Global exponential stability for a class of retarded functional differential equations with applications in neural networks

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Abstract

In this paper, the global exponential stability and asymptotic stability of retarded functional differential equations with multiple time-varying delays are studied by employing several Lyapunov functionals. A number of sufficient conditions for these types of stability are presented. Our results show that these conditions are milder and more general than previously known criteria, and can be applied to neural networks with a broad range of activation functions assuming neither differentiability nor strict monotonicity. Furthermore, the results obtained for neural networks with time-varying delays do not assume symmetry of the connection matrix.

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1. Introduction

In recent years, the stability problem of time-delay neural networks has been widely investigated [5–28] due to theoretical interest as well as application considerations. In particular, stability analysis is a powerful tool for practical systems and associative memory [1–3] since delays are often encountered in various engineering systems, such as the turbo-
jet engine, microwave oscillator, nuclear reactor, rolling mill, ship stabilization, chemical engineering systems, manual control and systems with lossless transmission lines as well as neural networks. Frequently, there is a source of generation of oscillations and a source of instability in many typical neural networks [4,21,22]. However, in many practical time-delay neural networks, the time delays appearing in the systems are time-varying or are only known to be bounded in a certain range. Typical time-delay neural networks with multiple time-varying delays include the Hopfield neural network model [5,6,8–10,14,15,18–20,23,24], cellular neural network model [11–13,16,17,27,28] and bi-directional associative memory [7,25,26]. Consequently, the stability analysis of time-delay systems has been a main concern of researchers. The stability criteria for time-delay systems can be classified into two categories, namely delay-independent criteria and delay-dependent criteria, depending on whether they contain the delay argument as a parameter. There have been a number of significant developments in searching the stability criteria for systems with constant delays. However, the criteria are mostly delay-independent ones for time-delay systems with constant delays [6–13,16–20,25,26,28] and only a few of them are for neural networks with time-varying delays; see, for example, [14,15,26,27]. It is the purpose of this paper to search for both delay-independent and delay-dependent criteria, under which the global exponential stability or the global asymptotic stability of a class of generalized neural networks with multiple time-varying delays is guaranteed.

In [15], the author has proposed the following retarded functional differential equations:

\[
\dot{x}(t) = -D(x(t)) + W_0(x(t))f(x(t)) + W_1(x(t))f(x(t - \tau(t))),
\]  

which generalize the Hopfield neural network model, hybrid network models of the cellular neural network type as well as the bi-directional associative memory, where \( D(x) = (d_1(x_1), d_2(x_2), \ldots, d_n(x_n))^T \) and \( f(x) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))^T \) are continuous maps from \( \mathbb{R}^n \) to \( \mathbb{R}^n \), with \( D(0) = 0, f(0) = 0 \), and the \( n \times n \) matrices \( W_0(x) \) and \( W_1(x) \) are continuous in \( x_1(t), x_2(t), \ldots, x_n(t) \); time delay \( \tau \) is time-varying.

The author studied the global asymptotic stability (GAS) for system (1). However, investigations on retarded functional differential equations involve not only the discussion of GAS, but also the global exponential stability and exponential convergence rate. In many applications such as optimization, neural control and signal processing by means of neural networks, the properties of global exponential stability are of great interest.

This paper is organized as follows. In Section 2, problem formulation and some preliminary analyses are given. In Section 3, some criteria are proposed to guarantee the global exponential stability for a class of retarded functional differential equations with multiple time-varying delays. In the meantime, the global asymptotic stability for the above functional differential equations is also obtained as a by-product. In Section 4, we apply the results obtained in Section 3 to neural networks with time-varying delays and derive some criteria for global exponential stability and global asymptotic stability that are either a generalization of those existing or new criteria. Finally, a conclusion is made in Section 5.
2. Problem formulation and some preliminaries

In this paper, we consider a modification of (1) by incorporating different delays \( \tau_{ij}(t) \geq 0, i, j = 1, 2, \ldots, n \) in different communication channels, namely,

\[
\dot{x}_i(t) = -d_i(x_i(t)) + \sum_{j=1}^{n} w_{ij}(x_1(t), x_2(t), \ldots, x_n(t)) f_j(x_j(t)) \\
+ \sum_{j=1}^{n} w_{ij}^T(x_1(t), x_2(t), \ldots, x_n(t)) f_j(x_j(t - \tau_{ij}(t))),
\]

\( i, j = 1, 2, \ldots, n, \quad (2) \)

where \( d_i(.) \) and \( f_i(.) \) are continuous maps from \( \mathbb{R} \) to \( \mathbb{R} \), with \( d_i(0) = 0, f_i(0) = 0 \). \( w_{ij}(.) \) and \( w_{ij}^T(.) \) are continuous in \( x_1(t), x_2(t), \ldots, x_n(t) \), \( d_i(.) \) is continuous in \( x_i(t) \). Clearly, (1) is a special case of (2).

Model (2) is very general. For example, if \( d_i(x_i(t)) = d_i x_i(t) \), \( d_i > 0 \) (\( i = 1, 2, \ldots, n \)) and \( w_{ij}(x_1(t), \ldots, x_n(t)) = w_{ij} \) (constant), \( w_{ij}^T(x_1(t), \ldots, x_n(t)) = w_{ij}^T \) (constant) (\( i, j = 1, 2, \ldots, n \)), then Eq. (2) becomes a model of Hopfield-type neural networks (HNN) with time-varying delays. Moreover, if \( \tau_{ij}(t) \) (\( i, j = 1, 2, \ldots, n \)) are constants, (2) reduces to the delay Hopfield neural network (DHNN) model which has been widely studied [5,6,8–10,14,15,18–20,23,24]. If \( w_{ij}^T = 0 \) (\( i, j = 1, 2, \ldots, n \)), then (2) becomes the standard HNN model without delay.

Accompanying equation (2) are initial values of the form

\[
x_i(s) = \phi_i(s), \quad s \in [-\tau, 0], \quad \tau = \max_{1 \leq i, j \leq n} \{\tau_{ij}\},
\]

\( \phi_i(.) \) denote real-valued continuous functions defined on \([-\tau, 0]\). A solution of (2) is denoted by \( x(t) \) for all \( t > 0 \), where

\[
x(t) = (x_1(t, \phi_1), x_2(t, \phi_2), \ldots, x_n(t, \phi_n))^T
\]

with \( T \) denotes the transpose of a matrix and \( \phi_i(.) \) defined in (3).

The following assumptions are made on system (2) throughout this paper.

(A1) There exist positive constants \( D_i \) and \( \overline{D}_i, \quad i = 1, 2, \ldots, n \), such that, for all arguments,

\[
0 < \overline{D}_i \leq \frac{d_i(x_i) - d_i(y_i)}{x_i - y_i} \leq D_i, \quad \text{for} \ x_i \neq y_i, \quad i = 1, 2, \ldots, n.
\]

(A2) There exist positive constants \( F_i, \quad i = 1, 2, \ldots, n \), such that, for all arguments,

\[
\left| f_i(x_i) \right| \leq F_i |x_i|, \quad i = 1, 2, \ldots, n.
\]

(A3) \( \tau_{ij} : [0, +\infty) \rightarrow [0, +\infty) \) is continuous, differentiable, and \( 0 \leq \tau_{ij}(t) \leq \bar{\tau}, \quad \tau'_{ij}(t) \leq R < 1 \).

Hence, the hypothesis (A3) ensures that \( t - \tau_{ij}(t) \) has differential inverse function denoted by \( \varphi_{ij}(t) \) and \( \inf_{t>0} \left\{ \varphi'_{ij}(t) \right\} > 0 \).
(A4) The following condition holds:

\[ A = \max_{1 \leq j \leq n} \sup_{a \leq t \leq 0} - \int_{t_j(0)}^{0} |w_{ij}(x_1(t_1), x_2(t_2), \ldots, x_n(t_n))| dt < +\infty. \] (6)

**Remark 1.** The assumption that \( \tau_{ij}'(t) \leq R < 1 \) stems from the need to bound the growth variations in the delay factor as a time function. It may be considered restrictive but in some applications it is considered realistic and holds for a wide class of retarded functional differential equations. Thus our results are applicable to the class of time-delay with bounded state-delay in the manner of (2).

