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Subdifferentials of Nonconvex Vector-Valued Functions

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The formula of Clarke's subdifferential for the sum of two real-valued locally Lipschitz functions has been extended by Rockafellar to the case where one of the two functions is directionally lipschitzian. We introduce in this paper a notion of cone related to the epigraph of a function and with the aid of this cone we extend the directionally lipschitzian behaviour to vector-valued functions and we study the sum of two vector-valued functions.

INTRODUCTION

In the last few years, the study of optimization problems with constraints defined by vector-valued functions has led many authors to introduce a notion of subdifferential or generalized differential for nondifferentiable vector-valued functions. The first ones to have considered such functions seem to be Raffin [18] and Valadier [31] who have extended the definition of convex subdifferential of Moreau and Rockafellar to convex vector-valued functions. This theory of subdifferentiation of convex vector-valued functions has been developed by Ioffe and Levin, Michel, Zowe, Kutateladze, Rubinov and others.

In [15] Penot has introduced the notion of upper and lower directional derivatives for nonconvex functions taking values in a Daniell topological vector space and with the aid of assumptions of infinitesimal convexity he has established a subdifferential calculus for tangentially convex vector-valued functions. For functions from $\mathbb{R}^n$ into $\mathbb{R}^m$ many authors have defined a notion of "generalized derivative" by using the famous theorem of Rachemacher for Lipschitz mappings (Clarke [5], Pourciau [17]) or by looking for a description of properties which secure the extension of implicit function theorems (Halkin [6], Warga [32]).

As there is no almost everywhere differentiability for a Lipschitz mapping defined on a nonseparable normed vector space E, we have introduced in

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[27, 28] the notion of strictly compactly lipschitzian vector-valued functions, a notion which generalizes the one of strictly differentiable functions of Bourbaki [1] and Leach [12] and which coincides in finite dimension with the one of Lipschitz functions (see [29] for the case where E is a separable Banach space). In order to study Pareto optimization problems defined by strictly compactly lipschitzian functions taking values in ordered topological vector spaces we have constructed in [27] a theory of subdifferentiation for such functions, a theory which generalizes the one introduced by Clarke [3, 4] for Lipschitz real-valued functions. Optimization problems defined by Lipschitz vector-valued functions have been also considered by Kusraev [10].

After recent works, for instance, those of Clarke [4], Hiriart–Urruty [7], Pourciau [17], and Thibault [26], which have shown that the generalized gradient of Clarke has many important applications, and after the two very interesting papers [21, 22] of Rockafellar which extends the known results for real-valued Lipschitz functions to real-valued non-Lipschitz functions, it is natural following the study of strictly compactly lipschitzian vector-valued functions to try to extend the results of Rockafellar to functions taking values in $F^{\bullet} = F \cup \{+\infty\}$, where F is an order complete topological vector space and $+\infty$ is a supremum adjoined to F.

We begin by recalling in Section 1 the notion of Clarke tangent cone which allows us to define in Section 2 the generalized directional derivative and the subdifferential of a function taking values in $F^{\bullet}$. We also prove that the subdifferential of a convex or strictly lipschitzian vector-valued function coincides with its subdifferential in the sense of convex or lipschitzian analysis.

If $F = \mathbb{R}$ (or more generally if $\text{int}(F_+) \neq \varnothing$) the notion of directionally lipschitzian functions introduced by Rockafellar can be interpreted geometrically (see [20, 21]) with the help of what he calls the hypertangent cone. In order to deal with spaces F for which the interior of the positive cone F_+ may be empty, we define in Section 3 for a mapping $f: E \to F^{\bullet}$ and a point $\bar{x} \in E$ with $f(\bar{x}) \neq +\infty$ a cone $Q(f; \bar{x})$ which is directly connected with the epigraph of f, a cone wich has the remarkable property to be convex whenever f is lower semi-continuous at $\bar{x}$ or whenever $\text{int}(F_+) \neq \varnothing$. Making use of this cone we introduce a tangential condition (T_2) which generalizes the notion of directionally lipschitzian behaviour of Rockafellar. This allows us to study the subdifferential of the sum of two vector-valued functions in Section 3 and the subdifferential of the composition of a function with a strictly differentiable mapping in Section 4.

1. TANGENT CONES

All topological vector spaces that we shall consider in this paper will be assumed to be Hausdorff.

Let E be a topological vector space, M a nonempty subset of E and $\bar{x}$ a point belonging to the closure of M.

The notion of tangent cone introduced by Clarke for normed vector spaces can be defined for topological vector spaces in terms of neighbourhoods (see [21]) or in terms of nets (see [28]). Here we shall adopt as definition the formulation given in [21].

1.1. DEFINITION (Rockafellar [21]). We shall call *tangent cone* to M at the point $\bar{x} \in \operatorname{cl} M$ and we shall denote by $T(M; \bar{x})$ the set of all points $v \in E$ such that for every neighbourhood V of v in E there exist a neighbourhood X of $\bar{x}$ in E and a real number $\varepsilon > 0$ such that

$$(x + tV) \cap M \neq \varnothing$$

for all $x \in X \cap M$ and $t \in]0, \varepsilon[$.

$T(M; \bar{x})$ is a closed convex cone (see [21] or [28]).

This tangent cone has also a characterization in terms of nets.

1.2. PROPOSITION (Thibault [28]). *The following two assertions are equivalent*:

(i) $v \in T(M; \bar{x})$;

(ii) *for every net* $(x_j)_{j \in J}$ *in* M *converging to* $\bar{x}$ *and every net* $(t_j)_{j \in J}$ *of positive real numbers converging to zero, there exist two subnets* $(x_{\alpha(i)})_{i \in I}$ *and* $(t_{\alpha(i)})_{i \in I}$ *and a net* $(v_i)_{i \in I}$ *in* E *converging to* v *such that*

$$x_{\alpha(i)} + t_{\alpha(i)} v_i \in M$$

for each $i \in I$.

Remarks. (1) If E is normed, we know (see Hiriart-Urruty [7]) that $v \in T(M; \bar{x})$ if and only if for every sequence $(x_n)_{n \in \mathbb{N}}$ in M converging to $\bar{x}$ and every sequence $(t_n)_{n \in \mathbb{N}}$ in $]0, +\infty[$ converging to zero there exist a sequence $(v_n)_{n \in \mathbb{N}}$ in E converging to v such that

$$x_n + t_n v_n \in M \qquad \text{for each } n \in \mathbb{N}.$$

(2) If M is convex, then (see [21]) the cone $T(M; \bar{x})$ is the closure of the set of all elements $v \in E$ for which there exist real numbers $\varepsilon_v > 0$ verifying $\bar{x} +]0, \varepsilon_v[v \subset M$.

2. Subdifferential

Let F be an ordered topological vector space, i.e., F is an ordered vector space and its positive cone $F_+ = \{x \in F \mid x \geqslant 0\}$ is *closed* in F. We shall assume that F is *an order complete vector lattice*, i.e., $\sup(x, y)$ exists for all $x, y \in F$ and every upper bounded nonempty subset of F has a supremum, and that the topology of F is *normal*, i.e., there exists a neighbourhood basis $\{W\}_W$ of the origin in F such that

$$W = (W + F_+) \cap (W - F_+).$$

By $F^\cdot = F \cup \{+\infty\}$ we shall mean the order space F with the adjunction of a supremum $+\infty$ and by $\bar{F} = F \cup \{-\infty, +\infty\}$ the space F with the adjunction of an infimum $-\infty$ and a supremum $+\infty$. Addition and multiplication by real numbers are extended in standard way to $F^\cdot$ and $\bar{F}$ with $(+\infty) + (-\infty) = +\infty$.

Let f be a mapping from E into $F^\cdot$ and let $\bar{x}$ be a point in E with $f(\bar{x}) \in F$.

We shall denote by epi f the epigraph of f,

$$\text{epi } f = \{(x, y) \in E \times F \mid y \geqslant f(x)\},$$

and by $L(E, F)$ the space of all continuous linear mappings from E into F.

In the sequel we shall put (for $f(\bar{x}) \in F$)

$$T(f; \bar{x}) = T(\text{epi } f; (\bar{x}; f(\bar{x}))).$$

2.1. Definitions. We shall call *directional subderivative* of f at $\bar{x}$ and we shall denote, as in [21], by $f^\uparrow(\bar{x}; \cdot)$ the mapping from E into $\bar{F}$ defined by

$$f^\uparrow(\bar{x}; v) = \inf\{l \in F \mid (v, l) \in T(f; \bar{x})\}$$

with the convention $\inf \varnothing = +\infty$.

