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The representation of generalized inverse $A_{T,S}^{(2,3)}$ and its applications

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ABSTRACT

This paper presents the explicit expression for matrix right symmetry factor with prescribed rang and null space. Moreover, the explicit expression for generalized inverse $A_{T,S}^{(2,3)}$, which is a {2,3}-inverse of A having the prescribed rang T and null space S, is derived. As an application, two numerical examples are given.

1. Introduction and preliminaries

It is well-known that the six important generalized inverses: the Moore–Penrose inverse A^+, the weighted Moore–Penrose inverse $A_{M,N}^+$, the Drazin inverse A^D, the group inverse Ag, the Bott–Duffin inverse $A_{(L)}^{(-1)}$ and the generalized Bott–Duffin inverse $A_{(L)}^{(+)}$ are all special generalized inverse $A_{T,S}^{(2)}$, which is {2}-inverse of A having the prescribed range T and the null space S. There are a great number of papers dealing with the representation of the generalized inverse $A_{T,S}^{(2)}$ and its applications [3–6]. However, there are hardly any discussions about generalized inverse $A_{T,S}^{(2,3)}$ in the literature. So we continue the work in this area.

The weighted Moore–Penrose inverse is a generalization of the inverse of a non-singular matrix. The weighted Moore–Penrose inverse is widely used for weighted linear least squares problem, statistics [1], etc. There are a good number of papers dealing with the computation of weighted Moore–Penrose inverse [5,11–13]. Here, by applying the generalized inverse $A_{T,S}^{(2,3)}$ to the weighted Moore–Penrose inverse $A_{M,N}^+$, we give a novel algorithm for the weighted Moore–Penrose inverse.

For the sake of convenience, we first present notations, definitions and lemmas needed in the discussions that will follow. A^*, rank(A), Ind (A), $A^{(i,j,\dots,k)}$, $R(A)$ and $N(A)$ denote the conjugate transpose, the rank, the index, the $\{i, j, \dots, k\}$-inverse, the range and the null space of a matrix A, respectively; $P_{L,M}$ and P_L denote the projector on the space L along the space M and the orthogonal projector on L, respectively; $\dim L$ and $L^\perp$ denote the dimension and the orthogonal complement of a space L; $C_r^{m\times n}$ denotes the set of all $m \times n$ matrices with rank r.

Definition 1 (*[7]*). Let $A \in C^{m\times n}$, the matrix X is a right symmetry factor of A, if AX is a Hermitian matrix.

According to the definition above, we know that the right symmetry factor of A is just $A^{(3)}$.

Lemma 1.1 ([1,2]). *For any* $A \in C^{m\times n}$,

(a) $P_{L,M}A = A \Leftrightarrow R(A) \subset L$; $AP_{L,M} = A \Leftrightarrow M \subset N(A)$.
(b) $N(A) = R(A^*)^\perp$, $N(A^*) = R(A)^\perp$.

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(c) *$AA^{(1)}$ and $A^{(1)}A$ are idempotent matrices;*

$$\mathrm{rank}(AA^{(1)}) = \mathrm{rank}(A^{(1)}A) = \mathrm{rank}(A).$$

(d) $R(AA^{(1)}) = R(A)$, $N(A^{(1)}A) = N(A)$, $R((A^{(1)}A)^*) = R(A^*)$.
(e) $B(AB)^{(1)}AB = B$ *if and only if* $\mathrm{rank}(AB) = \mathrm{rank}(B)$, *and* $AB(AB)^{(1)}A = A$ *if and only if* $\mathrm{rank}(AB) = \mathrm{rank}(A)$.