**Definition 1** [29]. If there exist \( k > 0 \) and \( \gamma(k) > 1 \), such that

\[ \|x(t)\| \leq \gamma(k)e^{-kt} \sup_{-\infty \leq \theta \leq 0} \|x(\theta)\|, \quad \forall t > 0, \] (7)

then system (2) is considered as exponentially stable, and \( k \) is called the degree of exponential stability.

3. Analysis of global exponential stability

Now we present our first main result, a delay-dependent criterion, for the global exponential stability of system (2).

**Theorem 1.** Consider the retarded functional differential equation (2) and assume that conditions (A1)–(A4) are satisfied. If there exist positive constants \( \lambda_i \) (i = 1, 2, \ldots, n), \( r_1 \in [0, 1], r_2 \in [0, 1], \varepsilon \) and the following condition holds:

\[ \alpha \equiv \max_{1 \leq i \leq n} \sup_{t \geq 0} \left\{ \frac{1}{(D_i - \varepsilon)\lambda_i} \left[ \lambda_i \sum_{j=1}^{n} \left( F_{ij}^{2r_1} |w_{ij}(x_1(t), \ldots, x_n(t))| \right) + \sum_{j=1}^{n} \lambda_j \left( F_{ij}^{2(1-r_1)} |w_{ji}(x_1(t), \ldots, x_n(t))| \right) \right] \right\} < 2, \] (8)

then the trivial solution for system (2) is globally exponentially stable. More precisely, we have

\[ \|x(t)\|_{2} \equiv \left( \sum_{i=1}^{n} \lambda_i \right)^{1/2} \leq K_{\alpha} \|\phi\|_{2} e^{-\varepsilon t} \] (9)
where
\[
K_{\alpha} = \left( \frac{\max_{1 \leq i \leq n} \{ \lambda_i \}}{\min_{1 \leq i \leq n} \{ \lambda_i \}} \right)^{1/2} \left( 1 + nA(1 - R)^{-1} \max_{1 \leq j \leq n} \left( F_j^{2(1 - r_j)} \right) \right)^{1/2}.
\]

Proof. Introducing the following nonnegative functional as the Lyapunov functional candidate.
\[
V(x_1, x_2, \ldots, x_n)(t) = \sum_{i=1}^{n} \lambda_i \left( x_i^2(t)e^{2\varepsilon t} + \sum_{j=1}^{n} F_j^{2(1-r_j)} \int_{t}^{\psi_j(t)} \left| w_{ij}^+ (x_1(s), x_2(s), \ldots, x_n(s)) \right| \right.
\]
\[
\times x_j^2(s - \tau_{ij}(s))e^{2\varepsilon(t-\tau_{ij}(t))} ds \right).
\]

By computing its derivative \( \dot{V} \) along the trajectories and make use of Eq. (2) and the Cauchy inequality (i.e., \( a^2 + b^2 \geq 2ab \)), we get for \( t \geq 0 \):
\[
\dot{V}(x_1, x_2, \ldots, x_n)(t) \leq \sum_{i=1}^{n} \lambda_i e^{2\varepsilon t} \left[ -2(D_i - \varepsilon)x_i^2(t) \right.
\]
\[
+ \sum_{j=1}^{n} \left| w_{ij} (x_1(t), x_2(t), \ldots, x_n(t)) \right| \left( F_j^{2(1-r_j)} \left| x_i(t) \right| \right)
\]
\[
\times \left( e^{\varepsilon \tau_{ij}(t)} F_j^{2(1-r_j)} \left| x_j(t - \tau_{ij}(t)) \right| \right)
\]
\[
\left. + \sum_{j=1}^{n} \frac{F_j^{2(1-r_j)}}{x_j^2(t)} \left| w_{ij} (x_1(t), x_2(t), \ldots, x_n(t)) \right| \right]
\]
\[
\left. \times \left( e^{-\varepsilon \tau_{ij}(t)} F_j^{2(1-r_j)} \left| x_j(t - \tau_{ij}(t)) \right| \right) \right].
\]
\[-\frac{n}{j=1}F_j^{2(1-r_j)}|w_{ij}^2(x_1(t), \ldots, x_n(t))|e^{-2e_i\tau_j(t)}x_{ij}^2(t - \tau_j(t))\}

\[\leq \sum_{j=1}^{n}\lambda_j e^{2rt} \left\{-2(D_i - \varepsilon)x_i^2(t) + \sum_{j=1}^{n}|w_{ij}(x_1(t), \ldots, x_n(t))|F_j^{2r_1}x_{ij}^2(t)\right\}

+ \sum_{j=1}^{n}|w_{ij}(x_1(t), \ldots, x_n(t))|F_j^{-2(1-r_j)}x_{ij}^2(t)

+ \sum_{j=1}^{n}|w_{ij}(x_1(t), \ldots, x_n(t))|e^{2e_i\tau_j(t)}F_j^{2r_2}x_{ij}^2(t)

+ \sum_{j=1}^{n}|w_{ij}(x_1(t), \ldots, x_n(t))|e^{-2e_i\tau_j(t)}F_j^{2(1-r_j)}x_{ij}^2(t - \tau_j(t))\]

\[\leq \sum_{i=1}^{n}e^{2rt} \left\{-2(D_i - \varepsilon)\lambda_i + \sum_{j=1}^{n}\lambda_j \left[F_j^{2r_1}|w_{ij}(x_1(t), \ldots, x_n(t))| + F_j^{2r_2}e^{2rt}|w_{ij}^2(x_1(t), \ldots, x_n(t))|\right]\right\}

+ \sum_{j=1}^{n}\lambda_j \left[F_j^{2(1-r_j)}|w_{ij}(x_1(t), \ldots, x_n(t))|\right]

+ \sum_{j=1}^{n}\lambda_j \left[F_j^{2(1-r_j)}|w_{ij}(x_1(t), \ldots, x_n(t))|\right] x_{ij}^2(t)\]

\[= \sum_{i=1}^{n}e^{2rt}\lambda_i(D_i - \varepsilon) \left\{-2 + \frac{1}{(D_i - \varepsilon)\lambda_i}\right\}

\times \left[\lambda_i \sum_{j=1}^{n}(F_j^{2r_1}|w_{ij}(x_1(t), \ldots, x_n(t))| + F_j^{2r_2}e^{2rt}|w_{ij}^2(x_1(t), \ldots, x_n(t))|\right]

+ \sum_{j=1}^{n}\lambda_j \left[F_j^{2(1-r_j)}|w_{ij}(x_1(t), \ldots, x_n(t))|\right]

+ \sum_{j=1}^{n}\lambda_j \left[F_j^{2(1-r_j)}|w_{ij}(x_1(t), \ldots, x_n(t))|\right] x_{ij}^2(t)\]

\[\leq \max_{1 \leq i \leq n} \left[\lambda_i(D_i - \varepsilon)(-2 + \alpha)\right] \sum_{i=1}^{n}x_i^2(t)e^{2rt} + B \sum_{i=1}^{n}x_i^2(t)e^{2rt} \quad \quad (12)\]
where $B$ is a positive constant given by $B \equiv -\max_{1 \leq i \leq n} \{ \lambda_i (D_i - \epsilon) (-2 + \alpha) \}$ and $\inf_{t \geq 0} \psi_i' (t) > 0$. By integration, we have, for $t \geq 0$,

$$V(x_1, x_2, \ldots, x_n)(t) - V(x_1, x_2, \ldots, x_n)(0) = \int_0^t \dot{V}(x_1, x_2, \ldots, x_n)(s) \, ds$$

$$\leq -B \sum_{i=1}^n \int_0^t x_i^2(s)e^{2\epsilon s} \, ds < 0$$

(13)

and, since $V(x_1, x_2, \ldots, x_n)(t)$ is a nonnegative functional, we get

$$B \sum_{i=1}^n \int_0^t x_i^2(s)e^{2\epsilon s} \, ds \leq V(x_1, \ldots, x_n)(0), \quad \text{for } t \geq 0.$$ 

(14)