So we can define the subdifferential $\partial f(\bar{x})$ of f at $\bar{x}$ by

$$\partial f(\bar{x}) = \{T \in L(E, F) \mid T(v) \leqslant f^\uparrow(\bar{x}; v), \forall v \in E\}.$$

In the sequel we shall be led to consider for mappings f the following *tangential condition* which excludes pathological situations.

$$f^\uparrow(\bar{x}; \cdot) \text{ takes values in } F^\cdot. \tag{T_1}$$

For a mapping f verifying condition (T_1) at $\bar{x}$ we shall put

$$\text{dom } f^\uparrow(\bar{x}; \cdot) = \{v \in E \mid f^\uparrow(\bar{x}; v) \in F\}.$$

Remark. If f is a mapping from E into $\bar{F}$ with $f(\bar{x}) \in F$, one easily sees that either $f^{\uparrow}(\bar{x}; 0) = -\infty$ or $f^{\uparrow}(\bar{x}; 0) = 0$, since $(0, 0) \in T(f; \bar{x})$. So if f verifies condition (T_1) at $\bar{x}$, one has $f^{\uparrow}(\bar{x}; 0) = 0$.

As a direct consequence of the convexity of the tangent cone and of the definition of the directional subderivative we have the following proposition.

2.2. PROPOSITION. *If f is a mapping from E into $F^{\cdot}$ and if $\bar{x}$ is a point in E with $f(\bar{x}) \in F$, the directional subderivative $f^{\uparrow}(\bar{x}; \cdot)$ is a sublinear mapping from E into $\bar{F}$, that is,*

$$f^{\uparrow}(\bar{x}; v_1 + v_2) \leqslant f^{\uparrow}(\bar{x}; v_1) + f^{\uparrow}(\bar{x}; v_2),$$
$$f^{\uparrow}(\bar{x}; \lambda v_1) = \lambda f^{\uparrow}(\bar{x}; v_1)$$

for all positive real numbers λ and all $v_1, v_2 \in E$, with the convention $(+\infty) + (-\infty) = +\infty$.

Now let us recall (Thibault [27, 28]) the following definition.

2.3. DEFINITION. A mapping f from E into F is said to be *strictly compactly lipschitzian* at a point $\bar{x} \in E$ if there exist a mapping K from E into the set Comp(F) of nonempty compact subsets of F, a mapping r of $]0, 1] \times E \times E$ into F and neighbourhoods X of $\bar{x}$ and V of zero in E verifying

(a) $\lim_{t \downarrow 0, x \to \bar{x}} r(t, x; v) = 0$ for each $v \in E$ and $\lim_{t \downarrow 0, x \to \bar{x}, v \to 0} r(t, x; v) = 0$;

(b) for all $x \in X$, $v \in V$ and $t \in]0, 1]$

$$t^{-1}[f(x + tv) - f(x)] \in K(v) + r(t, x; v);$$

(c) $K(0) = \{0\}$ and the set-valued mapping K is upper semi-continuous at the origin (that is, for every neighbourhood W of $K(0)$ in F there is a neighbourhood U of zero in E verifying $K(v) \subset W$ for every $v \in U$).

We recall (see Thibault [28]) that if f is strictly compactly lipschitzian at $\bar{x}$, then

$$\lim_{\substack{t \downarrow 0 \\ x \to \bar{x} \\ w \to v}} t^{-1}[f(x + tw) - f(x + tv)] = 0$$

for every $v \in E$, that for all nets $(x_j)_{j \in J}$ converging to $\bar{x}$ in E, $(t_j)_{j \in J}$ of positive real numbers converging to zero, there exists a subnet

$$(t_{\alpha(i)}^{-1}[f(x_{\alpha(i)} + t_{\alpha(i)} v) - f(x_{\alpha(i)})])_{i \in I}$$

which converges and that f is continuous at $\bar{x}$.

2.4. DEFINITIONS (see [27]). If f is a mapping which is strictly compactly lipschitzian at $\bar{x}$, we shall put for every $v \in E$

$$D_f(\bar{x}; v) = \bigcap_{\substack{\varepsilon > 0 \\ W \in \mathfrak{N}(\bar{x})}} \operatorname{cl}(q_f(]0, \varepsilon[\times W; v)),$$

where $q_f(t, x; v) = t^{-1}[f(x + tv) - f(x)]$, $\mathfrak{N}(\bar{x})$ is the neighbourhood basis of $\bar{x}$ in E, ε is a positive real number and cl denotes the closure in F. We also have

$$D_f(\bar{x}; v) = \{\lim_{j \in J} q_f(t_j, x_j; v) \mid t_j > 0, \lim_{j \in J} t_j = 0 \text{ and } \lim_{j \in J} x_j = \bar{x}\}.$$

This allows us to define the generalized directional derivative $f^\circ(\bar{x}; \cdot)$ by

$$f^\circ(\bar{x}; v) = \sup D_f(\bar{x}; v)$$

for every $v \in E$ and the lipschitzian subdifferential $\partial_l f(\bar{x})$ of f at $\bar{x}$ by

$$\partial_l f(\bar{x}) = \{T \in L(E, F) \mid T(v) \leqslant f^\circ(\bar{x}; v), \forall v \in E\}.$$

The following proposition shows that for strictly compactly lipschitzian mappings $f^\uparrow(\bar{x}; \cdot) = f^\circ(\bar{x}; \cdot)$ and hence generalizes Proposition 2.8 of Thibault [28].

2.5. PROPOSITION. *Let f be a mapping which is strictly compactly lipschitzian at $\bar{x}$. We have*

$$f^\uparrow(\bar{x}; v) = f^\circ(\bar{x}; v)$$

for every $v \in E$, and

$$\partial f(\bar{x}) = \partial_l f(\bar{x}).$$

Proof. Let $\bar{v}$ be a point in E. Let us begin by showing that $f^\circ(\bar{x}; \bar{v}) \leqslant f^\uparrow(\bar{x}; \bar{v})$. We may assume that $f^\uparrow(\bar{x}; \bar{v}) \neq +\infty$. Let $\bar{l}$ be any point in F such that $(\bar{v}, \bar{l}) \in T(f; \bar{x})$. Let us consider a net $(t_j, x_j)_{j \in J}$ in $]0, +\infty[\times E$ converging to $(0, \bar{x})$ and such that

$$\lim_{j \in J} t_j^{-1}[f(x_j + t_j \bar{v}) - f(x_j)]$$

exists. Then as the net $(x_j, f(x_j))_{j \in J}$ converges to $(\bar{x}, f(\bar{x}))$ there exist two subnets

$$(x_{\alpha(i)}, f(x_{\alpha(i)}))_{i \in I} \quad \text{and} \quad (t_{\alpha(i)})_{i \in I}$$

and a net $(v_i, l_i)_{i \in I}$ converging to $(\bar{v}, \bar{l})$ such that

$$(x_{\alpha(i)}, f(x_{\alpha(i)})) + t_{\alpha(i)}(v_i, l_i) \in \operatorname{epi} f$$

for every $i \in I$. Therefore we have

$$t_{\alpha(i)}^{-1}[f(x_{\alpha(i)} + t_{\alpha(i)} v_i) - f(x_{\alpha(i)})] \leqslant l_i$$

and hence (see the remark following Definition 2.3)

$$\begin{aligned}&\lim_{j \in J} t_j^{-1}[f(x_j + t_j \bar{v}) - f(x_j)]\\&\quad = \lim_{i \in I} t_{\alpha(i)}^{-1}[f(x_{\alpha(i)} + t_{\alpha(i)} v_i) - f(x_{\alpha(i)})] \leqslant \bar{l}.\end{aligned}$$

So we have $f^\circ(\bar{x}; \bar{v}) \leqslant f^\uparrow(\bar{x}; \bar{v})$.

