Compactification and local connectedness of frames

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Abstract


A classical result in the theory of Tychonoff spaces is that, for any such space X, its Stone Čech compactification βX is locally connected iff X is locally connected and pseudocompact. Since all concepts involved in this generalize from spaces to frames, it is natural to ask whether this result already holds for the latter, and the main purpose of this paper is to show this is indeed the case (Proposition 2.3). Further, for normal regular frames, we obtain the frame counterpart of an analogous result of Wallace in terms of a certain property of covers (Proposition 3.5). Finally, we establish a number of additional results concerning connectedness which seem to be of independent interest.

0. Preliminaries

Recall that a frame is a complete lattice L satisfying the distribution law

\[ a \lor \bigvee S = \bigvee \{ a \land x \mid x \in S \} \quad (a \in L, S \subseteq L) \]

and a frame homomorphism is a map \( h : M \to L \) between frames preserving arbitrary joins, including the zero 0, and all finitary meets, including the unit e. A frame homomorphism \( h : M \to L \) is called dense if \( h(x) = 0 \) implies \( x = 0 \) for all \( x \in M \).

A frame \( L \) is called regular if \( a = \bigvee\{ x \in L \mid x < a \} \) for each \( a \in L \), where \( x < a \) means that \( x \land y = 0 \) and \( a \lor y = e \) for some \( y \in L \). An element \( a \in L \) is called compact if \( a \leq \bigvee S \) implies \( a \leq \bigvee E \) for some finite \( E \subseteq S \), for all \( S \subseteq L \). \( L \) itself is called compact whenever the unit \( e \in L \) is compact. A compactification of a frame \( L \) is a compact regular frame \( M \) together with a dense onto homomorphism \( h : M \to L \).
For arbitrary frames $L$, one has the compact regular coreflection of $L$, realized by the largest regular subframe $\mathcal{R}L$ of the frame $\mathcal{Q}L$ of all ideals of $L$ and the homomorphism $\mathcal{V}: \mathcal{R}L \to L$ given by taking joins [5]. A frame $L$ has compactifications iff this homomorphism is onto; such $L$ are called compactifiable. For a general study of compactifications of frames see [4].

A frame $L$ is called completely regular if, for each $a \in L$, $a = \bigvee \{x \in L \mid x \ll a\}$ where $x \ll a$ means there exists a doubly indexed sequence $(x_{nk})_{n=0,1,...;k=0,1,...,2^n}$ such that

$$x = x_{n0}, \quad x_{nk} \ll x_{nk+1}, \quad x_{n2^n} = a, \quad x_{nk} = x_{n+1,2k}$$

for all $n=0,1,...$ and $k=0,1,...,2^n$. We note that, assuming the Axiom of Countably Dependent Choice (CDC), any compact regular frame is completely regular (since $x \ll y$ implies $x \ll z \ll y$ for some $z$ in any such frame), although in general this is believed not to be the case.

For any frame $L$, its compact completely regular coreflection is given by the join map $\mathcal{V}: \mathcal{CR}L \to L$ where $\mathcal{CR}L$ is the frame of all completely regular ideals $J \subseteq L$, that is, those ideals $J$ such that, for each $a \in J$, there exist $b \in J$ for which $a \ll b$. The dense homomorphism $\mathcal{V}: \mathcal{CR}L \to L$ is onto iff $L$ is completely regular. This, incidentally, leads to the characterization that a frame has completely regular compactifications iff it is completely regular.

Since a $T_0$-space $X$ is completely regular iff the frame $\mathcal{O}X$ of its open sets is completely regular, and $X$ is compact iff the frame $\mathcal{O}X$ is compact, it is clear that the compact completely regular coreflection of a completely regular frame $L$ is the exact frame counterpart of the Stone–Čech compactification $\beta X$ of a Tychonoff space $X$, and we shall therefore also refer to it as $\beta L$.

A frame $L$ is called normal whenever, for any $a, b \in L$ such that $a \lor b = e$, there exist $u, v \in L$ such that $u \lor b = e = a \lor v$ and $u \land v = 0$. Any normal regular frame $L$ is compactifiable, and its compact regular coreflection is given by the frame $\mathcal{R}L$ of all regular ideals $J \subseteq L$, with the join map $\mathcal{V}: \mathcal{R}L \to L$, where an ideal $J$ is called regular if, for each $a \in J$, there exist $b \in J$ for which $a \ll b$. Again, it should be noted that, with CDC, any normal regular frame $L$ is completely regular and $\mathcal{R}L = \beta L$, but without this axiom one expects this not to be the case.

For general background on frames we refer to [10], but it might be noted that our way of introducing $\beta L$ is different from Johnstone’s. The latter, although later on shown to be equivalent to ours, uses the frame translation of the original approach by Tychonoff and Čech whereas our choice is rooted in the treatment of complete regularity and compactifications which evolved later, eliminating the use of the real numbers from this context. It should also be pointed out that we follow [5] in the definition of complete regularity which differs slightly from that used in [10].

1. Connectedness

We follow Kříž and Pultr [11] in defining connectedness except that, for us, the
zero of a frame is always connected whereas the definition in [11] applies only to non-zero elements.

**Definition 1.1.** (i) For any frame $L$, $c \in L$ is called connected if $c = a \lor b$ and $a \land b = 0$ implies $c = a$ or $c = b$, for any $a, b \in L$.

(ii) A frame $L$ is called connected whenever the unit $e \in L$ is connected.

(iii) A frame $L$ is called locally connected if each element of $L$ is a join of connected elements.