Let $\xi = s - \tau_{ij}(s) \equiv \psi_{ij}^{-1}(s)$, we can easily obtain

$$\int_0^{\psi_{ij}(0)} \left| w_{ij}^T(\psi_{ij}(\xi), \ldots, \psi_{ij}(\xi)) \right| |x_j^2(\xi)| e^{2\epsilon \xi} \left( 1 - \frac{d\tau_{ij}(s)}{ds} \right)^{-1} d\xi$$

$$\leq (1 - R)^{-1} \sup_{-\tau_{ij}(0) \leq \xi \leq 0} \left\{ |x_j(\xi)|^2 \right\} \sup_{-\tau_{ij}(0) \leq \xi \leq 0} \left\{ e^{2\epsilon \xi} \right\} \times \int_{-\tau_{ij}(0)}^{0} \left| w_{ij}^T(x_1(\psi_{ij}(\xi)), \ldots, x_n(\psi_{ij}(\xi))) \right| d\xi$$

$$\leq (1 - R)^{-1} \sup_{-\tau_{ij}(0) \leq \xi \leq 0} \left\{ |x_j(\xi)|^2 \right\} \int_{-\tau_{ij}(0)}^{0} \left| w_{ij}^T(x_1(\psi_{ij}(\xi)), \ldots, x_n(\psi_{ij}(\xi))) \right| d\xi.$$ 

Hence, from Eq. (11) and the above inequality, we have

$$V(x_1, x_2, \ldots, x_n)(0)$$

$$= \sum_{i=1}^n \lambda_i x_i^2(0) + \sum_{i=1}^n \sum_{j=1}^n \lambda_i F_{ij}^{2(1-R)} \int_{-\tau_{ij}(0)}^{0} \left| w_{ij}^T(x_1(\psi_{ij}(\xi)), \ldots, x_n(\psi_{ij}(\xi))) \right| d\xi$$

$$\leq \max_{1 \leq i \leq n} \{ \lambda_i \} \sum_{i=1}^n \phi_i^2(0) + \max_{1 \leq i \leq n} \{ \lambda_i \} \max_{1 \leq j \leq n} \left\{ F_{ij}^{2(1-R)} \right\} (1 - R)^{-1} \| \phi \|_2^2$$
Thus, by Lemma 1 in [33], we get
\[
\sum_{i=1}^{n} \int_{0}^{\infty} x_i^2(t)e^{2\varepsilon t} dt < \infty \Rightarrow x_i^2(t) \to 0, \quad t \to \infty, \quad i = 1, 2, \ldots, n.
\] (16)

From inequality (13), we have
\[
V(x_1, \ldots, x_n)(t) \leq V(x_1, x_2, \ldots, x_n)(0), \quad \text{for all} \quad t \geq 0.
\] (17)

By means of the form of the Lyapunov functional (11), we obtain
\[
\min_{1 \leq i \leq n} \{\lambda_i\}e^{2\varepsilon t} \sum_{i=1}^{n} x_i^2(t) \leq \sum_{i=1}^{n} \lambda_i x_i^2(t)e^{2\varepsilon t} \leq V(x_1, x_2, \ldots, x_n)(t).
\] (18)

Combining inequalities (15), (17) and (18), we easily obtain
\[
\|x(t)\|_2 \equiv \left(\sum_{i=1}^{n} x_i^2(t)\right)^{1/2} \leq K_\alpha \|\phi\|_2 e^{-\varepsilon t},
\] (19)

where \(K_\alpha\) is given by Eq. (10). This completes the proof. \(\square\)

Note that if we let \(\varepsilon \to 0^+\), then the global exponential stability of Theorem 1 is reduced to the global asymptotic stability. Consequently, we have the following result, which is a delay-dependent criterion.

**Corollary 1.** System (2) satisfying (A1)–(A4) is globally asymptotically stable provided that there exist positive constants \(\alpha_i, \lambda_i (i = 1, 2, \ldots, n), r_1 \in [0, 1], r_2 \in [0, 1], \) and

\[
\alpha' \equiv \max\sup_{1 \leq i \leq n, t \geq 0} \left\{ \frac{1}{D_i \lambda_i} \left[ \lambda_i \sum_{j=1}^{n} \left( F_{ij}^2 x_i(t) \right) + F_{ij}^2 \left| w_{ij}(x_1(t), \ldots, x_n(t)) \right| \right] + \sum_{j=1}^{n} \lambda_j \left( F_{ij}^2 x_j(t) \right) + \sum_{j=1}^{n} \lambda_j \left( F_{ij}^2 x_j(t) \right) \right\} < 2.
\] (20)

**Remark 2.** In [15], the author considered the global asymptotic stability of system (1) and require \(\tau(t) = \tau\) (constant). Moreover, in the proof of Theorem 3.1 (see [15, p. 66]), the function \(f(x)\) satisfies condition (4) when \(D_i = 0\). However, by relaxing the limitation of \(\tau\) (constant) to \(\tau_{ij}(t)\), we not only obtain the global asymptotic stability, but also the global exponential stability. Hence, our results are more general than those of [15].
Now we present another main result, a delay-dependent criterion about the global exponential stability of system (2).

**Theorem 2.** Consider the retarded functional differential equation (2) and assume that conditions (A1)–(A4) are satisfied. If there exist positive constants $\lambda_i$ ($i = 1, 2, \ldots, n$), $\varepsilon$ and the following condition holds:

$$
\beta \equiv \max_{1 \leq i \leq n} \sup_{t \geq 0} \left\{ \frac{F_i}{\lambda_i (D_i - \varepsilon)} \sum_{j=1}^{n} \lambda_j \left( \left| w_{ji}(x_1(t), \ldots, x_n(t)) \right| \right) + e^{\varepsilon t} \left| w_{ji}^T(x_1(\phi_{ji}(t)), \ldots, x_n(\phi_{ji}(t))) \right| \right. \\
\left. \left| \phi_{ji}'(t) \right| \right\} < 1,
$$

(21) then the trivial solution for system (2) is globally exponentially stable. More precisely, we get

$$
\| x(t) \|_1 = \sum_{i=1}^{n} |x_i(t)| \leq K_\beta \| \phi \|_1 e^{-\varepsilon t},
$$

(22)

where

$$
K_\beta = \left( \frac{\max_{1 \leq i \leq n} \{ \lambda_i \}}{\min_{1 \leq i \leq n} \{ \lambda_i \}} \right) \left( 1 + n A (1 - R)^{-1} \max_{1 \leq i \leq n} \{ F_i \} \right).
$$

(23)

**Proof.** Introducing the following nonnegative functional as the Lyapunov functional candidate:

$$
V(x_1, x_2, \ldots, x_n)(t) = \sum_{i=1}^{n} \lambda_i \left[ |x_i(t)| e^{\varepsilon t} + \sum_{j=1}^{n} F_j \int_{t}^{\infty} w_{ji}^T(x_1(s), \ldots, x_n(s)) \left| x_j(s - \tau_{ji}(s)) \right| e^{\varepsilon s} ds \right].
$$

(24)

By computing its upper Dini derivative $D^+ V$ along the trajectories and utilizing Eq. (2), and note that $\phi_{ji}(t) - \tau_{ji}(\phi_{ji}(t)) = t$, then $e^{\varepsilon \phi_{ji}(t)} = e^{\varepsilon t + \tau_{ji}(\phi_{ji}(t))}$. Hence we get for $t \geq 0$:

$$
D^+ V(x_1, x_2, \ldots, x_n)
= \sum_{i=1}^{n} \lambda_i \left[ -d_i(x_i(t)) + \sum_{j=1}^{n} w_{ij} f_j(x_j(t)) \right. \\
+ \sum_{j=1}^{n} w_{ji}^T(x_1(t), \ldots, x_n(t)) f_j(x_j(t - \tau_{ij}(t))) \left. \right] e^{\varepsilon t} \operatorname{sgn}(x_i(t))
+ \varepsilon |x_i(t)| e^{\varepsilon t}
$$
\begin{align*}
&+ \sum_{j=1}^{n} F_j \left| w_{ij}^* (x_1 (t), \ldots, x_n (\psi_{ij} (t))) \right| \left| x_j (t) e^{\epsilon \psi_{ij} (t)} \psi_{ij}' (t) \right| \\
&- \sum_{j=1}^{n} F_j \left| w_{ij}^* (x_1 (t), \ldots, x_n (\psi_{ij} (t))) \right| \left| x_j (t - \tau_{ij} (t)) e^{\epsilon t} \right|
\end{align*}