To show the reverse inequality we may assume that $f^\circ(\bar{x}; \bar{v}) \in F$. Therefore it suffices to show that

$$(\bar{v}, f^\circ(\bar{x}; \bar{v})) \in T(f; \bar{x}).$$

Let $(x_j, y_j)_{j \in J}$ be a net in epi f converging to $(\bar{x}, f(\bar{x}))$ and $(t_j)_{j \in J}$ be a net in $]0, +\infty[$ converging to zero. Since f is strictly compactly lipschitzian at $\bar{x}$, there exists a subnet

$$(t_{\alpha(i)}^{-1}[f(x_{\alpha(i)} + t_{\alpha(i)} \bar{v}) - f(x_{\alpha(i)})])_{i \in I}$$

which converges. So there exist a point $k \in F$ with $k \leqslant f^\circ(\bar{x}; \bar{v})$ and a net $(r_i)_{i \in I}$ in F converging to zero in F such that

$$t_{\alpha(i)}^{-1}[f(x_{\alpha(i)} + t_{\alpha(i)} \bar{v}) - f(x_{\alpha(i)})] = k + r_i$$

for each $i \in I$ and hence

$$\begin{aligned}f(x_{\alpha(i)} + t_{\alpha(i)} \bar{v}) &= f(x_{\alpha(i)}) + t_{\alpha(i)}(k + r_i)\\&\leqslant y_{\alpha(i)} + t_{\alpha(i)}(f^\circ(\bar{x}; \bar{v}) + r_i).\end{aligned}$$

Therefore, there exists $(\bar{v}, f^\circ(\bar{x}; \bar{v}) + r_i)_{i \in I}$ converging to $(\bar{v}, f^\circ(\bar{x}; \bar{v}))$ and verifying

$$(x_{\alpha(i)}, y_{\alpha(i)}) + t_{\alpha(i)}(\bar{v}, f^\circ(\bar{x}; \bar{v}) + r_i) \in \operatorname{epi} f$$

and hence $(\bar{v}, f^\circ(\bar{x}; \bar{v})) \in T(f; \bar{x})$ and the proof of the proposition is finished. ∎

Remark. If E is normed and if f is strictly differentiable at $\bar{x}$ in the sense of Bourbaki [1] and Leach [12], that is, if there exists a continuous linear mapping $\nabla f(\bar{x})$ from E into F such that

$$f(x) - f(y) = \nabla f(\bar{x})(x - y) + \|x - y\| \, \varepsilon(x, y)$$

with $\lim_{x \to \bar{x}, y \to \bar{x}} \varepsilon(x, y) = 0$, it is not difficult to verify that

$$f^\circ(\bar{x}; \cdot) = \nabla f(\bar{x}) = f^\uparrow(\bar{x}; \cdot)$$

and hence that

$$\partial_l f(\bar{x}) = \nabla f(\bar{x}) = \partial f(\bar{x}). \quad \blacksquare$$

If we consider Hadamard differentiability instead of strict differentiability, then, as in Thibault [26], we have the following result.

2.6. Proposition. *Let us assume that E and F are normed vector spaces. If f is a mapping from E into $F^\bullet$ taking values in F on a neighborhood of $\bar{x}$ in E and if f is Hadamard differentiable at $\bar{x}$, that is, there exists a continuous linear mapping $\nabla f(\bar{x})$ from E into F such that for each compact subset K of E the relation*

$$f(\bar{x} + tv) - f(\bar{x}) = t \, \nabla f(\bar{x})v + t\varepsilon(t, v)$$

holds for t sufficiently small in $\mathbb{R}$, $v \in K$ with $\lim_{t \to 0} \varepsilon(t, v) = 0$ uniformly with respect to $v \in K$, then we have

$$\nabla f(\bar{x})(v) \leqslant f^\uparrow(\bar{x}; v)$$

for all $v \in E$ and

$$\nabla f(\bar{x}) \in \partial f(\bar{x}).$$

Proof. Consider $(v, l) \in T(f; \bar{x})$ and a sequence $(t_n)_{n \in \mathbb{N}}$ of positive real numbers converging to zero. According to the remark which follows Proposition 1.2 there exists a sequence $(v_n, l_n)_{n \in \mathbb{N}}$ converging to (v, l) such that

$$(\bar{x}, f(\bar{x})) + t_n(v_n, l_n) \in \operatorname{epi} f$$

or

$$f(\bar{x} + t_n v_n) - f(\bar{x}) \leqslant t_n l_n.$$

Let K be the compact subset $K = \{v\} \cup \{v_n \mid n \in \mathbb{N}\}$. We have

$$\nabla f(\bar{x})(v_n) + \varepsilon(t_n, v_n) \leqslant l_n$$

and hence

$$\nabla f(\bar{x})(v) \leqslant l,$$

for $\lim_{n\to\infty} \varepsilon(t_n, v_n) = 0$ since $\lim_{t\to 0} \varepsilon(t, w) = 0$ uniformly with respect to $w \in K$. Therefore it follows that

$$\nabla f(\bar{x})(v) \leqslant f^{\uparrow}(\bar{x}; v)$$

for each $v \in E$, and the proof is finished. ∎

Remark. If, as Rockafellar has made for read-valued functions in [21], we suppose that f is strictly Hadamard differentiable at $\bar{x}$, that is, for every compact subset K of E the relation

$$f(x + tv) - f(x) = t\, \nabla f(\bar{x})(v) + t\varepsilon(t, x; v)$$

holds for x in a neighbourhood of $\bar{x}$, t sufficiently small, with

$$\lim_{t\to 0, x\to\bar{x}} \varepsilon(t, x; v) = 0$$

uniformly with respect to $v \in K$, then it is not difficult to verify that

$$(v, \nabla f(\bar{x})(v)) \in T(f; \bar{x})$$

for every $v \in E$. Therefore, in this case one has

$$f^{\uparrow}(\bar{x}; v) = \nabla f(\bar{x})(v) \qquad \text{and} \qquad \partial f(\bar{x}) = \nabla f(\bar{x}). \quad ∎$$

Before closing this section, let us consider the case of a convex mapping from E into $F^{\cdot}$, with $f(\bar{x}) \in F$.

For such a mapping, one defines (see Valadier [31]) the *directional derivative* $f'(\bar{x}; \cdot)$ by

$$f'(\bar{x}; v) = \inf_{t>0} t^{-1}[f(\bar{x} + tv) - f(\bar{x})]$$

and the subdifferential in the sense of convex analysis of f at $\bar{x}$ by

$$\partial_c f(\bar{x}) = \{T \in L(E, F) \mid T(v) \leqslant f'(\bar{x}; v), \forall v \in E\}.$$

We are going to study relationship between directional derivative and directional subderivative for such a mapping f and to show that the subdifferential of f at $\bar{x}$ coincides with its convex subdifferential.

2.7. PROPOSITION. *Let f be a convex mapping from E into $F^{\cdot}$ with $f(\bar{x}) \in F$. Then*

$$\sup\{Av \mid A \in \partial_c f(\bar{x})\} \leqslant f^{\uparrow}(\bar{x}; v) \leqslant f'(\bar{x}; v)$$

for all $v \in E$, with the convention $\sup \varnothing = -\infty$, *and*

$$\partial f(\bar{x}) = \partial_c f(\bar{x}).$$

Proof. Since epi f is a convex set, according to Remark 2 which follows proposition 1.2, $T(f; \bar{x})$ is the closure in $E \times F$ of the set of all elements (v, l) for which there exists $\varepsilon > 0$ with

$$(\bar{x}, f(\bar{x})) + \,]0, \varepsilon[\,(v, l) \subset \operatorname{epi} f.$$

Thus consider any point (v, l) of this set. There exists a positive real number ε such that for every $t \in \,]0, \varepsilon[$

$$t^{-1}[f(\bar{x} + tv) - f(\bar{x})] \leqslant l$$

and hence $f'(\bar{x}; v) \leqslant l$. Therefore, if $(\bar{v}, \bar{l}) \in T(f; \bar{x})$, there exists a net $(v_j, l_j)_{j \in J}$ in the above set such that $(\bar{v}, \bar{l}) = \lim_{j \in J} (v_j, l_j)$ and hence for every $A \in \partial_c f(\bar{x})$ we have

$$A(\bar{v}) = \lim_{j \in J} A(v_j) \leqslant \lim_{j \in J} l_j = \bar{l}$$

and the first inequality of the proposition is proved. Now to show that $f^{\uparrow}(\bar{x}; v) \leqslant f'(\bar{x}; v)$, we may assume that there exist a real number $\beta > 0$ such that $f(\bar{x} + \beta\bar{v}) \in F$ and hence that $f(\bar{x} + t\bar{v}) \in F$ for all $t \in \,]0, \beta]$. Therefore, for every $\alpha \in \,]0, \beta]$ and every $t \in \,]0, \alpha]$ we have

$$f(\bar{x} + t\bar{v}) \leqslant \alpha^{-1} t f(\bar{x} + \alpha\bar{v}) + (1 - \alpha^{-1} t)\, f(\bar{x}),$$

that is,

$$(\bar{x}, f(\bar{x})) + \,]0, \alpha[\,(\bar{v}, \alpha^{-1}[f(\bar{x} + \alpha\bar{v}) - f(\bar{x})]) \subset \operatorname{epi} f.$$

So we derive that

$$(\bar{v}, \alpha^{-1}[f(\bar{x} + \alpha\bar{v}) - f(\bar{x})]) \in T(f; \bar{x})$$

for every $\alpha \in \,]0, \beta]$ and hence that

$$f^{\uparrow}(\bar{x}; v) \leqslant f'(\bar{x}; v),$$

and the proof of the proposition is finished, for the equality concerning the subdifferentials is a direct consequence of the two preceding inequalities. ∎

Remarks. (1) If $\operatorname{int}(F_+) \neq \varnothing$, then according to the results of Penot and Thera [25], the two inequalities of the proposition shows that $f^{\uparrow}(\bar{x};\cdot)$ lies between $f'(\bar{x};\cdot)$ and its lower semi-continuous hull.