Lemma 1.2. (1) *For any matrix $A \in C^{m\times n}$, the Moore–Penrose inverse A^+ and the weighted Moore–Penrose inverse $A^+_{M,N}$ satisfy:*

(a) [1] $A^+ = A^{(2)}_{R(A^*),N(A^*)}$;

(b) [8] $A^+_{M,N} = A^{(2)}_{R(A^\#),N(A^\#)} = A^{(2)}_{N^{-1}R(A^*),M^{-1}N(A^*)}$, *where M and N are Hermitian positive definite matrices of order m and n, and $A^\# = N^{-1}A^*M$.*

(2) *For any $A \in C^{n\times n}$, the Drazin inverse A^D, the group inverse Ag, the Bott–Duffin inverse $A^{(-1)}_{(L)}$ and the generalized Bott–Duffin inverse $A^{(+)}_{(L)}$ satisfy:*

(c) [9] $A^D = A^{(2)}_{R(A^k),N(A^k)}$, *where $k = \mathrm{Ind}\,(A)$; in particular, when $\mathrm{Ind}\,(A) = 1$, we have $Ag = A^{(2)}_{R(A),N(A)}$;*

(d) [10] $A^{(-1)}_{(L)} = A^{(2)}_{L,L^\perp}$, *where L is a subspace of C^n satisfying $AL \oplus L^\perp = C^n$;*

(e) [10] $A^{(+)}_{(L)} = A^{(2)}_{S,S^\perp}$, *where L is a subspace of C^n, $S = R(P_LA)$, A is an L-p.s.d matrix, which means that A is a Hermitian matrix such that P_LAP_L is nonnegative definite, and $N(P_LAP_L) = N(AP_L)$.*

In this paper, we firstly propose the explicit expressions for the matrix right symmetry factor with prescribed rang T and null spaces S. Then, we derive the representation of the generalized inverse $A^{(2,3)}_{T,S}$, which is a {2,3}-inverse of A having the prescribed rang T and null space S. Finally, we give the application of generalized inverse $A^{(2,3)}_{T,S}$ and two numerical examples.

2. Main results

In this section, we firstly derive the explicit expression for matrix right symmetry factor with prescribed rang and null space.

Theorem 2.1. *Let $A \in C^{m\times n}$, T and S be subspaces of C^n and C^m, respectively, with $\dim(AT) = \dim T = t$. Suppose that E is any matrix satisfying $R(E) = T$ and M is an arbitrary Hermitian matrix satisfying $R(M) = AT$, then $X = E(AE)^{(1)}M$ is a right symmetry factor of matrix A and $R(X) = T$, $N(X) = S$ if and only if $AT = S^\perp$. Moreover X is independent of the choice of {1}-inverse $(AE)^{(1)}$.*

Proof. IF: Sine $R(M) = AT = AR(E) = R(AE)$, then

$$\begin{aligned} AX = AE(AE)^{(1)}M &= P_{R(AE(AE)^{(1)}),N(AE(AE)^{(1)})}M, \quad \text{by Lemma 1.1(c)},\\ &= P_{R(AE),N(AE(AE)^{(1)})}M, \quad \text{by Lemma 1.1(d)},\\ &= M, \quad \text{by Lemma 1.1(a)}. \end{aligned} \tag{1}$$

Hence AX is a Hermitian matrix, i.e., X is right symmetry factor of A.

From $X = E(AE)^{(1)}M$ we get $R(X) \subset R(E)$. Furthermore, we have

$$t = \mathrm{rank}(E) \ge \mathrm{rank}(X) \ge \mathrm{rank}(AX) = \mathrm{rank}(M) = \dim(AT) = t, \tag{2}$$

so we have $\mathrm{rank}(X) = \mathrm{rank}(E) = \mathrm{rank}(AX) = t$ and $R(X) = R(E) = T$.