\begin{align*}
&\leq \sum_{i=1}^{n} \lambda_i e^{\epsilon t} \left\{ -(D_i - \epsilon) \left| x_i (t) \right| + \sum_{j=1}^{n} F_j \left| w_{ij} (x_1 (t), \ldots, x_n (t)) \right| \left| x_j (t) \right| \right.
\end{align*}

\begin{align*}
&+ \sum_{j=1}^{n} F_j \left| w_{ij}^* (x_1 (t), \ldots, x_n (\psi_{ij} (t))) \right| \left| x_j (t - \tau_{ij} (t)) \right| e^{\epsilon t}
\end{align*}

\begin{align*}
&\leq \sum_{i=1}^{n} \lambda_i e^{\epsilon t} \left\{ -(D_i - \epsilon) \left| x_i (t) \right| + \sum_{j=1}^{n} F_j \left| w_{ij} (x_1 (t), \ldots, x_n (t)) \right| \left| x_j (t) \right| \right.
\end{align*}

\begin{align*}
&+ \sum_{j=1}^{n} \lambda_j F_i e^{\epsilon t} \left| w_{ji}^* (x_1 (\psi_{ji} (t)), \ldots, x_n (\psi_{ji} (t))) \right| \left| \psi'_{ji} (t) \right| \left| x_i (t) \right|
\end{align*}

\begin{align*}
&= \sum_{i=1}^{n} \lambda_i (D_i - \epsilon) e^{\epsilon t} \left\{ -1 + \frac{F_i}{\lambda_i (D_i - \epsilon)} \sum_{j=1}^{n} \lambda_j \left( \left| w_{ji} (x_1 (t), \ldots, x_n (t)) \right| \right) \right.
\end{align*}

\begin{align*}
&+ \epsilon t \left| w_{ji}^* (x_1 (\psi_{ji} (t)), \ldots, x_n (\psi_{ji} (t))) \right| \left| \psi'_{ji} (t) \right| \left| x_i (t) \right|
\end{align*}

\begin{align*}
&\leq \{ -1 + \beta \} \sum_{i=1}^{n} \lambda_i (D_i - \epsilon) \left| x_i (t) \right| e^{\epsilon t} < 0. \tag{25}
\end{align*}

It follows from Eq. (25) that

\begin{align*}
V(x_1, \ldots, x_n)(t) &\leq V(x_1, \ldots, x_n)(0), \quad \text{for all } t \geq 0 \tag{26}
\end{align*}

and from Eq. (24) that

\begin{align*}
\min_{1 \leq i \leq n} \{ \lambda_i \} e^{\epsilon t} \sum_{i=1}^{n} \left| x_i (t) \right| &\leq \sum_{i=1}^{n} \lambda_i \left| x_i (t) \right| e^{\epsilon t} \leq V(x_1, \ldots, x_n)(t). \tag{27}
\end{align*}

Let \( \xi = s - \tau_{ij} (s) = \psi_{ij}^{-1} (s) \), we have

\begin{align*}
\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i F_j \int_{0}^{\psi_{ij}(0)} \left| w_{ij}^* (x_1(s), \ldots, x_n(s)) \right| \left| x_j (s - \tau_{ij} (s)) \right| e^{\epsilon s} \, ds
\end{align*}
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i F_j \int_{-\tau_{ij}(0)}^{0} |w_{ij}^T(x_1(\psi_{ij}(\xi)), \ldots, x_n(\psi_{ij}(\xi)))| |x_j(\xi)| \left(1 - \frac{d\tau_{ij}(s)}{ds}\right)^{-1} \times e^{\varepsilon(\xi - \tau_{ij}(s))} d\xi \]

\[ \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i F_j (1 - R)^{-1} \sup_{-\tau_{ij}(0) \leq \xi \leq 0} \left\{ |x_j(\xi)| \right\} \]

\[ \times \int_{-\tau_{ij}(0)}^{0} |w_{ij}^T(x_1(\psi_{ij}(\xi)), \ldots, x_n(\psi_{ij}(\xi)))| |x_j(s - \tau_{ij}(s))| d\xi \]

\[ \leq nA(1 - R)^{-1} \max_{1 \leq i \leq n} \lambda_i \max_{1 \leq j \leq n} \{F_j\} \|\psi\|_1. \]

Hence, by Eq. (24) and the above inequality, we also have

\[ V(x_1, \ldots, x_n)(0) \]

\[ = \sum_{i=1}^{n} \lambda_i \left\{ |x_i(0)| + \sum_{j=1}^{n} F_j \int_{0}^{\psi_i(0)} |w_{ij}^T(x_1(s), \ldots, x_n(s))| e^{\varepsilon(s - \tau_{ij}(s))} |x_j(s - \tau_{ij}(s))| ds \right\} \]

\[ \leq \max_{1 \leq i \leq n} \lambda_i \left(1 + nA(1 - R)^{-1} \max_{1 \leq j \leq n} \{F_j\} \|\phi\|_1 \right). \]

Combining (26)–(28), we have

\[ \|x(t)\|_1 = \sum_{i=1}^{n} |x_i(t)| \leq K\beta \|\phi\|_1 e^{-\varepsilon t}, \]

where \(K\beta\) is given by Eq. (23). Hence, the proof of Theorem 2 is completed. \(\square\)

Under similar circumstances, the global exponential stability of Theorem 2 is reduced to the global asymptotic stability if we let \(\varepsilon \to 0^+\). Consequently, we have the following result, which is a delay-dependent criterion.

**Corollary 2.** Consider the retarded functional differential equation (2) and assume that conditions (A1)–(A4) are satisfied. If there exist positive constants \(\lambda_i (i = 1, 2, \ldots, n)\), and the following condition holds:

\[ \beta' \equiv \max_{1 \leq i \leq n} \sup_{\tau, \sigma \geq 0} \left\{ \frac{F_i}{\lambda_i D_i} \sum_{j=1}^{n} \lambda_j \left( |w_{ji}(x_1(t), \ldots, x_n(t))| \right) 
\]

\[ + |w_{ji}^T(x_1(\psi_{ji}(t)), \ldots, x_n(\psi_{ji}(t)))|\psi'_{ji}(t)\right\} \leq 1 \] (30)

then the trivial solution for system (2) is globally asymptotically stable.
Remark 3. By constructing the following Lyapunov functional

\[ V(x_1, \ldots, x_n) = \sum_{i=1}^{n} p_i \left( |x_i(t)| + \sum_{j=1}^{n} |w_{ij}^r| k_j \int_{t-\tau_j}^{t} |x_j(s)| \, ds \right), \]  

(31)

where \( p_i > 0, k_i > 0, i = 1, 2, \ldots, n \). Lu [18] consider the following retarded differential equations:

\[ \frac{du_i}{dt} = -g_i(u_i) + \sum_{j=1}^{n} w_{0ij}^r f_j(u_j(t)) + \sum_{j=1}^{n} w_{ij}^r f_j(u_j(t-\tau_j)) + I_i, \quad i = 1, 2, \ldots, n, \]  

(32)

which generalized the Hopfield neural networks with constant delays, where \( w_{0ij}^r, w_{ij}^r, I_i, \tau_j \) are real constant numbers. Some assumptions are made.

(L1) \( g_i: \mathbb{R} \rightarrow \mathbb{R} \) is differentiable and strictly monotonically increasing, i.e., \( m_i = \inf_{x \in \mathbb{R}} \{ g'_i(x) \} > 0, i = 1, 2, \ldots, n \), where \( g'_i(x) \) represents the derivative of \( g_i(x) \).

(L2) \( f_i: \mathbb{R} \rightarrow \mathbb{R} \) is global Lipschitz with Lipschitz constant \( k_i > 0, \) i.e., \( |f_i(x_1) - f_i(x_2)| \leq k_i |x_1 - x_2|, \) for any \( x_1, x_2 \in \mathbb{R} \) and \( i = 1, 2, \ldots, n \).

