, there exists $a \in Y_x$ such that

$$(x, a) \in \bigl\{ \bigl(x', p(x')\bigr) \mid x' \in \Lambda(n) \bigr\}^d.$$

Since $\mathcal{U}(n)$ covers Z, there exists $U \in \mathcal{U}(n)$ with $(x, a) \in U$. This implies that U contains both $(x', p(x'))$ and $(x', q(x'))$ for some $x' \in \Lambda(n)$. This is a contradiction to the definition of $\Lambda(n)$. □

Proposition 2. *If X and all Y_x, $x \in X$, have a G_δ-diagonal and Λ is F_σ-discrete in X, then the resolution Z has a G_δ-diagonal.*

Proof. Let $\{\mathcal{U}(n) \mid n \in \mathbb{N}\}$, $\{\mathcal{U}(x, n) \mid n \in \mathbb{N}\}$ be G_δ-diagonal sequences of X, Y_x, respectively. Let $\Lambda = \bigcup\{\Lambda(n) \mid n \in \mathbb{N}\}$, where each $\Lambda(n)$ is discrete and closed in X. Let $n, m \in \mathbb{N}$. There exists a family $\{V(p) \mid p \in \Lambda(n)\}$ of open subsets of X such that $p \in V(p)$ and $V(p) \cap (\Lambda(n) \setminus \{p\}) = \emptyset$ for each $p \in \Lambda(n)$. Define open covers $\mathcal{V}(n, m)$ and $\mathcal{V}(n)$ as follows:

$$\mathcal{V}(n, m) = \left\{V(x) \otimes U \mid U \in \mathcal{U}(x, m),\ x \in \Lambda(n)\right\} \cup \left\{\pi^{-1}\left(X \setminus \Lambda(n)\right)\right\},$$
$$\mathcal{V}(n) = \left\{\pi^{-1}(U) \mid U \in \mathcal{U}(n)\right\}.$$

Then it is easy to see that $\{\mathcal{V}(n, m), \mathcal{V}(n) \mid n, m \in \mathbb{N}\}$ is a G_δ-diagonal sequence of Z. □

Theorem 3. *Let $\pi : Z \to X$ be a closed projection and let X, Y_x, $x \in X$, have a G_δ-diagonal. Then the resolution Z has a G_δ-diagonal if and only Λ is F_σ-discrete in X.*

Proof. If part follows from Proposition 2. Only if part: Let $\Lambda(n)$, $n \in \mathbb{N}$, be the same as in the proof of Proposition 1. Suppose that $\Lambda(n)$ is not discrete or not closed in X. Then $\Lambda(n)^d \neq \emptyset$. Note that $X = \Omega$ by the assumption. By the same discussion as there, we can reach a contradiction to the definition of $\Lambda(n)$. Hence each $\Lambda(n)$ is discrete and closed in X. □

Remark. The closedness of π is necessary. Indeed, let Z be the resolution of a real line $\mathbb{R}$ at each point $x \in \mathbb{R}$ into $Y_x = \mathbb{R} \setminus \{x\}$ by an identity mapping $f_x : \mathbb{R} \setminus \{x\} \to \mathbb{R} \setminus \{x\}$. In this case, $Z = \bigoplus\{Y_x \mid x \in \mathbb{R}\}$ is metrizable, but $\Lambda = \mathbb{R}$ is not F_σ-discrete.

Corollary 4. *Let Y_x, $x \in X$, be compact metrizable spaces and X a space with a G_δ-diagonal. Then the resolution Z has a G_δ-diagonal if and only if Λ is F_σ-discrete in X.*

A space X is *developable* if there exists a sequence $\{\mathcal{U}(n) \mid n \in \mathbb{N}\}$ of open covers of X such that for each $x \in X$, $\{S(x, \mathcal{U}(n)) \mid n \in \mathbb{N}\}$ forms a local base at x in X. This sequence is called the development for X. Note that the concept of developments is stronger than that of G_δ-diagonal sequences. But the treatment for resolutions are similar. Thus as the corollary to the above theorem we have the following:

Theorem 5. *Let X be a developable space and Y_x, $x \in X$, compact metrizable spaces. Then the resolution Z is developable if and only if Λ is F_σ-discrete in X.*

For brevity, let CP stand for the term "closure-preserving". As for the definitions of M_1-, M_3-spaces, refer to [3, Definition 5.1, 5.24].