Further, from $\mathrm{rank}(X) = \mathrm{rank}(E) = \mathrm{rank}(AX)$, we have

$$\dim(N(AX)) = m - \mathrm{rank}(AX) = m - \mathrm{rank}(X) = \dim(N(X)), \tag{3}$$

which, together with $N(AX) \supset N(X)$, implies that

$$\begin{aligned} N(X) &= N(AX) = R(AX)^\perp, \quad \text{by Lemma 1.1(b)},\\ &= (AT)^\perp, \quad \text{by } R(X) = R(E) = T,\\ &= S. \end{aligned} \tag{4}$$

ONLY IF: note that X is a right symmetry factor of matrix A and $R(X) = T$, $N(X) = S$. From Lemma 1.1(b), we have

$$AT = AR(X) = R(AX) = N(AX)^\perp \subset N(X)^\perp = S^\perp. \tag{5}$$

On the other hand, from $R(X) = T$ and $\dim(AT) = \dim T = t$, we have

$$t = \dim(AT) = \mathrm{rank}(AX) \le \mathrm{rank}(X) = \dim T = t. \tag{6}$$

Combining (5) and (6) we get $AT = S^\perp$.

Now, we prove that X is independent of the choice of {1}-inverse $(AE)^{(1)}$.

Since $R(M) = AT = R(AE)$, there exists some matrix Y such that $M = AEY$, therefore, we have

$$X = E(AE)^{(1)}M = E(AE)^{(1)}AEY = EP_{R((AE)^{(1)}AE),N((AE)^{(1)}AE)}Y = EP_{R((AE)^{(1)}AE),N(AE)}Y. \quad (7)$$

From $N(AE) \supset N(E)$ and $\operatorname{rank}(E) = t = \dim(AT) = \operatorname{rank}(AE)$, we have

$$N(AE) = N(E). \quad (8)$$

Combining (7) and (8), and Lemma 1.1(a) we get $X = EY$. Obviously, X is independent of the choice of $\{1\}$-inverse $(AE)^{(1)}$. □

In the following theorems, we consider the construction for the generalized inverse $A^{(2,3)}_{T,S}$, which is a {2,3}-inverse of A having the prescribed rang T and null space S.

Theorem 2.2. *Let $A \in C^{m\times n}$, T and S be subspaces of C^n and C^m, respectively, with $\dim(AT) = \dim T = t$. Suppose that E is any matrix satisfying $R(E) = T$ and M is an arbitrary Hermitian idempotent matrix satisfying $R(M) = AT$, then $X = E(AE)^{(1)}M$ is a $\{2, 3\}$-inverse of A and $R(X) = T$, $N(X) = S$ if and only if $AT = S^{\perp}$. Moreover X is independent of the choice of $\{1\}$-inverse $(AE)^{(1)}$.*

Proof. IF: From Theorem 2.1, it is obvious that we need only prove that X is $\{2\}$-inverse of the matrix A.

Since M is an idempotent matrix and $AX = M$ by Theorem 2.1, we have

$$XAX = XM = E(AE)^{-}MM = E(AE)^{-}M = X, \quad (9)$$

namely X is $\{2\}$-inverse of the matrix A.

ONLY IF: since X is a {2,3}-inverse of matrix A with $R(X) = T$, $N(X) = S$, we obtain from Lemma 1.1(b)

$$AT = AR(X) = R(AX) = N(AX)^{\perp} \subset N(X)^{\perp} = S^{\perp}. \quad (10)$$

Furthermore, for X is $\{2\}$-inverse of matrix A, we know

$$\operatorname{rank}(X) = \operatorname{rank}(XAX) \le \operatorname{rank}(AX) \le \operatorname{rank}(X). \quad (11)$$

Therefore, we have $N(AX)^{\perp} = N(X)^{\perp}$, which leads to $AT = S^{\perp}$.

The proof that X is independent of the choice of $\{1\}$-inverse $(AE)^{(1)}$ is analogous to Theorem 2.1. □

Theorem 2.3. *Let $A \in C^{m\times n}$, T and S be subspaces of C^n and C^m, respectively, with $\dim(AT) = \dim T = t$. Suppose that E is a arbitrary matrix satisfying $R(E) = T$, then the matrix A has a unique $A^{(2,3)}_{T,S} = E(AE)^{(1)}P_{AT}$ if and only if $AT = S^{\perp}$.*

Proof. IF: From Theorem 2.2, we know that $X = E(AE)^{(1)}M$ is $\{2, 3\}$-inverse of matrix A and $R(X) = T$, $N(X) = S$, where M is a Hermitian idempotent matrix satisfying $R(M) = AT$ and X is independent of the choice of $\{1\}$-inverse$(AE)^{(1)}$.