Hence, the global asymptotic stability condition is

\[ \max_{1 \leq i \leq n} \sum_{j=1}^{n} \left( \frac{k_i d_j}{m_i d_i} \right) \left( |w_{ij}^0| + |w_{ij}^r| \right) < 1. \]  

(33)

Obviously, condition (33) coincides with (30) if \( w_{ij}(x_1(t), \ldots, x_n(t)) = w_{ij}(constant), \) \( w_{ij}^r(x_1(t), \ldots, x_n(t)) = w_{ij}^r(constant), \) and \( \tau_{ij}(t) = \tau_{ij}(constant) \). We also note that the requirement for a differentiable function \( g_i(x) \) is relaxed to (A1) and the time delay can be time-varying. Hence, our results are more general than those reported in [18].

In Theorems 1 and 2, we require the time-varying delays satisfy \( \tau_{ij}'(t) \leq R < 1 \). However, in the following theorem, these delays are not necessarily continuous and differentiable. They only need to satisfy the condition \( 0 \leq \tau_{ij}(t) \leq \bar{\tau} \). We construct another Lyapunov functional and obtain the following criteria for global exponential stability.

Theorem 3. Consider the retarded functional differential equation (2) and assume there exist positive constants \( a_i, b_i, i = 1, 2, \ldots, n, \rho > 1 \) and conditions (A1), (A2), (A4), and \( 0 \leq \tau_{ij}(t) \leq \bar{\tau} \) hold. If the following requirement is satisfied:

\[ \gamma \equiv \max_{1 \leq j \leq n} \sup_{t \geq 0} \left\{ \frac{1}{(D_j - \varepsilon)} \left[ \sum_{i=1}^{n} \left| w_{ij}(x_1(t), \ldots, x_n(t)) \frac{\partial c_j}{\partial t} \right| + \rho \sum_{j=1}^{n} \sum_{s=1}^{n} F_{ij} \left| \frac{\partial x_s}{\partial t} (x_1(t), \ldots, x_n(t)) \right| \right] \right\} < 1, \]  

(34)
where
\[ c_i = \max_{1 \leq i \leq n} \{ a_i, b_i \}, \quad d_i = \min_{1 \leq i \leq n} \{ a_i, b_i \}, \quad \psi_i(x_i) = \begin{cases} a_i, & x_i \geq 0, \\ -b_i, & x_i < 0, \end{cases} \] (35)
then the trivial solution of system (2) is globally exponentially stable. More precisely, we have
\[ \| x(t) \|_1 = \sum_{i=1}^{n} |x_i(t)| \leq \max_{1 \leq i \leq n} \{ c_i \} \| \phi \|_1 e^{-\varepsilon t}. \] (36)

Proof. Define a Lyapunov functional as follows:
\[ V(x_1(t), \ldots, x_n(t)) = \sum_{s=1}^{n} \psi_s(x_s(t)) x_s(t) e^{\varepsilon t}. \] (37)
It is easy to calculate the upper right derivative of \( V(x_1, \ldots, x_n)(t) \) along the solution of Eq. (2).
\[ D^+ V(x_1, \ldots, x_n)(t) \]
\[ = \sum_{s=1}^{n} \psi_s(x_s(t) \pm 0) \left[ -d_s(x_s) + \sum_{j=1}^{n} w_{sj}(x_1(t), \ldots, x_n(t)) f_j(x_j(t)) \\
+ \sum_{j=1}^{n} w^T_{sj}(x_1(t), \ldots, x_n(t)) f_j(x_j(t - \tau_{sj}(t))) + \varepsilon x_s(t) \right] e^{\varepsilon t} \]
\[ \leq \sum_{j=1}^{n} \psi_j(x_j(t) \pm 0) \left[ -d_j(x_j(t)) + \varepsilon x_j(t) \right] e^{\varepsilon t} \\
+ \left[ \sum_{s=1}^{n} \sum_{j=1}^{n} F_j \left| w_{sj}(x_1(t), \ldots, x_n(t)) \right| |\psi_s(x_s(t) \pm 0)| |x_j(t)| \right] e^{\varepsilon t} \]
\[ \leq \sum_{j=1}^{n} \psi_j(x_j(t)) x_j(t) \left[ -(D_j - \varepsilon) + F_j \sum_{s=1}^{n} \left| w_{sj}(x_1(t), \ldots, x_n(t)) \psi_s(x_s(t) \pm 0) \right| \right] e^{\varepsilon t} \]
\[ + \sum_{s=1}^{n} \sum_{j=1}^{n} F_j |\psi_s(x_s(t) \pm 0)| w^T_{sj}(x_1(t), \ldots, x_n(t)) |x_j(t - \tau_{sj}(t))| e^{\varepsilon t}. \] (38)
We can choose \( \rho > 1 \) such that
\[ V(x_1(t + \theta), \ldots, x_n(t + \theta)) \leq \rho V(x_1(t), \ldots, x_n(t)), \quad \theta \in (-\bar{\tau}, 0]. \] (39)
Hence, we have
\[
|x_j(t + \theta)| \leq \frac{\rho}{|\psi_j(x_j(t + \theta))|} \sum_{s=1}^{n} \psi_s(x_s(t))x_s(t).
\] (40)

Therefore,
\[
D^+ V(x_1, \ldots, x_n)(t)
\leq \sum_{j=1}^{n} \psi_j(x_j(t))x_j(t)
\times \left( -(D_j - \epsilon) + F_j \sum_{s=1}^{n} |w_{xs}(x_1(t), \ldots, x_n(t)) \frac{\psi_s(x_s(t) \pm 0)}{\psi_j(x_j(t))}| \right) e^{\epsilon t}
+ \sum_{s=1}^{n} \sum_{j=1}^{n} F_j |\psi_s(x_s(t) \pm 0)w^T_{js}(x_1(t), \ldots, x_n(t))| \frac{\rho}{|\psi_j(x_j(t - \tau_{sj}(t))|)}
\times \sum_{s=1}^{n} \psi_s(x_s(t))x_s(t)e^{\epsilon t}
\leq \sum_{j=1}^{n} \psi_j(x_j(t))x_j(t)
\times \left( -(D_j - \epsilon) + F_j \sum_{s=1}^{n} |w_{xs}(x_1(t), \ldots, x_n(t)) \frac{\psi_s(x_s(t) \pm 0)}{\psi_j(x_j(t))}| \right) e^{\epsilon t}
+ \sum_{s=1}^{n} \sum_{j=1}^{n} F_i |\psi_s(x_s(t) \pm 0)w^T_{si}(x_1(t), \ldots, x_n(t))| \frac{\rho}{|\psi_i(x_i(t - \tau_{si}(t))|)}
\times \sum_{s=1}^{n} \psi_j(x_j(t))x_j(t)e^{\epsilon t}
\leq \sum_{j=1}^{n} \psi_j(x_j(t))x_j(t)
\times \left( -(D_j - \epsilon) + F_j \sum_{s=1}^{n} |w_{xs}(x_1(t), \ldots, x_n(t)) \frac{\psi_s(x_s(t) \pm 0)}{\psi_j(x_j(t))}| \right) e^{\epsilon t}
+ \rho \sum_{i=1}^{n} \sum_{s=1}^{n} F_i |\psi_{i}(x_i(t))w^T_{si}(x_1(t), \ldots, x_n(t))| e^{\epsilon t}
\leq (-1 + \gamma) \sum_{j=1}^{n} (D_j - \epsilon) \psi_j(x_j(t))x_j(t)e^{\epsilon t} < 0.
\] (41)

By inequality (41), we can easily obtain
\[
V(x_1, \ldots, x_n)(t) \leq V(x_1, \ldots, x_n)(0), \quad \text{for all } t \geq 0
\] (42)

and from the form of Lyapunov functional (37), we get
\[
\min_{1 \leq i \leq n} [d_i] e^{\epsilon t} \sum_{j=1}^{n} |x_j(t)| \leq \sum_{i=1}^{n} \psi_i(x_i)e^{\epsilon t} = V(x_1, \ldots, x_n)(t).
\] (43)
Similarly, by (37) we get

\[ V(x_1, \ldots, x_n)(0) = \sum_{i=1}^{n} \psi_i(x_i(0))x_i(0) \leq \max_{1 \leq i \leq n} \{c_i\} \sum_{i=1}^{n} |\phi_i(0)|. \tag{44} \]

By (42)–(44), we have

\[ \|x(t)\|_1 = \sum_{i=1}^{n} |x_i(t)| \leq \max_{1 \leq i \leq n} \{c_i\} \|\phi\|_1 e^{-\varepsilon t}. \tag{45} \]

This completes the proof. \( \square \)

In the following, if \( a_s = a^s, b_s = b^s \) (where \( a^s \) is the \( s \)th power of \( a \)) in Theorem 3, then we have the following result which is easily verified.