Proposition 6. *Let X, Y_x, $x \in X$ be M_3-spaces and let Λ be F_σ-discrete in X. Then the resolution Z is an M_3-space.*

Proof. Let $\Lambda = \bigcup\{\Lambda(n) \mid n \in \mathbb{N}\}$, where each $\Lambda(n)$ is discrete and closed in X. For each $n \in \mathbb{N}$, let $\{U(p) \mid p \in \Lambda(n)\}$ be a discrete open expansion of $\Lambda(n)$ in X. For each $p \in \Lambda(n)$, there exists a CP closed neighborhood base $\mathcal{B}(p)$ of p in X such that $\bigcup \mathcal{B}(p) \subset U(p)$. Let $\bigcup\{\mathcal{B}(n) \mid n \in \mathbb{N}\}$ be a quasi-base for X, where each $\mathcal{B}(n)$ is a CP family of closed subsets of X. For each $x \in \Lambda$, there exists a quasi-base $\bigcup\{\mathcal{B}(x,n) \mid n \in \mathbb{N}\}$ for Y_x, where each $\mathcal{B}(x,n)$ is a CP family of closed subsets of Y_x. Note that for each $n, k \in \mathbb{N}$,

$$\mathcal{P}(n,k) = \left\{B \otimes B' \mid B' \in \mathcal{B}(x,n),\ B \in \mathcal{B}(x),\ x \in \Lambda(k)\right\}$$

is a CP family of closed subsets of Z. It is easy to check that $\bigcup\{\mathcal{P}(n,k) \mid n,k \in \mathbb{N}\} \cup \{\pi^{-1}(B) \mid B \in \mathcal{B}(n),\ n \in \mathbb{N}\}$ is a quasi-base for Z, proving that Z is an M_3-space. □

As for the M_1-ness of resolutions, we know that Ceder notes that Saalfrank line, that is the resolution of [0, 1] at each element into a copy of [0, 1] by the constant zero mapping, [8, Example 3.1.77], is an M_1-space. With respect to the M_3 vs. M_1 problem, Heath and Junnila [4] showed that every M_3-space is a perfect retraction of an M_1-space. We can use resolutions for the same purpose.

Proposition 7. *Every M_3-space X is a perfect retraction of an M_1-resolution of X.*

Proof. Let X be an M_3-space. Let $\mathcal{B} = \bigcup\{\mathcal{B}(n) \mid n \in \mathbb{N}\}$ be a quasi-base for X, where each $\mathcal{B}(n)$ is a CP family of closed subsets of X. There exists a subset $D = \bigcup\{D_n \mid n \in \mathbb{N}\}$ with each D_n discrete and closed in X such that $\overline{B \cap D} = B$ for each $B \in \mathcal{B}$. For each $n \in \mathbb{N}$, let $\{U(x) \mid x \in D_n\}$ be a discrete open expansion of D_n in X. For each $x \in D_n$, there exists a mapping g_x of X onto [0, 1] such that $f\big(X \setminus U(x)\big) = \{1\}$ and $f(x) = 0$.

Resolve X at each $x \in D$ into $Y_x = [0,1]$ by a mapping $f_x = g_x \mid (X \setminus \{x\})$. Let

$$Z = \bigcup\left\{\{x\} \times Y_x \mid x \in X\right\},$$

where $Y_x = \{0\}$ if $x \in X \setminus D$.

By [7, Lemma 6] π is a perfect mapping. Since X is homeomorphic to the subspace $\{(x,0) \mid x \in X\}$ of Z, it suffice to show that Z is an M_1-space. Let

$$\mathcal{P} = \left\{\{x\} \times [a,b] \mid 0 < a < b \le 1,\ a,b \in \mathbb{Q},\ x \in D\right\},$$

where $\mathbb{Q}$ is the set of all rationals in [0, 1]. Then obviously $\mathcal{P}$ is a σ-CP family of regular closed subsets of Z and forms a neighborhood base of each point of $\bigcup\left\{\{x\} \times (Y_x \setminus \{0\}) \mid x \in D\right\}$ in Z.