So we only need to prove that X is unique and $E(AE)^{(1)}M = E(AE)^{(1)}P_{AT}$.

Since M is Hermitian idempotent matrix satisfying $R(M) = AT$, we know M is unique and $M = P_{AT}$. Furthermore, as $E(AE)^{(1)}M$ is independent of the choice of $\{1\}$-inverse$(AE)^{(1)}$, we have a unique

$$X = E(AE)^{(1)}M = E(AE)^{(1)}P_{AT}.$$

ONLY IF: this result can easily be obtained analogous to Theorem 2.2. □

Corollary 2.4. *Let $A \in C^{m\times n}$, T and S be subspaces of C^n and C^m, respectively, with $\dim T = \dim S^{\perp} = t$. Suppose that E is any matrix satisfying $R(E) = T$ and M is an arbitrary Hermitian idempotent matrix satisfying $R(M) = S^{\perp}$, then $X = E(AE)^{(1)}M$ is a $\{2, 3\}$-inverse of the matrix A and $R(X) = T$, $N(X) = S$ if and only if $AT = S^{\perp}$. Moreover X is independent of the choice of $\{1\}$-inverse $(AE)^{(1)}$.*

Proof. The "**IF**" part can easily be obtained from Theorem 2.2. And we can prove "**ONLY IF**" part and that X is independent of the choice of $\{1\}$-inverse$(AE)^{(1)}$ by the same method as in Theorem 2.2. □

Corollary 2.5. *Let $A \in C^{m\times n}$, T and S be subspaces of C^n and C^m, respectively, with $\dim T = \dim S^{\perp} = t$. Suppose that E is an arbitrary matrix satisfying $R(E) = T$, then the matrix A has a unique $A^{(2,3)}_{T,S} = E(AE)^{(1)}P_{AT}$ if and only if $AT = S^{\perp}$.*

Proof. The proof of Corollary 2.5 is analogous to that of Theorem 2.3. □

3. Applications and examples

The six important generalized inverses: the Moore–Penrose inverse A^+, the weighted Moore–Penrose inverse $A^+_{M,N}$, the Drazin inverse A^D, the group inverse Ag, the Bott–Duffin inverse $A^{(-1)}_{(L)}$ and the generalized Bott–Duffin inverse $A^{(+)}_{(L)}$ are all special generalized inverse $A^{(2)}_{T,S}$, which is {2}-inverse of A having the prescribed range T and the null space S. Combining Lemma 1.2 and Theorem 2.3 or Corollary 2.5, we can get the corresponding explicit expressions for the six important generalized inverses mentioned above. In this section, we shall give the explicit expressions for the Moore–Penrose inverse $A^+$and the weighted Moore–Penrose inverse $A^+_{M,N}$.

Theorem 3.1. *For $A \in C^{m\times n}$, we have $A^+ = A^*(AA^*)^{(1)}P_{R(A)} = A^*Y$, where Y is a matrix satisfying $P_{R(A)} = AA^*Y$.*

Proof. Let $T = R(A^*), S = N(A^*), E = A^*$, we can get

$$\dim(AT) = \operatorname{rank}(AA^*) = \operatorname{rank}(A^*) = \dim(T), \tag{12}$$

$$AT = AR(A^*) = R(AA^*) = R(A) = N(A^*)^\perp = S^\perp. \tag{13}$$

Using Theorem 2.3 and Lemma 1.2(a), we have

$$A^+ = A^*(AA^*)^{(1)}P_{AT} = A^*(AA^*)^{(1)}P_{R(A)}. \tag{14}$$

Since $R(P_{R(A)}) = R(A) = R(AA^*)$ there exists some matrix Y such that

$$P_{R(A)} = AA^*Y. \tag{15}$$

Therefore, combining (14) and (15) and Lemma 1.1(e), we have

$$A^+ = A^*Y,$$

where Y is a matrix satisfying $P_{R(A)} = AA^*Y$. □

Theorem 3.2. *For $A \in C^{m\times n}$, we have*

(1) $A^+_{M,N} = N^{-1}A^*(AN^{-1}A^*)^{-1}$, if A is full row rank matrix.
(2) $A^+_{M,N} = N^{-1}A^*M^{1/2}YM^{1/2}$, where Y is matrix satisfying $P_{R(M^{1/2}A)} = M^{1/2}AN^{-1}A^*M^{1/2}Y$.