**Corollary 3.** If the conditions of Theorem 3 are satisfied and there exist constants \( a > 0 \) and \( \rho > 1 \) such that

(i) \( D_j - \varepsilon - F_j |w_{jj}(x_1(t), \ldots, x_n(t))| > 0, \quad j = 1, 2, \ldots, n, \tag{46} \)

(ii) \( \frac{F_j|w_{jj}(x_1(t), \ldots, x_n(t))|}{D_j - \varepsilon - F_j |w_{jj}(x_1(t), \ldots, x_n(t))|} \leq \frac{a^{1-s}}{n}, \quad s \neq j, \quad j = 1, 2, \ldots, n, \tag{47} \)

(iii) \( \frac{D_j - \varepsilon - F_j |w_{jj}(x_1(t), \ldots, x_n(t))|}{n \sum_{i=1}^{n} n \sum_{s=1}^{n} F_i a^{s-1} |w_{ss}^i(x_1(t), \ldots, x_n(t))|} \geq \rho > 1, \quad j = 1, 2, \ldots, n, \tag{48} \)

then the trivial solution of system (2) is globally exponentially stable.

**Proof.** By Theorem 3, we have

\[ D^+ V(x_1, \ldots, x_n)(t) \leq \sum_{j=1}^{n} \psi_j(x_j(t))x_j(t) \]

\[ \times \left\{ \left( -(D_j - \varepsilon) + F_j \sum_{s=1}^{n} |w_{sj}(x_1(t), \ldots, x_n(t))| \psi_s(x_s(t) \pm 0) \psi_j(x_j) \right) \right\} \]

\[ + \rho \sum_{i=1}^{n} \sum_{s=1}^{n} F_i \frac{c_s}{d_i} |w_{si}^r(x_1(t), \ldots, x_n(t))| e^{\varepsilon t} \]

\[ = \sum_{j=1}^{n} \psi_j(x_j(t))x_j(t) \left\{ \left( -(D_j - \varepsilon) + F_j |w_{jj}(x_1(t), \ldots, x_n(t))| \right) \right\} \]

\[ + F_j \sum_{s=1}^{n} |w_{sj}(x_1(t), \ldots, x_n(t))| \frac{\psi_s(x_s(t) \pm 0)}{\psi_j(x_j)}. \]
\[ + \rho \sum_{i=1}^{n} \sum_{x=1}^{n} F_i \left| \frac{d_c}{d_{j}} w_{ji}^T (x_1(t), \ldots, x_n(t)) \right| \left| \frac{d_d}{d_{j}} w_{ji}^T (x_1(t), \ldots, x_n(t)) \right| e^{\varepsilon t} \]

\[ \leq \sum_{j=1}^{n} w_j (x_j(t)) x_j(t) \left\{ ((D_j - \varepsilon) - F_j \left| w_{jj} (x_1(t), \ldots, x_n(t)) \right| \right\} \]

\[ \times \left( -1 + F_j \sum_{x=1}^{n} \frac{|w_{xj} (x_1(t), \ldots, x_n(t))|}{(D_j - \varepsilon) - F_j \sum_{x=1}^{n} |w_{jj} (x_1(t), \ldots, x_n(t))|} \right) \]

\[ + \rho \sum_{i=1}^{n} \sum_{x=1}^{n} F_i \left| \frac{d_c}{d_{j}} w_{ji}^T (x_1(t), \ldots, x_n(t)) \right| \left| \frac{d_d}{d_{j}} w_{ji}^T (x_1(t), \ldots, x_n(t)) \right| \]

\[ \leq \sum_{j=1}^{n} w_j (x_j(t)) x_j(t) \left\{ ((D_j - \varepsilon) - F_j \left| w_{jj} (x_1(t), \ldots, x_n(t)) \right| \right\} \]

\[ \times \left( -1 + \frac{n - 1}{n} + \frac{1}{n} \right) = 0. \quad (49) \]

Similar to the approach used in proving the above theorem, we can easily obtain this result.

Under similar circumstances, if we let \( \varepsilon \to 0^+ \), then the global exponential stability of Theorem 3 and Corollary 3 is reduced to the global asymptotic stability. Consequently, we have the following result, which is a delay-independent criterion.

**Corollary 4.** Consider the functional differential equation (2) and assume there exist positive constants \( a_i, b_i, i = 1, 2, \ldots, n \), \( \rho > 1 \) and the conditions (A1), (A2), (A4), and \( 0 \leq \tau_j(t) \leq \bar{\tau} \) hold. If the following condition holds:

\[ \gamma' \equiv \max_{1 \leq j \leq n} \sup_{t \geq 0} \left\{ \frac{F_j}{D_j} \sum_{i=1}^{n} \left| w_{ij} (x_1(t), \ldots, x_n(t)) \right| \frac{c_i}{d_{ij}} \right\} \]

\[ + \rho \sum_{i=1}^{n} \sum_{x=1}^{n} F_i \left| \frac{d_c}{d_{i}} w_{ji}^T (x_1(t), \ldots, x_n(t)) \right| \left| \frac{d_d}{d_{i}} w_{ji}^T (x_1(t), \ldots, x_n(t)) \right| < 1, \quad (50) \]

where \( c_i, d_i \) and \( \psi_j(x_j) \) are similar to Eq. (43), then the trivial solution of system (2) is globally asymptotically stable.

**Corollary 5.** If the conditions of Corollary 4 is satisfied and there exist constants \( a > 0 \) and \( \rho > 1 \) such that conditions (46)–(48) under \( \varepsilon = 0 \) are satisfied, then the trivial solution of system (2) is globally asymptotically stable.

**Remark 4.** In [15], the author consider system (1) and the corresponding global asymptotic stability criterion is that there exists a positive diagonal matrix \( P \) such that

\[ PW_0 (x(t)) + W_0 (x(t))^T P + PW_1 (x(t)) \left[ PW_1 (x(t)) \right]^T + I \]
is negative definite. However, in practical applications, it is difficult to ensure that the
determination of the above matrix is negative definite for large order of matrix. However,
our results (see Theorems 1–3 as well as their corollaries) can be easily verified by simple
inequality computation.

4. Applications to neural networks with time-varying delays

Let

\[ d_i(t) = d_i x_i(t), \quad d_i > 0, \quad w_{ij}(x_1(t), \ldots, x_n(t)) = w_{ij} \text{(constant)}, \]
\[ w_{ij}^T(x_1(t), \ldots, x_n(t)) = w_{ij}^T \text{(constant)} \]

in Eq. (2), then the system is reduced to the following Hopfield-type neural networks with
time-varying delays:

\[
\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} w_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} w_{ij}^T f_j(x_j(t - \tau_{ij}(t))) + I_i,
\]

\( t > 0, \ i = 1, 2, \ldots, n, \) (51)

where \( x_i(t) \) corresponds to the membrane potential of the unit \( i \) at time \( t \); \( f_j(.) \) denotes
a measure of response or activation to its incoming potentials; \( w_{ij} \) and \( w_{ij}^T \) denote the
synaptic connection weights of unit \( j \) to unit \( i \); \( \tau_{ij}(t) \) corresponds to the transmission
delay along the axon of unit \( j \) to unit \( i \); the constant \( I_i \) corresponds to the external bias or
input from outside to unit \( i \); the coefficient \( d_i \) is the rate with which unit \( i \) self-regulates or
resets its potential when isolated from other units and inputs.

There exists an extensive literature on various aspects of systems in the form of (51) with
and without time delays. For more details of literature related to models of the form (51),
we refer to [4–28] and the references cited therein.