(2) If $f'(\bar{x};\cdot)$ is continuous, then $f^{\uparrow}(\bar{x};\cdot)=f'(\bar{x};\cdot)$, since in this case one has

$$f'(\bar{x};v)=\sup\{Av \mid A\in\partial_c f(\bar{x})\}$$

for all $v\in E$.

3. Finite Sum

In order to study the sum of two mappings, we shall introduce the following tangential condition (T_2).

Let f be a mapping from E into $F^{\cdot}$ and let $\bar{x}$ be a point in E with $f(\bar{x})\in F$. We define the cone $Q(f;\bar{x})$ of $E\times F$ (the term of cone is justified by the first part of the proof of Proposition 3.2) to be the set of all $(\bar{v},\bar{l})$ in $E\times F$ such that for each neighbourhood L of $\bar{l}$ in F, there exists a neighbourhood X of $\bar{x}$ in E, a neighbourhood Y of $f(\bar{x})$ in F, a real number $\varepsilon>0$ and a neighbourhood V of $\bar{v}$ in E such that

$$[(x,y)+t(\{v\}\times L)]\cap \operatorname{epi} f\neq\varnothing$$

for all $(x,y)\in(X\times Y)\cap\operatorname{epi} f$, $t\in\,]0,\varepsilon[$ and $v\in V$.

If $f(\bar{x})\in F$, we shall say that f verifies the tangential condition (T_2) at $\bar{x}$ if

$$T(f;\bar{x})=\operatorname{cl}_{E\times F}(Q(f;\bar{x})). \qquad (T_2)$$

Remarks. (1) The following inclusion is always true: $Q(f;\bar{x})\subset T(f;\bar{x})$.

(2) If $I(f;\bar{x})$ denotes the *interior pseudo-tangent cone* (see Thibault [28] and Rockafellar [21] where the terminology "hypertangent" is used), that is, the set of all $(v,l)\in E\times F$ for which there exist a neighbourhood Ω of $(\bar{x},f(\bar{x}))$, a real number $\varepsilon>0$ and a neighbourhood U of (v,l) such that

$$\Omega\cap\operatorname{epi} f+\,]0,\varepsilon[\,U\subset\operatorname{epi} f,$$

then the following relation holds

$$I(f;\bar{x})\subset Q(f;\bar{x})\subset T(f;\bar{x}).$$

Therefore, if $F=\mathbb{R}$ and if f is directionally lipschitzian at $\bar{x}$ in the sense of Rockafellar [21] ($T(f;\bar{x})=\operatorname{cl}_{E\times F}[I(f;\bar{x})]$), then f verifies at $\bar{x}$ the tangential condition (T_2).

The reader will note that the preceding inclusions can be strict. A simple example is given by $f: \mathbb{R} \to \mathbb{R}^{\cdot}$ with $f(x)=0$ if $x \geqslant 0$ and $f(x)=+\infty$ otherwise and $\bar{x}=0$. Indeed one has

$$I(f;0)=\,]0,+\infty[\,\times\,]0,+\infty[,\qquad Q(f;0)=\,]0,+\infty[\,\times\,]0,+\infty[$$

and

$$T(f;0)=[0,+\infty[\,\times\,[0,+\infty[.$$

(3) It is easy to verify that for each $v \in E$ the set

$$L=\{l \in F \mid (v,l) \in Q(f;\bar{x})\}$$

is closed in F and operator-convex in the following sense:

$$\alpha L+(Id_F-\alpha)L \subset L$$

for every $\alpha \in L(F,F)$ with $0 \leqslant \alpha \leqslant Id_F$, i.e., $0 \leqslant \alpha y \leqslant y$ for all $y \in F_+$.

(4) If F is a Daniell topological lattice, i.e., the application $y \mapsto \inf(y,0)$ is continuous and for every decreasing net $(y_j)_{j\in J}$ which is bounded below one has $\lim_{j\in J} y_j=\inf_{j\in J} y_j$, then $Q(f;\bar{x})$ is the epigraph of a mapping from E into $\bar{F}$. Indeed, let v be a point in E such that the set

$$L=\{l \in F \mid (v,l) \in Q(f;\bar{x})\}$$

is nonempty and bounded below. Using the continuity of the mapping $(y_1,y_2) \mapsto \inf(y_1,y_2)$ one easily sees that $\inf(l_1,l_2) \in L$ whenever l_1 and l_2 are in L. Therefore, considering the decreasing net of infimums of finite subsets of L we derive from what precedes that $\inf(L) \in L$ since L is closed.

(5) If f is continuous at $\bar{x}$, then

$$(-\bar{v},\bar{l}) \in Q(f;\bar{x}) \Leftrightarrow (\bar{v},\bar{l}) \in Q(-f;\bar{x}).$$

Indeed, let $(-\bar{v},\bar{l})$ be a point in $Q(f;\bar{x})$ and let L be a neighbourhood of $\bar{l}$ in F. There exists a neighbourhood X_1 of $\bar{x}$ in E, a neighbourhood Y_1 of $f(\bar{x})$ in F, a neighbourhood V_1 of $-\bar{v}$ in E and a real number $\varepsilon_1>0$ such that

$$((x,y)+t(\{w\}\times L)) \cap \operatorname{epi} f \neq \varnothing$$

or

$$t^{-1}[f(x+tw)-y] \in L-F_+$$

for all $(x,y) \in (X_1 \times Y_1) \cap \operatorname{epi} f$, $t \in\,]0,\varepsilon[$, $w \in V_1$. By arguments of continuity, there exist a neighbourhood X of $\bar{x}$ in E with $X \subset X_1$, a neighbourhood V of $\bar{v}$ in E with $-V \subset V_1$, and a real number $\varepsilon>0$ with $\varepsilon<\varepsilon_1$ such that

$$X+\,]0,\varepsilon[\,V \subset X_1 \qquad \text{and} \qquad f(X+\,]0,\varepsilon[\,V) \subset Y_1.$$

If we put $Y = -Y_1$, we obtain for all $(x, y) \in (X \times Y) \cap \operatorname{epi}(-f)$, $t \in]0, \varepsilon[$ and $v \in V$ that

$$\begin{aligned} & t^{-1}[(-f)(x+tv) - y] \\ & \quad = t^{-1}[(-y) - f(x+tv)] \\ & \quad \in t^{-1}[f(x) - f(x+tv)] - F_+ \\ & \quad = t^{-1}[f((x+tv) + t(-v)) - f(x+tv)] - F_+ \\ & \quad \subset L - F_+ \end{aligned}$$

or

$$((x, y) + t(\{v\} \times L)) \cap \operatorname{epi}(-f) \neq \varnothing$$

and hence $(\bar{v}, \bar{l}) \in Q(-f; \bar{x})$. So, the assertion is proved for the reverse implication derives from arguments of symmetry. ▮

In order to give a condition under which the set $Q(f; \bar{x})$ is convex, we recall the extension of Penot and Thera of the notion of lower semi-continuity to functions taking values in ordered topological vector spaces (see [25]).

3.1. Definition. Let f be a mapping from E into $F^{\cdot}$ and let $\bar{x}$ be a point in E with $f(\bar{x}) \in F$. One says that f is *lower semi-continuous* at $\bar{x}$ if for each neighbourhood Y of $f(\bar{x})$ in F there is a neighbourhood X of $\bar{x}$ in E such that

$$f(X) \subset Y + F^{\cdot}_+,$$

where $F^{\cdot}_+ = F_+ \cup \{+\infty\}$.