Proof. (1) Let $T = N^{-1}R(A^*), S = M^{-1}N(A^*), E = N^{-1}A^*$. Since A is full row rank matrix, we can get

$$\dim(AT) = \operatorname{rank}(AN^{-1}A^*) = \operatorname{rank}(N^{-1/2}A^*) = \operatorname{rank}(N^{-1}A^*) = \dim(T) \tag{16}$$

$$AT = AN^{-1}R(A^*) = R(AN^{-1}A^*) = R(A) = N(A^*)^\perp = (M^{-1}N(A^*))^\perp = S^\perp. \tag{17}$$

Using Theorem 2.3 and Lemma 1.2(b), we have

$$A^+_{M,N} = N^{-1}A^*(AN^{-1}A^*)^{(1)}P_{AT} = N^{-1}A^*(AN^{-1}A^*)^{(1)}P_{R(A)}. \tag{18}$$

Furthermore, since A is full row rank matrix, we obtain from Lemma 1.1(a)

$$A^+_{M,N} = N^{-1}A^*(AN^{-1}A^*)^{-1}P_{R(A)} = N^{-1}A^*(AN^{-1}A^*)^{-1}. \tag{19}$$

(2) From Theorem 1.4.4 of [2], we have

$$A^+_{M,N} = N^{-1/2}(M^{1/2}AN^{-1/2})^+M^{1/2}. \tag{20}$$

Using Theorem 3.1 we get

$$A^+_{M,N} = N^{-1}A^*M^{1/2}(M^{1/2}AN^{-1}A^*M^{1/2})^{(1)}P_{R(M^{1/2}A)}M^{1/2} = N^{-1}A^*M^{1/2}YM^{1/2}, \tag{21}$$

where Y is a matrix satisfying $P_{R(M^{1/2}A)} = M^{1/2}AN^{-1}A^*M^{1/2}Y$. □

Example 1. Let

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Compute the Moore–Penrose inverse A^+.

Solution. From Theorem 3.1, we know $A^+ = A^*Y$, where Y is a matrix satisfying $P_{R(A)} = AA^*Y$. Clearly, we only need compute the matrix Y. From the Lemma 5.3.3 of [8] we have

$$P_{R(A)} = A(A^*A)^{(1)}A^*, \tag{22}$$

which, together with $P_{R(A)} = AA^*Y$, gives

$$\begin{aligned} AA^*Y = A(A^*A)^{(1)}A^* &\Rightarrow A^*AA^*Y = A^*A(A^*A)^{(1)}A^*, \\ &\Rightarrow A^*AA^*Y = A^*, \quad \text{by Lemma 1.1(e)}. \end{aligned} \tag{23}$$

Therefore, Y can be obtained from the matrix equation $A^*AA^*Y = A^*$, namely

$$\begin{pmatrix} 17 & 0 & 12 & -5 \\ 0 & -10 & 0 & 0 \\ 47 & 0 & 33 & -14 \\ 0 & 5 & 0 & 0 \end{pmatrix} Y = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 3 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{24}$$

Using Gaussian elimination, we can easily get one solution of matrix equation (24), namely

$$Y = \begin{pmatrix} -1/3 & 0 & 4/3 & 5/3 \\ 0 & 1/5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4/3 & 0 & 13/3 & 17/3 \end{pmatrix}.$$

Bring Y to $A^+ = A^*Y$, we give

$$A^+ = \begin{pmatrix} -1/3 & 0 & 4/3 & 5/3 \\ 0 & -2/5 & 0 & 0 \\ 1/3 & 0 & -1/3 & -2/3 \\ 0 & 1/5 & 0 & 0 \end{pmatrix}.$$

The following example comes form [13], and the approximate value of the weighted Moore–Penrose inverse $A^+_{M,N}$ was obtained by using the successive matrix squaring algorithm. However, the precision value can be obtained by using Theorem 3.2. Now we illustrate the calculation procedure by the same example.