Let

$$\mathcal{B}(p) = \left\{B \in \mathcal{B} \mid p \in \operatorname{Int} B \subset B \subset U(p)\right\}, \quad p \in D.$$

Then $\mathcal{B}(p)$ is a σ-CP closed neighborhood base of p in X. For each $B \in \mathcal{B}(p)$, $p \in D$ and each $a \in \mathbb{Q}$ with $0 < a$, let

$$[B, p, a] = \bigcup\left\{\{x\} \times Y_x \mid x \in B \setminus \{p\}\right\} \cup \{p\} \times [0,a],$$

and let

$$\mathcal{Q} = \{[B, p, a] \mid B \in \mathcal{B}(p),\ p \in D,\ 0 < a,\ a \in \mathbb{Q}\}.$$

Then obviously $\mathcal{Q}$ is a σ-CP family of closed subsets of Z and forms a neighborhood base of each point of $\{(p, 0) \mid p \in D\}$ in Z. Moreover, each $[B, p, a]$ is regular closed in Z because

$$[B, p, a] = \Big\{\mathrm{Cl}_Z\Big[\bigcup\big\{\{x\} \times \big(Y_x \setminus \{0\}\big) \mid x \in B \setminus \{p\}\big\} \cup \{p\} \times (0, a)\Big]\Big\}.$$

Similarly,

$$\mathcal{R} = \{\pi^{-1}(B) \mid B \in \mathcal{B}\}$$

is a σ-CP family of regular closed subsets of Z and forms a neighborhood base of each point of $\bigcup\{\{x\} \times Y_x \mid x \in X \setminus D\}$ in Z. Thus we can construct a σ-CP quasi-base $\mathcal{P} \cup \mathcal{Q} \cup \mathcal{R}$ for Z consisting of regular closed subsets of Z. Hence Z is an M_1-space. □

An M_0-space X is a space which has a σ-CP base consisting of clopen subsets of X. Then the next follows from the same method as Proposition 6.

Proposition 8. *Let Y_x, $x \in X$, be M_0-spaces and let X be an M_0-space. Let Λ be F_σ-discrete in X. Then the resolution Z is an M_0-space.*

We do not know what kind of additional conditions on X, Y_x or f_x is needed for the resolution Z to be an M_1-space. Finally, we consider the class of M_3–μ-spaces. A space X is called a *μ-space* in [6] if X is embedded into the countable product of paracompact F_σ-metrizable spaces. But here, we use the characterization of M_3–μ-spaces in [5]. To state it, we need the following term: Let $\mathcal{U}, \mathcal{F}$ be families of subsets of a space X and let $p \in X$. We call that $\mathcal{U}$ is *$\mathcal{F}$-preserving in both sides at p* if for any $\mathcal{U}_0 \subset \mathcal{U}$, the following two conditions are satisfied:

(i) If $p \in \bigcap \mathcal{U}_0$, then there exists $F \in \mathcal{F}$ such that $p \in F \subset \bigcap \mathcal{U}_0$;
(ii) if $p \in X \setminus \bigcup \mathcal{U}_0$, then there exists $F \in \mathcal{F}$ such that $p \in F \subset X \setminus \bigcup \mathcal{U}_0$.

We call that $\mathcal{U}$ is $\mathcal{F}$-preserving in both sides *in* X if $\mathcal{U}$ is so at each point of X. The characterization is as follows: An M_3-space X is a *μ-space* if and only if there is a pair $\langle \mathcal{U}, \mathcal{F} \rangle$ of families of subsets of X satisfying the following:

(i) $\mathcal{F}$ is a σ-discrete closed network for X;
(ii) $\mathcal{U} = \bigcup\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ is a base for X such that for each $n, \mathcal{U}_n$ is $\mathcal{F}$-preserving in both sides in X.