In this section, we assume that each activation function \( f_i(.) \), \( i = 1, 2, \ldots, n \), in (51)
satisfies the following conditions:

(H1) There exist constants \( F_j, 0 < F_j < +\infty, j = 1, 2, \ldots, n \), such that the incremental
ratio for \( f_j : \mathbb{R} \rightarrow \mathbb{R} \) satisfies

\[ 0 \leq \frac{f_j(x_j) - f_j(y_j)}{x_j - y_j} \leq F_j, \quad x_j \neq y_j. \] (52)

(H2) \[ |f_j(x)| \leq M_j, \quad x \in \mathbb{R}^n \quad \text{and} \quad M_j > 0, \quad j = 1, 2, \ldots, n. \] (53)

This type of activation functions is clearly more general than the usual sigmoid activa-
tion functions [6–28,30–32]. It has been used by Van den Driessche and Zou [8] in their
stability analysis of system (51) and sufficient conditions for the global attractivity of the
equilibrium of (51) are obtained.
An equilibrium of (51) is 
\[ x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \]
where \( T \) denotes the transpose of a matrix (in other words \( x^* \) is a column vector) and

\[ d_i x_i^* = \sum_{j=1}^{n} (w_{ij} + w_{ij}^r) f_j(x_j^*) + I_i, \quad i = 1, 2, \ldots, n. \]  
(54)

It is not difficult to prove that if conditions (H1) and (H2) are satisfied then there exists at least an equilibrium for system (51) and that any solution of system (51) is bounded in \([0, +\infty]\), see, for example, Lemma 1 and 2 of Refs. [17–19,24] for details. Now suppose that \( x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \) is an equilibrium of system (51). By using the transformation

\[ y_i(t) = x_i(t) - x_i^*, \quad i = 1, 2, \ldots, n, \]

we can transform system (51) to

\[ \dot{y}_i(t) = -d_i y_i(t) + \sum_{j=1}^{n} w_{ij} g_j(y_j(t)) + \sum_{j=1}^{n} w_{ij}^r g_j(y_j(t - \tau_{ij})), \quad i = 1, 2, \ldots, n, \]  
(55)

where \( g_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*), \quad j = 1, 2, \ldots, n. \) We know that the stability of system (51) around \( x^* \) corresponds to that of system (55) around the origin, so we only consider system (55).

In the following theorems, we will omit the proof of the existence and uniqueness of the equilibrium for system (51). Interested readers may refer to [24,25].

Similar to the approach adopted in Theorem 1, we have the following result for system (51).

**Theorem 4.** Consider system (51) and assume that conditions (H1), (H2), and (A3) are satisfied. If there exist positive constants \( \lambda_i (i = 1, 2, \ldots, n), r_1 \in [0, 1], r_2 \in [0, 1], \) and the following condition holds

\[ \alpha_{NN} \equiv \max_{1 \leq i \leq n} \sup_{t \geq 0} \left\{ \frac{1}{(d_i - \varepsilon) \lambda_i} \left[ \lambda_i \sum_{j=1}^{n} (F_j^{2r_1} |w_{ij}| + F_j^{2r_2} e^{2\varepsilon \tau} |w_{ij}^r|) + \sum_{j=1}^{n} \lambda_j \left( F_i^{2(1-r_1)} |w_{ji}| + F_i^{2(1-r_2)} |w_{ji}^r| |\psi_{ji}(t)| \right) \right] \right\} < 2, \]  
(56)

then system (51) is globally exponentially stable.

**Remark 5.** If we derive \( \tau_{ij}(t) = \tau_{ij}(\text{real constant numbers}) \) and \( \varepsilon \) is sufficiently small, then the sufficient condition for global exponential stability is equivalent to

\[ \max_{1 \leq i \leq n} \left\{ \frac{1}{d_i \lambda_i} \left[ \lambda_i \sum_{j=1}^{n} (F_j^{2r_1} |w_{ij}| + F_j^{2r_2} |w_{ij}^r|) + \sum_{j=1}^{n} \lambda_j \left( F_i^{2(1-r_1)} |w_{ji}| + F_i^{2(1-r_2)} |w_{ji}^r| \right) \right] \right\} < 2. \]  
(57)

Condition (57) is similar to the result of Theorem 3 in Ref. [19]. If we let \( \lambda_i = 1 \) \( (i = 1, 2, \ldots, n) \) in Eq. (57), then this condition becomes
\[
\max_{1 \leq i \leq n} \left\{ \frac{1}{d_i} \left[ \sum_{j=1}^{n} \left( F^2_{j1} |w_{ij}| + F^2_{j2} |w^T_{ij}| \right) \right]
\right. \\
+ \sum_{j=1}^{n} \left( F^2_{i1} |w_{ji}| + F^2_{i2} |w^T_{ji}| \right) \left\} < 2. \quad (58)
\]

Condition (58) is similar to the result of the Theorem in Ref. [17]. But condition (56) is less restricted than existing ones since we consider time-varying delays. Moreover, global exponential stability is ensured.

By means of the approach in Corollary 1, we immediately have

**Corollary 6.** System (51) satisfying (H1), (H2), and (A3) is globally asymptotically stable provided that there exist positive constants \(\lambda_i (i = 1, 2, \ldots, n)\), \(r_1 \in [0, 1]\), \(r_2 \in [0, 1]\), and \(\alpha'_{\text{NN}} \equiv \max_{1 \leq i \leq n} \sup_{t \geq 0} \left\{ \frac{1}{d_i \lambda_i} \left[ \sum_{j=1}^{n} \left( |w_{ij}| + |w^T_{ij}| \right) \right] + \sum_{j=1}^{n} \left( |w_{ji}| + |w^T_{ji}| \right) \left| \psi'_{ji}(t) \right| \left\} < 2. \quad (59)\]

**Remark 6.** If \(d_i = 1\) and \(f_i(x_i) = 0.5(|x_i + 1| - |x_i - 1|)\) in Eq. (51), then Eq. (51) corresponds to cellular neural networks with time-varying delays. Therefore, if \(r_1 = r_2 = 0.5\), condition (51) is reduced to

\[
\max_{1 \leq i \leq n} \sup_{t \geq 0} \left\{ \frac{1}{\lambda_i} \left[ \sum_{j=1}^{n} \left( |w_{ij}| + |w^T_{ij}| \right) \right] + \sum_{j=1}^{n} \left( |w_{ji}| + |w^T_{ji}| \right) \left| \psi'_{ji}(t) \right| \left\} < 2. \quad (60)\]

which is similar to the result of Ref. [27].

If \(d_i > 0\), \(\tau_{ij}\) are real constants, \(f_i(x_i) = 0.5(|x_i + 1| - |x_i - 1|)\) and \(r_1 = r_2 = 0.5\) in Eq. (51), then (59) becomes

\[
\max_{1 \leq i \leq n} \sup_{t \geq 0} \left\{ \frac{1}{d_i \lambda_i} \left[ \sum_{j=1}^{n} \left( |w_{ij}| + |w^T_{ij}| \right) \right] + \sum_{j=1}^{n} \left( |w_{ji}| + |w^T_{ji}| \right) \left| \psi'_{ji}(t) \right| \left\} < 2. \quad (61)\]

Condition (61) corresponds to the results of Ref. [16]. If \(w_{ij} = 0\), and \(\tau_{ij}\) are positive real constants, then this condition becomes

\[
\max_{1 \leq i \leq n} \sup_{t \geq 0} \left\{ \frac{1}{d_i} \left[ \sum_{j=1}^{n} \left( F_j |w^T_{ij}| + F_i |w^T_{ji}| \right) \right] \right\} < 2 \quad (62)
\]

when \(r_1 = r_2 = 0.5\) and \(\lambda_i = 1\) in (59). Condition (62) corresponds to the result of Theorem 2.2 in Ref. [5]. Hence, the global asymptotic stability is ensured.

Similar to the approach in Theorem 2, we can easily obtain the following result for system (51).
Theorem 5. Consider system (51) and assume that conditions (H1), (H2), and (A3) are satisfied. If there exist positive constants $\lambda_i$ $(i = 1, 2, \ldots, n)$, $\varepsilon$ and the following condition holds:

$$\beta_{NN} \equiv \max_{1 \leq i \leq n} \sup_{t \geq 0} \left\{ \frac{F_i}{\lambda_i(d_i - \varepsilon)} \sum_{j=1}^{n} \lambda_j \left( |w_{ji}| + e^{\varepsilon t} |w^r_{ji}(t)| \right) \right\} < 1$$

(63)

then system (51) is globally exponentially stable.