Example 2. The martix A and the weighted matrices M and N are as follows:

$$A = \begin{pmatrix} 22 & 10 & 2 & 3 & 7 \\ 14 & 7 & 10 & 0 & 8 \\ -1 & 13 & -1 & -11 & 3 \\ -3 & -2 & 13 & -2 & 4 \\ 9 & 8 & 1 & -2 & 4 \\ 9 & 1 & -7 & 5 & -1 \\ 2 & -6 & 6 & 5 & 1 \\ 4 & 5 & 0 & -2 & 2 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 8 \end{pmatrix}_{8\times 8} \quad \text{and} \quad N = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 5 \end{pmatrix}_{5\times 5}.$$

Compute the weighted Moore–Penrose inverse $A^+_{M,N}$.

Solution. From Theorem 3.2 we know $A^+_{M,N} = N^{-1}A^*M^{1/2}YM^{1/2}$, where Y is matrix satisfying $P_{R(M^{1/2}A)} = M^{1/2}AN^{-1}A^*M^{1/2}Y$. Clearly, we only need compute the matrix Y. From (22), we have

$$P_{R(M^{1/2}A)} = M^{1/2}A(A^*MA)^{(1)}A^*M^{1/2}, \tag{25}$$

which, together with $P_{R(M^{1/2}A)} = M^{1/2}AN^{-1}A^*M^{1/2}Y$, gives

$$\begin{aligned} M^{1/2}A(A^*MA)^{(1)}A^*M^{1/2} = M^{1/2}AN^{-1}A^*M^{1/2}Y &\Rightarrow A^*MA(A^*MA)^{(1)}A^*M^{1/2} = A^*MAN^{-1}A^*M^{1/2}Y \\ &\Rightarrow A^*M^{1/2} = A^*MAN^{-1}A^*M^{1/2}Y, \quad \text{by Lemma 1.1(e)}. \end{aligned} \tag{26}$$

Therefore, Y can be obtained from the matrix equation $A^*MAN^{-1}A^*M^{1/2}Y = A^*M^{1/2}$, namely

$$\begin{pmatrix} 194135/4 & 62333\sqrt{2}/2 & 13283\sqrt{3}/4 & -13699 & 42923\sqrt{5}/2 & 74267\sqrt{6}/4 & 6345\sqrt{7}/4 & 10131\sqrt{8} \\ 540787/20 & 176649\sqrt{2}/10 & 223903\sqrt{3}/20 & -44703/5 & 146647\sqrt{5}/10 & 166519\sqrt{6}/20 & -80019\sqrt{7}/20 & 39283\sqrt{8}/5 \\ -26669/20 & 36217\sqrt{2}/10 & -28961\sqrt{3}/20 & 71041/5 & -7529\sqrt{5}/10 & -87113\sqrt{6}/20 & 68493\sqrt{7}/20 & -3381\sqrt{8}/5 \\ 69197/20 & 12623\sqrt{2}/10 & -143869\sqrt{3}/20 & -10531/5 & -6601\sqrt{5}/10 & 71723\sqrt{6}/20 & 71157\sqrt{7}/20 & -5124\sqrt{8}/5 \\ 291609/20 & 111753\sqrt{2}/10 & 60441\sqrt{3}/20 & 4359/5 & 71229\sqrt{5}/10 & 70833\sqrt{6}/20 & 9687\sqrt{7}/20 & 17466\sqrt{8}/5 \end{pmatrix} Y$$