3.2. Proposition. *Let f be a mapping from E into $F^{\cdot}$ with $f(\bar{x}) \in F$. If f is lower semi-continuous at $\bar{x}$, then $Q(f; \bar{x})$ is a convex cone in $E \times F$.*

Proof. Let us begin by proving that $Q(f; \bar{x})$ is a cone. Let λ be a positive real number, let $(\bar{v}, \bar{l})$ be a point in $Q(f; \bar{x})$ and let L be a neighbourhood of $\lambda\bar{l}$ in F. The set $L_0 = \lambda^{-1}L$ is a neighbourhood of $\bar{l}$ and hence there exist a neighbourhood X of $\bar{x}$ in E, a neighbourhood Y of $f(\bar{x})$ in Y, a real number $\varepsilon_0 > 0$, and a neighbourhood V_0 of $\bar{v}$ in E such that

$$((x, y) + t(\{v\} \times L_0)) \cap \operatorname{epi} f \neq \varnothing$$

for all $(x, y) \in (X \times Y) \cap \operatorname{epi} f$, $t \in]0, \varepsilon_0[$ and $v \in V_0$. Therefore, if we put $V = \lambda V_0$ and $\varepsilon = \lambda^{-1}\varepsilon_0$, we obtain

$$((x, y) + t(\{v\} \times L)) \cap \operatorname{epi} f \neq \varnothing$$

for all $(x, y) \in (X \times Y) \cap \operatorname{epi} f$, $t \in]0, \varepsilon[$, $v \in V$ and hence $\lambda(\bar{v}, \bar{l}) \in Q(f; \bar{x})$, that is, $Q(f; \bar{x})$ is a cone. Now let $(\bar{v}_1, \bar{l}_1)$ and $(\bar{v}_2, \bar{l}_2)$ be two points in $Q(f; \bar{x})$ and let L be a neighbourhood of $\bar{l}_1 + \bar{l}_2$ in F. There exist a neighbourhood L_1 of $\bar{l}_1$ in F and a neighbourhood L_2' of $\bar{l}_2$ in F such that $L_1 + L_2' \subset L$. So there are a neighbourhood X_1 of $\bar{x}$ in E, a normal neighbourhood

$$Y_1 = (Y_1 + F_+) \cap (Y_1 - F_+)$$

of $f(\bar{x})$ in F, a real numer $\varepsilon > 0$, and a neighbourhood V_1 of $\bar{v}_1$ in E such that

$$((x, y) + t(\{v_1\} \times L_1)) \cap \operatorname{epi} f \neq \emptyset,$$

that is,

$$t^{-1}[f(x + tv_1) - y] \in L_1 - F_+$$

for all $(x, y) \in (X_1 \times Y_1) \cap \operatorname{epi} f$, $t \in]0, \varepsilon_1[$ and $v_1 \in V_1$. Consider a neighbourhood L_2 of $\bar{l}_2$ in F, a real number $\varepsilon_2 > 0$ and a neighbourhood Y_2 of $f(\bar{x})$ in F verifying

$$L_2 \subset L_2' \qquad \text{and} \qquad Y_2 +]0, \varepsilon_2[\, L_2 \subset Y_1. \tag{1}$$

Then, since f is lower semi-continuous at $\bar{x}$ and $(\bar{v}_2, \bar{l}_2) \in Q(f; \bar{x})$, there exist a neighbourhood X of $\bar{x}$ in E, a neighbourhood Y of $f(\bar{x})$ in F, a positive real number $\varepsilon < \inf(\varepsilon_1, \varepsilon_2)$ and a neighbourhood V_2 of $\bar{v}$ in E such that

$$\begin{gathered} X +]0, \varepsilon[\, V_2 \subset X_1, \qquad Y \subset Y_2, \\ f(X +]0, \varepsilon[\, V_2) \subset Y_1 + F_+ \end{gathered} \tag{2}$$

and

$$t^{-1}[f(x + tv_2) - y] \in L_2 - F_+ \tag{3}$$

for all $(x, y) \in (X \times Y) \cap \operatorname{epi} f$, $t \in]0, \varepsilon[$ and $v_2 \in V_2$. Therefore, for every $(x, y) \in (X \times Y) \cap \operatorname{epi} f$, every $v_2 \in V_2$ and every $t \in]0, \varepsilon[$ relations (1), (2), and (3) imply that

$$f(x + tv_2) \in Y_1 + F_+$$

and

$$f(x + tv_2) \in y + tL_2 - F_+ \subset Y_1 - F_+,$$

which implies that $f(x + tv_2) \in Y_1$ and hence

$$(x + tv_2, f(x + tv_2)) \in (X_1 \times Y_1) \cap \operatorname{epi} f.$$

Therefore, for all $(x, y) \in (X \times Y) \cap \operatorname{epi} f$, $t \in]0, \varepsilon[$ and $v_1 + v_2 \in V_1 + V_2$ with $v_1 \in V_1$ and $v_2 \in V_2$, we derive from the relation

$$t^{-1}[f(x + tv_1 + tv_2) - y] = t^{-1}[f((x + tv_2) + tv_1) - f(x + tv_2)] + t^{-1}[f(x + tv_2) - y]$$

that

$$t^{-1}[f(x + tv_1 + tv_2) - y] \in L_1 + L_2 - F_+ \subset L - F_+$$

and hence $(\bar{v}_1 + \bar{v}_2, \bar{l}_1 + \bar{l}_2) \in Q(f; \bar{x})$ and the proof of the proposition is finished. ∎

Remark. If $F = \mathbb{R}$ and if f is directionally lipschitzian at $\bar{x}$ in the sense of Rockafellar (see Remark 2 following the definition of $Q(f; \bar{x})$), then even if f is not lower semi-continuous at $\bar{x}$ the cone $Q(f; \bar{x})$ lies between the tangent cone $T(f; \bar{x})$ and its interior for in this case (see Rockafellar [21])

$$I(f; \bar{x}) = \operatorname{int}_{E \times F}(T(f; \bar{x}))$$

and

$$I(f; \bar{x}) \subset Q(f; \bar{x}) \subset T(f; \bar{x}). \quad ∎$$

More generally we can establish that if $\operatorname{int}(F_+) \neq \varnothing$, then the cone $Q(f; \bar{x})$ is convex and it can be derived from $I(f; \bar{x})$.

3.3. Proposition. *If* $\operatorname{int}(F_+) \neq \varnothing$, *then for every* $v \in E$ *we have*

$$I_v(f; \bar{x}) = \operatorname{int}[Q_v(f; \bar{x})],$$

where

$$I_v(f; \bar{x}) = \{l \in F \mid (v, l) \in I(f; \bar{x})\}$$

and

$$Q_v(f; \bar{x}) = \{l \in F \mid (v, l) \in Q(f; \bar{x})\}.$$

Proof. First of all let us note that the set $I_v(f; \bar{x})$ is open since $I(f; \bar{x})$ is open. Therefore, since $I(f; \bar{x}) \subset Q(f; \bar{x})$, we have

$$I_{\bar{v}}(f; \bar{x}) \subset \operatorname{int}[Q_{\bar{v}}(f; \bar{x})] \qquad \text{for every } \bar{v} \in E.$$

Now let $\bar{l}$ be a point in $\operatorname{int}[Q_{\bar{v}}(f; \bar{x})]$ and let W be a circled neighbourhood of zero in F verifying $W = (W + F_+) \cap (W - F_+)$ and such that $(\bar{v}, l) \in Q(f; \bar{x})$ for every $l \in \bar{l} + W$. Choose two points

$$a \in (\bar{l} + W) \cap (\bar{l} - \operatorname{int}(F_+)) \qquad \text{and} \qquad b \in (\bar{l} + W) \cap (\bar{l} + \operatorname{int}(F_+)).$$

If we put $c = 2^{-1}(a + \bar{l})$ and $d = 2^{-1}(a + c)$, then we have $d \in]a, c[\subset \bar{l} + W$, where $]a, c[$ denotes

$$]a, c[= (a + \operatorname{int}(F_+)) \cap (c - \operatorname{int}(F_+)).$$

So $(\bar{v}, d) \in Q(f; \bar{x})$ and hence, since $]a, c[$ is a neighbourhood of d in F, there exists a neighbourhood V of $\bar{v}$ in E and a real number $\varepsilon > 0$ such that

$$t^{-1}[f(x + tv) - y] \in]a, c[- F_+$$

for all $(x, y) \in (X \times Y) \cap \operatorname{epi} f$, $v \in V$ and $t \in]0, \varepsilon[$. Therefore we have

$$t^{-1}[f(x + tv) - y] \in l - F_+$$

or

$$(x, y) + t(v, l) \in \operatorname{epi} f$$

for all $(x, y) \in (X \times Y) \cap \operatorname{epi} f$, $t \in]0, \varepsilon[$ and $(v, l) \in V \times]c, b[$ and hence $(\bar{v}, \bar{l}) \in I(f; \bar{x})$, which proves the proposition. ∎

3.4. COROLLARY. *Let f be a mapping from E into $F^{\bullet}$ and let $\bar{x}$ be a point in E with $f(\bar{x}) \in F$. If $\operatorname{int}(F_+) \neq \varnothing$, then we have*

(i) $I_v(f; \bar{x}) \neq \varnothing$ *if and only if* $Q_v(f; \bar{x}) \neq \varnothing$;

(ii) $Q_v(f; \bar{x}) = \operatorname{cl}_F(I_v(f; \bar{x}))$ *for each* $v \in E$;

(iii) $Q(f; \bar{x})$ *is a convex cone.*

Proof. Since $I(f; \bar{x}) \subset Q(f; \bar{x})$, $I_v(f; \bar{x}) \neq \varnothing$ implies that $Q_v(f; \bar{x}) \neq \varnothing$. Now let us suppose that $Q_v(f; \bar{x}) \neq \varnothing$. As

$$Q_v(f; \bar{x}) + \operatorname{int}(F_+) \subset Q_v(f; \bar{x}),$$

we obtain that $\operatorname{int}(Q_v(f; \bar{x})) \neq \varnothing$ and hence according to Proposition 3.3 the cone $I_v(f; \bar{x})$ is nonempty, and assertion (i) is verified.