Remark 7. If $\tau_{ij}$ are positive real constants, then the sufficient condition for global exponential stability (63) becomes

$$\max_{1 \leq i \leq n} \left\{ \frac{F_i}{\lambda_i(d_i - \varepsilon)} \sum_{j=1}^{n} \lambda_j \left( |w_{ji}| + e^{\tau_{ij} t} |w^r_{ji}(t)| \right) \right\} < 1.$$  

(64)

This condition is similar to the result of Theorem 4 in Ref. [19]. In [10], a special case of a network model with the intraneural signal transmission delays ignored was studied. It is a special case of model (51) when $w_{ij} = 0$ for any $i \neq j$, $w^r_{ii} = 0$ for any $i$, and $d_i = F_i = 1$.

A number of sufficient conditions ensuring global exponential stability were established by the author as (in our notations)

$$w_{ii} < 0, \quad i = 1, 2, \ldots, n,$$

and

$$\max_{1 \leq i \leq n} \left\{ w_{ii} + n \sum_{j \neq i} \lambda_j |w^r_{ji}| \right\} < 1.$$  

(65)

It is obvious that our condition is less restrictive than those of [10].

By means of the approach in Corollary 2, we can easily get

Corollary 7. Consider system (51) and assume that conditions (H1), (H2), and (A3) are satisfied. If there exist positive constants $\lambda_i$ $(i = 1, 2, \ldots, n)$, and the following condition holds:

$$\beta'_{NN} \equiv \max_{1 \leq i \leq n} \sup_{t \geq 0} \left\{ \frac{F_i}{\lambda_id_i} \sum_{j=1}^{n} \lambda_j \left( |w_{ji}| + |w^r_{ji}| \right) \right\} < 1$$

(66)

then system (51) is globally asymptotically stable.

Remark 8. If $\tau_{ij}$ are positive real constants, then the sufficient condition for global asymptotic stability (66) becomes

$$\max_{1 \leq i \leq n} \left\{ \frac{F_i}{\lambda_id_i} \sum_{j=1}^{n} \lambda_j \left( |w_{ji}| + |w^r_{ji}| \right) \right\} < 1.$$  

(67)

Condition (67) corresponds to the result of [17]. If $w_{ij} = 0$ and $\lambda_i = 1$ in (67), then that equation becomes

$$\max_{1 \leq i \leq n} \left\{ \frac{F_i}{d_i} \sum_{j=1}^{n} (|w^r_{ji}|) \right\} < 1.$$  

(68)
In fact, (67) is the result of Gopalsamy and He [6]. Hence, our results are less conservative than the previously known results.

Similar to the approach in Theorem 3, we immediately have

**Theorem 6.** Assume that there exist positive constants \(a_i, b_i\), \(i = 1, 2, \ldots, n\), \(\rho > 1\) and conditions (H1), (H2), and \(0 \leq \tau_{ij}(t) \leq \bar{\tau}\) hold. If the following condition holds:

\[
\gamma_{NN} \equiv \max_{1 \leq i \leq n} \sup_{t \geq 0} \left\{ \frac{1}{(d_j - \varepsilon)} \left( F_j \sum_{i=1}^{n} w_{ij} \frac{c_i}{d_j} + \rho \sum_{i=1}^{n} \sum_{s=1}^{n} F_i \left| \frac{c_s}{d_i} w_{s\tau} \right| \right) \right\} < 1, \tag{69}
\]

where \(c_i, d_i, \) and \(\varphi_i(x_i)\) are similar to Eq. (35), then system (51) is globally exponentially stable.

Let \(a_s = a^s, b_s = b^s\) (where \(a^s\) is the \(s\)th power of \(a\)) in Theorem 6, we have the following easily verifiable result.

**Corollary 8.** If the conditions of Theorem 6 are satisfied and there exist constants \(a > 0\) and \(\rho > 1\) such that

(i) \(d_j - \varepsilon - F_j |w_{jj}| > 0, \quad j = 1, 2, \ldots, n,\) \hspace{2cm} (70)

(ii) \(\frac{F_j |w_{sj}|}{d_j - \varepsilon - F_j |w_{jj}|} \leq \frac{a^{j-s}}{n}, \quad s \neq j, \quad j = 1, 2, \ldots, n,\) \hspace{2cm} (71)

(iii) \(\frac{d_j - \varepsilon - F_j |w_{jj}|}{n \sum_{i=1}^{n} \sum_{s=1}^{n} |F_i a^{s-i} | |w_{si}|} \geq \rho > 1, \quad j = 1, 2, \ldots, n,\) \hspace{2cm} (72)

then system (51) is globally exponentially stable.

By means of the approach in Corollary 4, we can easily obtain

**Corollary 9.** Assume that there exist positive constants \(a_i, b_i\), \(i = 1, 2, \ldots, n\), \(\rho > 1\) and conditions (H1), (H2), and \(0 \leq \tau_{ij}(t) \leq \bar{\tau}\) hold. If the following condition holds:

\[
\gamma'_{NN} \equiv \max_{1 \leq i \leq n} \sup_{t \geq 0} \left\{ \frac{1}{d_j} \left( F_j \sum_{i=1}^{n} w_{ij} \frac{c_i}{d_j} + \rho \sum_{i=1}^{n} \sum_{s=1}^{n} F_i \left| \frac{c_s}{d_i} w_{s\tau} \right| \right) \right\} < 1, \tag{73}
\]

where \(c_i, d_i, \) and \(\varphi_i(x_i)\) are similar to Eq. (35), then system (51) is globally asymptotically stable.

**Corollary 10.** If the conditions of Corollary 9 are satisfied and there exist constants \(a > 0\) and \(\rho > 1\) such that conditions (70)–(72) under \(\varepsilon = 0\) are satisfied, then system (51) is globally asymptotically stable.

**Remark 9.** If \(d_i = 1\) and \(f_i(x) = 0.5(|x + 1| - |x - 1|)\), then the results of Corollaries 9 and 10 are similar to the result of [27].
**Remark 10.** Joy [14] considered the following neural network with a single time-varying delay $r(t) > 0$:

$$
\dot{x}(t) = -Dx(t) + T_0 g(x(t)) + T_1 g(x(t - r(t))) + I
$$

(74)

and assume that $r$ is differentiable, nonnegative and bounded: $0 \leq r(t) \leq \tau$, and $r'(t) \leq \rho < 1$. If there exists a positive diagonal matrix $P$ such that

$$
\beta \left( PT_0 + T_0^T P \right) + \beta^2 \sigma^{-1} \left[ PT_1 \right]^T PT_1 - 2 \beta D \Gamma^{-1} + I,
$$

(75)

is negative definite, where $\sigma = 1 - \rho$, then (74) is absolutely stable. As pointed out in Remark 4, it is difficult to determine the negative definiteness of (75) if the above matrix has a sufficiently large order. In Theorem 6, we only need the bounded time-varying delay. Therefore, this result is less conservative than the results in [14]. Moreover, the results of Theorems 4–6 as well as their corollaries can be easily verified by simple algebraic inequality computation.

**5. Conclusions**

It is well known that delays often appear in artificial neural networks, though to identify them is not easy. Moreover, apparently constant delay is only an ideal and simplified case. In most situations, delays are time-varying.

In this paper, by employing several Lyapunov functionals, we have obtained a number of sets of easily verifiable delay-dependent sufficient conditions for the global exponential stability of retarded functional differential equations with multiple time-varying delays, which generalize both the Hopfield neural network model as well as hybrid network models of the cellular neural network type. The results obtained generalize those by Joy [14, 15] and extend the previously known results for the hybrid neural networks. Without considering the symmetry of the connection weights, the differentiability and the monotonicity of the activation functions, we are provided with wider options in designing the circuitry of the hybrid neural networks for certain computational tasks. The results on global exponential stability provide us with relevant estimates on how fast such networks can perform during real-time computations. As a by-product, some delay-independent sufficient conditions for global asymptotic stability are also obtained. For the delayed Hopfield-type networks, some of our results are improvements on previous works reported by other researchers.

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