$$= \begin{pmatrix} 22 & 14\sqrt{2} & -\sqrt{3} & -6 & 9\sqrt{5} & 9\sqrt{6} & 2\sqrt{7} & 4\sqrt{8} \\ 10 & 7\sqrt{2} & 13\sqrt{3} & -4 & 8\sqrt{5} & \sqrt{6} & -6\sqrt{7} & 5\sqrt{8} \\ 2 & 10\sqrt{2} & -\sqrt{3} & 26 & \sqrt{5} & -7\sqrt{6} & 6\sqrt{7} & 0 \\ 3 & 0 & -11\sqrt{3} & -4 & -2\sqrt{5} & 5\sqrt{6} & 5\sqrt{7} & -2\sqrt{8} \\ 7 & 8\sqrt{2} & 3\sqrt{3} & 8 & 4\sqrt{5} & -\sqrt{6} & \sqrt{7} & 2\sqrt{8} \end{pmatrix}.$$

Using Gaussian elimination and Maple, we can easily get one solution of matrix equation above, namely

$$Y = \begin{pmatrix} \frac{836322}{329166541} & \frac{157187\sqrt{2}}{498198008} & -\frac{125915659\sqrt{3}}{9216663148} & -\frac{12945357}{9216663148} & -\frac{7715501\sqrt{5}}{2633332328} & \frac{10958599\sqrt{6}}{2304165787} & \frac{31263257\sqrt{7}}{4608331574} & -\frac{50329593\sqrt{2}}{9216663148} \\ -\frac{2738575\sqrt{2}}{1974999246} & \frac{376655}{1494594024} & \frac{171132267\sqrt{6}}{18433326296} & \frac{68155639\sqrt{2}}{55299978888} & \frac{33825703\sqrt{10}}{15799993968} & -\frac{44369021\sqrt{3}}{6912497361} & -\frac{124753589\sqrt{14}}{27649989444} & \frac{212217827}{27649989444} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1072541}{987499623} & \frac{4000679\sqrt{2}}{5978376096} & -\frac{224351749\sqrt{3}}{36866652592} & \frac{118318855}{110599957776} & -\frac{40954169\sqrt{5}}{31599987936} & \frac{43958287\sqrt{6}}{27649989444} & \frac{185761465\sqrt{7}}{55299978888} & -\frac{273060709\sqrt{2}}{110599957776} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Bring Y to $A^+_{M,N} = N^{-1}A^*M^{1/2}YM^{1/2}$, we get

$$A^+_{M,N} = \begin{pmatrix} \frac{10421956}{987499623} & \frac{2756141}{213513432} & \frac{-32370117}{2633332328} & \frac{-22197971}{3949998492} & \frac{257680295}{15799993968} & \frac{10332517}{329166541} & \frac{76365037}{3949998492} & \frac{16825001}{1974999246} \\ \frac{543157}{658333082} & \frac{39899}{17792786} & \frac{17477253}{658333082} & \frac{-352263}{329166541} & \frac{19278775}{1316666164} & \frac{-3683352}{329166541} & \frac{-20189813}{658333082} & \frac{5921374}{329166541} \\ \frac{1839416}{987499623} & \frac{2925151}{213513432} & \frac{344281}{2633332328} & \frac{130757791}{3949998492} & \frac{85873085}{15799993968} & \frac{-8781881}{329166541} & \frac{32776261}{1316666164} & \frac{1438289}{658333082} \\ \frac{1618367}{1974999246} & \frac{-92357}{106756716} & \frac{-16435203}{1316666164} & \frac{-8386681}{1974999246} & \frac{-35606675}{7899996984} & \frac{3904936}{329166541} & \frac{11950981}{987499623} & \frac{-6425201}{987499623} \\ \frac{846898}{987499623} & \frac{409015}{106756716} & \frac{3338145}{1316666164} & \frac{14572295}{1974999246} & \frac{26671765}{7899996984} & \frac{-2108858}{329166541} & \frac{6004523}{1974999246} & \frac{2705707}{987499623} \end{pmatrix}.$$

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