Since $Q_v(f; \bar{x})$ is convex and closed for each $v \in E$ (see Remark 3 following the definition of $Q(f; \bar{x})$) assertion (ii) is a direct consequence of (i) and of Proposition 3.3.

To show (iii) we may suppose that $Q(f; \bar{x})$ is nonempty. Let (v, k) and (w, l) be two points in $Q(f; \bar{x})$ and let α and β be two non-negative real numbers with $\alpha + \beta = 1$. According to (ii) there exist two nets $(k_j)_{j \in J}$ and $(l_j)_{j \in J}$ in F such that

$$(v, k_j) \in I(f; \bar{x}) \quad \text{and} \quad (w, l_j) \in I(f; \bar{x}) \quad \text{for every } j \in J$$

and

$$\lim_{j \in J} k_j = k \quad \text{and} \quad \lim_{j \in J} l_j = l.$$

Moreover it is not difficult to see that $I(f;\bar{x})$ is convex. So we obtain

$$(\alpha v+\beta w, \alpha k_j+\beta l_j)\in I(f;\bar{x}) \qquad \text{for every } j\in J$$

and hence according to (ii)

$$(\alpha v+\beta w, \alpha k+\beta l)\in Q(f;\bar{x})$$

which finishes the proof. ∎

Taking Proposition 3.2 and Corollary 3.4 into account we are naturally led to define the directional pseudo-subderivative in the following way.

3.5. DEFINITION. If f is a mapping from E into $F^{\cdot}$ with $f(\bar{x})\in F$, we shall call *directional pseudo-subderivative* of f at $\bar{x}$ the mapping $f^{\square}(\bar{x};\cdot)$ from E into $\bar{F}$ defined by

$$f^{\square}(\bar{x};v)=\inf\{l\in F \mid (v,l)\in Q(f;\bar{x})\}.$$

Remarks. (1) The relation $f^{\uparrow}(\bar{x};v)\leqslant f^{\square}(\bar{x};v)$ always holds and hence if f verifies tangential condition (T_2), $f^{\square}(\bar{x};\cdot)$ takes its values in $F^{\cdot}$.

(2) If f is lower semi-continuous at $\bar{x}$ or if $\operatorname{int}(F_+)\neq\varnothing$, then according to Proposition 3.2 and Corollary 3.4, $f^{\square}(\bar{x};\cdot)$ is a sublinear mapping from E into $\bar{F}$. ∎

3.6. LEMMA. *Let f and g be two mappings from E into $F^{\cdot}$. If f and g are semi-continuous at $\bar{x}\in \operatorname{dom} f\cap \operatorname{dom} g$ or if f or g is continuous at $\bar{x}$ with respect to $\operatorname{dom} f\cap\operatorname{dom} g$, then for all $(\bar{v},\bar{l}_1)\in Q(f;\bar{x})$ and $(\bar{v},\bar{l}_2)\in T(f;\bar{x})$ we have $(\bar{v},\bar{l}_1+\bar{l}_2)\in T(f+g;\bar{x})$.*

Proof. Let V be a neighbourhood of $\bar{v}$ in E and let L be a neighbourhood of $\bar{l}_1+\bar{l}_2$ in F. There exist neighbourhoods L_i of $\bar{l}_i$, $i=1, 2$, in F verifying $L_1+L_2\subset L$. Then, according to the definition of $Q(f;\bar{x})$ there exist a neighbourhood X_1 of $\bar{x}$ in E, a neighbourhood Y_1 of $f(\bar{x})$ in F, a real number $\varepsilon_1>0$ and a neighbourhood V_1 of $\bar{v}$ in E with $V_1\subset v$ such that

$$((x,y_1)+t(\{v\}\times L_1))\cap \operatorname{epi} f\neq\varnothing \tag{4}$$

for all $(x,y_1)\in(X\times Y_1)\cap\operatorname{epi} f$, $t\in\,]0,\varepsilon_1[$ and $v\in V_1$. Taking now the definition of $T(g;\bar{x})$ into account we can find a neighbourhood X_2 of $\bar{x}$ with $X_2\subset X_1$, a neighbourhood Y_2 of $g(\bar{x})$, and a positive real number ε with $\varepsilon<\varepsilon_1$ such that

$$((x,y_2)+t(V_1\times L_2))\cap\operatorname{epi} f\neq\varnothing \tag{5}$$

for all $(x,y_2)\in(X\times Y_2)\cap\operatorname{epi} g$ and $t\in\,]0,\varepsilon[$. With the help of the assumptions it is not difficult to see there exist a neighbourhood X of $\bar{x}$ with

$X \subset X_1 \cap X_2$, a neighbourhood Y of $f(\bar{x}) + g(\bar{x})$ such that for each $(x, y) \in (X \times Y) \cap \operatorname{epi}(f+g)$ we can write $y = y_1 + y_2$ with $(x, y_1) \in (X \times Y_1) \cap \operatorname{epi} f$ and $(x, y_2) \in (X \times Y_2) \cap \operatorname{epi} g$. So for each $(x, y) \in (X \times Y) \cap \operatorname{epi}(f+g)$ and $t \in]0, \varepsilon[$, according to relation (5), there exist $v \in V_1 \subset V$ and $l_2 \in L_2$ such that

$$g(x + tv) \leqslant y_2 + tl_2.$$

But relation (4) implies that there exists $l_1 \in L_1$ such that

$$f(x + tv) \leqslant y_1 + tl_1$$

and hence

$$(f + g)(x + tv) \leqslant y + t(l_1 + l_2)$$

with $l_1 + l_2 \in L_1 + L_2 \subset L$. Thus it follows that

$$((x, y) + t(V \times L)) \cap \operatorname{epi}(f + g) \neq \varnothing$$

and the lemma is proved. ∎

Let us also establish another lemma which will be used in the sequel.

3.7. LEMMA. *Let f be a mapping from E into $F^{\cdot}$ with $f(\bar{x}) \in F$ and verifying tangential condition* (T_2) *at $\bar{x}$. We have*

$$\sup\{Av \mid A \in L(E, F), A(\cdot) \leqslant f^{\square}(\bar{x}; \cdot)\} \leqslant f^{\uparrow}(\bar{x}; v) \leqslant f^{\square}(\bar{x}; v)$$

for all $v \in E$.

Proof. Let A be an element in $L(E, F)$ verifying $A(w) \leqslant f^{\square}(\bar{x}; w)$ for all $w \in E$ and let $(\bar{v}, \bar{l})$ be any point in $T(f; \bar{x})$. There exists a net $(v_j, l_j)_{j \in J}$ in $Q(f; \bar{x})$ converging to $(\bar{v}, \bar{l})$ in $E \times F$. Also for each $j \in J$ we have

$$A(v_j) \leqslant f^{\square}(\bar{x}; v_j) \leqslant l_j$$

and hence

$$A(\bar{v}) = \lim_{j \in J} A(v_j) \leqslant \lim_{j \in J} l_j = \bar{l}.$$

Thus it follows that

$$\sup\{A\bar{v} \mid A \in L(E, F), A(\cdot) \leqslant f^{\square}(\bar{x}; \cdot)\} \leqslant f^{\uparrow}(\bar{x}; \bar{v})$$

and the lemma is proved. ∎

Let us recall now the following result of Zowe [33].

3.8. PROPOSITION. *Let f and g be two convex mappings from E into $F^{\cdot}$.*

If there is a point $z \in \operatorname{dom} f \cap \operatorname{dom} g$ *such that* f *is continuous at* z, *then for each* $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ *we have*

$$\partial(f+g)(x_0) = \partial f(x_0) + \partial g(x_0),$$

with the convention $\phi + B = \varnothing$ *for every subset* B *of* $L(E, F)$.

We can now establish our result concerning the subdifferential of the sum of two mappings.

3.9. Proposition. *Let* f *and* g *be two mappings from* E *into* $F^{\cdot}$ *such that* f *and* g *are semi-continuous at* $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} g$ *or such that* f *or* g *is continuous at* $\bar{x}$ *with respect to* $\operatorname{dom} f \cap \operatorname{dom} g$.