We call the pair $\langle \mathcal{U}, \mathcal{F} \rangle$ an *M-structure* of X.

Proposition 9. *Let X, Y_x, $x \in X$ be M_3–μ-spaces and let Λ be F_σ-discrete in X. Then the resolution Z is an M_3–μ-space.*

Proof. By Proposition 6, Z is an M_3-space. Thus it remains to show that Z has an M-structure. Let $\langle\bigcup\{\mathcal{U}(n) \mid n \in \mathbb{N}\}, \mathcal{F}\rangle$ be the M-structure of X such that for each $n, \mathcal{U}(n)$ is $\mathcal{F}$-preserving in both sides in X. Assume that $\mathcal{F}$ is closed under finite intersections. For each $x \in X$, let $\langle\bigcup\{\mathcal{V}(x, n) \mid n \in \mathbb{N}\}, \mathcal{F}(x)\rangle$ be the M-structure of Y_x such that for each n, $\mathcal{V}(x, n)$ is $\mathcal{F}(x)$-preserving in both sides in Y_x. Let $\Lambda = \bigcup\{\Lambda_n \mid n \in \mathbb{N}\}$, where each Λ_n is discrete and closed in X. For each n, let $\{U(x) \mid x \in \Lambda_n\}$ be the discrete open expansion of Λ_n in X. For each $n, m, k \in \mathbb{N}$, Define $\mathcal{W}(n, m, k)$ as follows:

$$\mathcal{U}(n, m\colon x) = \bigl\{U \in \mathcal{U}(n) \mid x \in U \subset U(x)\bigr\}, \quad x \in \Lambda_m,$$
$$\mathcal{W}(n, m, k\colon x) = \bigl\{U \otimes V \mid U \in \mathcal{U}(n, m\colon x),\ V \in \mathcal{V}(x, k)\bigr\}, \quad x \in \Lambda_m,$$
$$\mathcal{W}(n, m, k) = \bigcup\bigl\{\mathcal{W}(n, m, k : x) \mid x \in \Lambda_m\bigr\}.$$

Then it is easy to see that

$$\mathcal{W} = \bigcup\bigl\{\mathcal{W}(n, m, k) \mid n, m, k \in \mathbb{N}\bigr\}$$

is a local base at each point of $\bigcup\bigl\{\{x\} \times Y_x \mid x \in \Lambda\bigr\}$ in Z. For each $m \in \mathbb{N}$, let

$$\mathcal{H}(m) = \bigl\{\{x\} \times F \mid F \in \mathcal{F}(x),\ x \in \Lambda_m\bigr\}$$
$$\cup \bigl\{\pi^{-1}\bigl(f_x^{-1}(F) \cap F'\bigr) \mid F \in \mathcal{F}(x),\ F' \in \mathcal{F},\ F' \subset U(x),\ x \in \Lambda_m\bigr\}.$$

Then $\mathcal{H}(m)$ is a σ-discrete family of closed subsets of Z and obviously $\mathcal{W}(m, n, k)$ is $\mathcal{H}(m)$-preserving in both sides at each point in $\bigcup\{U(x) \mid x \in \Lambda_m\}$. Note that for each $n, \pi^{-1}\bigl(\mathcal{U}(n)\bigr)$ is $\pi^{-1}(\mathcal{F})$-preserving in both sides in Z. Hence we have the M-structure

$$\Bigl\langle \mathcal{W} \cup \bigcup\bigl\{\pi^{-1}\bigl(\mathcal{U}(n)\bigr) \mid n \in \mathbb{N}\bigr\},\ \bigcup\bigl\{\mathcal{H}(m) \mid m \in \mathbb{N}\bigr\} \cup \pi^{-1}(\mathcal{F})\Bigr\rangle$$

for Z. □

Combining this with Corollary 4, we have the following:

Theorem 10. *Let $Y_x, x \in X$, be compact metrizable spaces and let X be an M_3–μ-space. Then the resolution Z is an M_3–μ-space if and only if Λ is F_σ-discrete in X.*

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