(i) *For every* $v \in E$ *we have*

$$(f+g)^{\uparrow}(\bar{x}; v) \leqslant f^{\square}(\bar{x}; v) + g^{\uparrow}(\bar{x}; v).$$

(ii) *If in addition* f *and* g *verify* (T_1) *at* $\bar{x}$, *if* f *verifies* (T_2) *at* $\bar{x}$ *and is lower semi-continuous at* $\bar{x}$ *and if there is a point* $\bar{v} \in \operatorname{dom} f^{\square}(\bar{x}; \cdot) \cap \operatorname{dom} g^{\uparrow}(\bar{x}; \cdot)$ *such that* $f^{\square}(\bar{x}; \cdot)$ *or* $g^{\uparrow}(\bar{x}; \cdot)$ *is continuous at* $\bar{v}$, *then*

$$\partial(f+g)(\bar{x}) \subset \partial f(\bar{x}) + \partial g(\bar{x}).$$

Proof. Part (i) is an immediate consequence of Lemma 3.6. To show (ii) we may suppose that $(f+g)^{\uparrow}(\bar{x}; \cdot)$ takes its values in $F^{\cdot}$, for otherwise $\partial(f+g)(\bar{x}) = \varnothing$. If for every mapping s from E into $F^{\cdot}$ we denote by s_0 the mapping from E into $F^{\cdot}$ defined by $s_0(0) = 0$ and $s_0(v) = s(v)$ if $v \neq 0$, then the mapping s_0 is sublinear whenever s is sublinear. Therefore, it follows that the mappings $(f+g)_0^{\uparrow}(\bar{x}; \cdot)$, $f_0^{\square}(\bar{x}; \cdot)$ and $g_0^{\uparrow}(\bar{x}; \cdot)$ are sublinear. Thus using assertion (i), Propositions 3.2 and 3.8 and Lemma 3.7, one obtains the result of (ii). ∎

Remarks. (1) The result of (ii) is still true if, instead of assuming f is lower semi-continuous at $\bar{x}$, we assume the cone $Q(f; \bar{x})$ is convex.

(2) If $\operatorname{int}(F_+) \neq \varnothing$, then according to Corollary 3.4 the assumption that f is lower semi-continuous at $\bar{x}$ is superfluous. So if f is a function taking values in $\mathbb{R}^{\cdot}$ and if f is directionally lipschitzian at $\bar{x}$, we find the inclusion formula about the subdifferential of a sum of two extented real-valued functions of Theorem 2 of Rockafellar [22]. ∎

Consider now the case where f is strictly compactly lipschitzian at $\bar{x}$.

3.10. Proposition. *Let* f *be a mapping from* E *into* F *which is strictly compactly lipschitzian at* $\bar{x}$. *Then* f *verifies tangential conditions* (T_1) *and* (T_2) *at* $\bar{x}$ *and*

$$f^{\uparrow}(\bar{x}; v) = f^{\square}(\bar{x}; v) \qquad \text{for all} \quad v \in E.$$

Proof. Condition (T_1) is a direct consequence of equality $f^\circ(\bar{x};\cdot) = \hat{f}(\bar{x};\cdot)$ in Proposition 2.5. To show that condition (T_2) and the equality of the proposition are verified it suffices to show $T(f;\bar{x}) \subset Q(f;\bar{x})$, for the reverse inclusion is always true. Suppose there exists a point $(\bar{v},\bar{l})$ in $T(f;\bar{x})$ which does not belong to $Q(f;\bar{x})$. According to the definition of $Q(f;\bar{x})$ there are an open neighbourhood L of $\bar{l}$ in F, nets $(x_j, y_j)_{j\in J}$ in epi f, $(t_j)_{j\in J}$ in $]0,\infty[$ and $(v_j)_{j\in J}$ in E converging respectively to $(\bar{x}, f(\bar{x}))$, zero and $\bar{v}$ such that

$$(x_j + t_j v_j, y_j + t_j l) \notin \operatorname{epi} f$$

for every $l \in L$ and every $j \in J$. So we have

$$t_j^{-1}[f(x_j + t_j v_j) - y_j] \notin L - F_+$$

for every $j \in J$ and hence

$$t_j^{-1}[f(x_j + t_j v_j) - f(x_j)] \notin L - F_+$$

for every $j \in J$ since $y_j \geqslant f(x_j)$ and $t_j > 0$. Since f is strictly compactly lipschitzian at $\bar{x}$, there exists a subnet

$$(t_{\alpha(i)}^{-1}[f(x_{\alpha(i)} + t_{\alpha(i)} v_i) - f(x_{\alpha(i)})])_{i\in I}$$

which converges to some point $z \in F$. So using the remark following Definition 2.3 we obtain that

$$z = \lim_{i\in I} t_{\alpha(i)}^{-1}[f(x_{\alpha(i)} + t_{\alpha(i)}\bar{v}) - f(x_{\alpha(i)})]$$

$$= \lim_{i\in I} t_{\alpha(i)}^{-1}[f(x_{\alpha(i)} + t_{\alpha(i)} v_{\alpha(i)}) - f(x_{\alpha(i)})]$$

and hence that $z \notin L - F_+$. But since $(\bar{v},\bar{l}) \in T(f;\bar{x})$, according to Proposition 2.5 we have $f^\circ(\bar{x};\bar{v}) \leqslant \bar{l}$. So it follows that

$$z = \lim_{i\in I} t_{\alpha(i)}^{-1}[f(x_{\alpha(i)} + t_{\alpha(i)}\bar{v}) - f(x_{\alpha(i)})] \notin f^\circ(\bar{x};\bar{v}) - F_+,$$

which is in contradiction with the definition of $f^\circ(\bar{x};\bar{v})$. ∎

Let us study the case where f is convex.

3.11. Proposition. *If f is a convex mapping from E into $F^\cdot$ which is continuous on a neighbourhood of $\bar{x}$ in E, then f verifies tangential conditions* (T_1) *and* (T_2) *at* $\bar{x}$.

Proof. Denote by $R(f;\bar{x})$ the set of all $(v,l) \in E \times F$ for which there exists a real number $\varepsilon > 0$ verifying $(\bar{x}, f(\bar{x})) +]0,\varepsilon[(v,l) \in \operatorname{epi} f$. Since condition ($T_1$) is an immediate consequence of Proposition 2.7 and of the nonvacuity of the subdifferential $\partial_c f(\bar{x})$ (see Corollary 1 of Theorem 1 of Zowe [33] or Theorem 6 of Valadier [31]), then according to Remark 2 following Proposition 1.2 it is enough to show

$$R(f; \bar{x}) \subset Q(f; \bar{x}).$$

Let $(\bar{v}, \bar{l})$ be a point in $R(f; \bar{x})$ and let L be a neighbourhood of $\bar{l}$ in F. There exists a real number $\varepsilon > 0$ such that

$$t^{-1}[f(\bar{x} + t\bar{v}) - f(\bar{x})] \in \bar{l} - F_+$$

for every $t \in]0, \varepsilon[$. Since f is continuous on a neighbourhood of $\bar{x}$, there is $\alpha \in]0, \varepsilon[$ such that f is continuous at the point $\bar{x} + \alpha\bar{v}$. So, as

$$\alpha^{-1}[f(\bar{x} + \alpha\bar{v}) - f(\bar{x})] \in L - F_+,$$

there exist a neighbourhood X of $\bar{x}$ in E and a neighbourhood V of $\bar{v}$ in E verifying

$$\alpha^{-1}[f(x + \alpha v) - f(x)] \in L - F_+$$

for all $x \in X$ and $v \in V$. Therefore, for every $(x, y) \in (X \times Y) \cap \operatorname{epi} f$, every $t \in]0, \alpha[$ and every $v \in V$ we have

$$\begin{aligned} t^{-1}[f(x + tv) - y] &\in t^{-1}[f(x + tv) - f(x)] - F_+ \\ &\subset \alpha^{-1}[f(x + \alpha v) - f(x)] - F_+ \\ &\subset L - F_+ \end{aligned}$$

and hence $(\bar{v}, \bar{l}) \in Q(f; \bar{x})$.

Remark. If the order intervals of F are compact and if the lattice mapping of F defined by $x \mapsto \sup(x, 0)$ is continuous, then the preceding proposition can also be seen as a consequence of Proposition 3.10 above and of Proposition 1.9 of Thibault [27].

4. Composition with a Strictly Differentiable Mapping

We shall begin by studying the case $f \circ A$ where f is a convex mapping and A is a continuous linear mapping.

4.1. Proposition. *Let A be a continuous linear mapping from a topological vector space G into E and let f be a convex mapping from E into*

$F^{\cdot}$ with $f(A\bar{x}) \in F$. If there is a point $\bar{v} \in E$ such that f is continuous at $A\bar{v}$, then

$$\partial(f \circ A) = \partial f(A\bar{x}) \circ A = \{T \circ A \mid T \in \partial f(A\bar{x})\}.$$

Proof. The proof follows an idea of Rockafellar [22] which consists to derive composition formula from sum formula. Define two convex mappings g_1 and g_2 from $G \times E$ into $F^{\cdot}$ by putting

$$g_1(x, y) = 0 \quad \text{if} \quad y = Ax \quad \text{and} \quad g_1(x, y) = +\infty \quad \text{otherwise}$$

and

$$g_2(x, y) = f(y) \quad \text{for all} \quad (x, y) \in G \times E.$$

Set $g = g_1 + g_2$. Since f is continuous at $A\bar{v}$, then g_2 is continuous at $(\bar{v}, A\bar{v})$ and $(\bar{v}, A\bar{v}) \in \operatorname{dom} g_1 \cap \operatorname{dom} g_2$. So as $(\bar{x}, A\bar{x}) \in \operatorname{dom} g_1 \cap \operatorname{dom} g_2$, according to Proposition 3.6 we have

$$\partial g(\bar{x}, A\bar{x}) = \partial g_1(\bar{x}, A\bar{x}) + \partial g_2(\bar{x}, A\bar{x}). \tag{6}$$

Now let us characterize the three sets that appear in relation (6). For each continuous linear mapping T from $G \times E$ into F we shall denote by T' and T'' the continuous linear mappings from G into F and from E into F defined by $T(x, y) = T'(x) + T''(y)$ for all $(x, y) \in G \times E$ and we shall write $T = (T', T'')$. We claim

$$\partial g(\bar{x}, A\bar{x}) = \{T \in L(G \times E, F) \mid T' + T'' \circ A \in \partial(f \circ A)(\bar{x})\}.$$

Indeed, we have $T \in \partial g(\bar{x}, A\bar{x})$ if and only if

$$T(x, y) - T(\bar{x}, A\bar{x}) \leqslant g(x, y) - g(\bar{x}, A\bar{x})$$

for all $(x, y) \in G \times E$, hence if and only if

$$T'(x) + T'' \circ A(x) - T'(\bar{x}) - T'' \circ A(\bar{x}) \leqslant f \circ A(x) - f \circ A(\bar{x})$$

and hence if and only if

$$T' + T'' \circ A \in \partial(f \circ A)(\bar{x}).$$

One shows in a similar way

$$\partial g_1(\bar{x}, A\bar{x}) = \{T \in L(G \times E, F) \mid T' + T'' \circ A = 0\}$$

and

$$\partial g_2(\bar{x}, A\bar{x}) = \{T \in L(G \times E, F) \mid T' = 0 \text{ and } T'' \in \partial f(A\bar{x})\}.$$

Let T' be an element in $\partial(f \circ A)(\bar{x})$. According to relation (6) and the above characterizations of the subdifferentials appearing in relation (6), there exist $T_1' \in L(G, F)$, $T_1'' \in L(E, F)$ with $T_1' + T_1'' \circ A = 0$ and $T_2'' \in \partial f(A\bar{x})$ verifying

$$(T', 0) = (T_1', T_1'' + T_2'').$$

It follows $T' = T_2'' \circ A$ and hence

$$\partial(f \circ A)(\bar{x}) \subset \partial f(A\bar{x}) \circ A.$$

As the reverse inclusion is obvious, we obtain the desired equality. ∎

Now let us extend the definition in the remark following Proposition 2.5 of a strictly differentiable mapping to the case where the space of definition is not necessarily normed (see [22]).

4.2. DEFINITION. A mapping g from a topological vector space G into E is said to be *strictly differentiable* at a point $\bar{x} \in G$ if there are a continuous linear mapping $\nabla g(\bar{x}) \in L(G, E)$ and a mapping r from $]0, +\infty[\times G \times G$ into E such that

$$g(x + tv) = g(x) + t\nabla g(\bar{x})v + \mathrm{tr}(t, x; v)$$

for all $(t, x; v) \in]0, \infty[\times G \times G$ and

$$\lim_{\substack{t \downarrow 0 \\ x \to \bar{x} \\ w \to v}} r(t, x; w) = 0 \qquad \text{for all} \quad v \in G.$$

Remark. It is not difficult to see that a mapping g is strictly differentiable at a point $\bar{x}$ if and only if it is strictly compactly lipschitzian at $\bar{x}$ with a mapping K (see Definition 2.3) taking values in the set of singletons of E.

4.3. LEMMA. *Let g be a mapping from a topological vector space G into E which is strictly differentiable at $\bar{x}$ with derivative $\nabla g(\bar{x}) = A$ and let f be a mapping from E into $F^{\cdot}$ with $f(g(\bar{x})) \in F$. For every $(\bar{v}, \bar{l}) \in G \times F$ such that $(A\bar{v}, \bar{l}) \in Q(f; g(\bar{x}))$ we have $(\bar{v}, \bar{l}) \in Q(f \circ g; \bar{x})$.*

Proof. Suppose $(A\bar{v}, \bar{l}) \in Q(f; g(\bar{x}))$ and let L be a neighbourhood of $\bar{l}$ in F. There exist a neighbourhood $Z \times Y$ of $(g(\bar{x}), f \circ g(\bar{x}))$ in $E \times F$, a real number $\varepsilon_1 > 0$ and a neighbourhood U of $A\bar{v}$ in E such that

$$((z, y) + t(\{u\} \times L)) \cap \mathrm{epi}\, f \neq \emptyset,$$

that is,

$$t^{-1}[f(z + tu) - y] \in L - F_+ \tag{7}$$

for all $(z, y) \in (Z \times Y) \cap \operatorname{epi} f$, $t \in]0, \varepsilon[$ and $u \in U$. As (see Definition 4.2)

$$\lim_{\substack{t \downarrow 0 \\ x \to \bar{x} \\ w \to \bar{v}}} r(t, x; w) = 0$$

and that g is continuous at $\bar{x}$, there exist a neighbourhood X of $\bar{x}$ in G, a positive real number $\varepsilon < \varepsilon_1$ and a neighbourhood V of $\bar{v}$ in G such that

$$g(X) \subset Z \qquad \text{and} \qquad A(V) + r(]0, \varepsilon[\times X \times V) \subset U. \tag{8}$$

So for all $(x, y) \in (X \times Y) \cap \operatorname{epi}(f \circ g)$, $t \in]0, \varepsilon[$ and $v \in V$, since $(g(x), y) \in (Z \times Y) \cap \operatorname{epi} f$, we have according to relations (7) and (8)

$$t^{-1}[f \circ g(x + tv) - y] = t^{-1}[f(g(x) + t(Av + r(t, x; v))) - y] \in L - F_+$$

and hence $(\bar{v}, \bar{l}) \in Q(f \circ g; \bar{x})$. ∎

We can now state the following result.

4.4. Proposition. *Let g be a mapping from a topological vector space G into E which is strictly differentiable at $\bar{x}$ with derivative $\nabla g(\bar{x}) = A$ and let f be a mapping from E into $F^{\cdot}$ with $f(g(\bar{x})) \in F$. Then*

(i) *for every $v \in G$*

$$(f \circ g)^{\uparrow}(\bar{x}; v) \leqslant (f \circ g)^{\square}(\bar{x}; v) \leqslant f^{\square}(g(\bar{x}); Av).$$

(ii) *If in addition f verifies* (T_2) *at $g(\bar{x})$, if f is lower semi-continuous at $g(\bar{x})$ and if there is a point $\bar{v} \in G$ such that $f^{\square}(g(\bar{x}); \cdot)$ is continuous at $A\bar{v}$, then*

$$\partial(f \circ g)(\bar{x}) \subset \partial f(g(\bar{x})) \circ A.$$

Proof. Part (i) is a direct consequence of the preceding lemma and of Definition 3.5. For (ii) it suffices to repeat the arguments of the proof of Proposition 3.9 and to apply Proposition 4.1 to the mapping $\varphi \circ A$ at zero where φ is defined by $\varphi(y) = f^{\square}(g(\bar{x}); y)$ if $y \neq 0$ and $\varphi(0) = 0$ by using Proposition 3.2 and Lemma 3.7. ∎

Remarks. (1) Assertion (ii) still holds if instead of assuming f is lower semi-continuous at $g(\bar{x})$ we suppose the cone $Q(f; g(\bar{x}))$ is convex.

(2) If $\operatorname{int}(F_+) \neq \varnothing$, then (as in Remark 2 following Proposition 3.7) the assumption that f is lower semi-continuous at $g(\bar{x})$ is superfluous. Moreover in this case condition (T_2) at $g(\bar{x})$ is according to Corollary 3.4 equivalent to

$$\operatorname{cl}_{E \times F}(I(f; g(\bar{x}))) = T(f; g(\bar{x})).$$

So if f takes values in $\mathbb{R}^{\cdot}$ and if it is directionally lipschitzian at $\bar{x}$, we find inclusion formula of Theorem 3 of Rockafellar [22].

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