(iv) For any frame $L$ and $a \in L$, a (connected) component of $a$ is a maximal connected $c \leq a$.

In the following, we call a subset $S$ of a frame chained if, for any $a, b \in S$, there exist $t_0, t_1, \ldots, t_n \in S$ such that $a = t_0$, $t_k \land t_{k+1} \neq 0$ for all $k = 0, 1, \ldots, n - 1$, and $b = t_n$.

**Lemma 1.2.** In any frame $L$, the join of any chained set $S$ of connected elements is connected.

**Proof.** Let $c = \bigvee S$ and consider any $a, b \in L$ such that $c = a \lor b$ and $a \land b = 0$. Then, for any $t \in S$, $t \leq a$ or $t \leq b$ since $t$ is connected. Now, suppose there exist $u, v \in S$ such that $u \leq a$ and $v \leq b$. Then, by hypothesis, there exist $t_0, t_1, \ldots, t_n \in S$ for which $u = t_0$, $t_k \land t_{k+1} \neq 0$ for all $k = 0, 1, \ldots, n - 1$, and $b = t_n$. Here, if $i$ is the first index such that $t_i \leq b$ then $0 < i$ and $0 < t_{i-1} \land t_i \leq a \land b = 0$, a contradiction. It follows that $t \leq a$ for all $t \in S$ or $t \leq b$ for all $t \in S$, and therefore $c = a$ or $c = b$, as desired. □

**Corollary 1.3.** In any locally connected frame $L$, different components of any $a \in L$ are disjoint.

**Proof.** If $c \land d \neq 0$ for components $c$ and $d$ of $a$ then $c \lor d$ is connected by Lemma 1.2, and hence $c = c \lor d = d$ by the maximality of $c$ and $d$. □

**Corollary 1.4.** In any locally connected frame $L$, each element is the join of its components.

**Proof.** For any $a \in L$, let $C$ be the set of all connected elements $\neq 0$ below $a$, and put $x - y$ iff there exist $z_0, z_1, \ldots, z_n \in C$ such that $x = z_0$, $z_k \land z_{k+1} \neq 0$ for all $k = 0, 1, \ldots, n - 1$ and $y = z_n$. This defines an equivalence relation on $C$ whose blocks $B \subseteq C$ are chained so that $\bigvee B \in C$. Moreover, for any $x \in C$ and any such block $B$, if $x \land y \neq 0$ for some $y \in B$ then $x \in B$. It follows that each $\bigvee B$ is maximal in $C$ and thus a component of $a$. This proves the result since $a = \bigvee C$. □

**Remark.** If $S$ is a set of non-zero, pairwise disjoint connected elements in a locally connected frame then $S$ is the set of components of $\bigvee S$.

In the following, we characterize connectedness of frames in a manner which cor-
responds to the result that a topological space X is connected iff each continuous map from X into a discrete space is constant. Here, the relevant comparison frame is the Boolean algebra \( \mathbb{4} \) of four elements. Also, 2 is the two-element frame.

**Lemma 1.5.** A frame \( L \) is connected iff each homomorphism \( \mathbb{4} \to L \) factors through the unique homomorphism \( 2 \to L \).

**Proof.** (\( \Rightarrow \)) If \( a \) and \( b \) are the non-zero elements of \( \mathbb{4} \) then, for any \( h: \mathbb{4} \to L \), \( h(a)\vee h(b) = e \) and \( h(a)\wedge h(b) = 0 \), hence \( h(a) = e \) or \( h(b) = e \) so that we can define \( \overline{h}: \mathbb{4} \to 2 \) by \( \overline{h}(a) = 1 \) or \( \overline{h}(a) = 0 \), giving the desired factorization.

(\( \Leftarrow \)) If \( u \vee v = e \) and \( u \wedge v = 0 \) we obtain \( h: \mathbb{4} \to L \) by letting \( h(a) = u \) and \( h(b) = v \), and since this factors through \( 2 \to L \) we must have \( u = e \) or \( v = e \), as claimed. \( \Box \)

**Lemma 1.6.** Any dense homomorphism \( h: M \to L \) reflects connectedness, that is, \( c \in M \) is connected whenever \( h(c) \) is connected.

**Proof.** Let \( c = a \vee b \) and \( a \wedge b = 0 \). Then \( h(c) = h(a) \vee h(b) \) and \( h(a) \wedge h(b) = 0 \), hence \( h(c) = h(a) \) or \( h(c) = h(b) \), therefore \( h(b) = 0 \) or \( h(a) = 0 \), consequently \( b = 0 \) or \( a = 0 \) by denseness, and finally \( c = a \) or \( c = b \). \( \Box \)

**Proposition 1.7.** A frame \( L \) is connected iff its compact (completely) regular co-reflection is connected.

**Proof.** Let \( u: K \to L \) be either of the coreflections. Then \( u \) is dense, and hence \( K \) is connected whenever \( L \) is, by Lemma 1.6. Conversely, assume that \( K \) is connected and consider any \( h: \mathbb{4} \to L \). Now, since \( \mathbb{4} \) is compact and completely regular, there exists \( f: \mathbb{4} \to K \) such that \( uf = h \). On the other hand, \( f: \mathbb{4} \to K \) factors through \( 2 \to K \) by Lemma 1.5, and therefore \( h: \mathbb{4} \to L \) factors through \( 2 \to L \). This shows \( L \) is connected, again by Lemma 1.5. \( \Box \)

For the following, recall that any frame homomorphism \( h: M \to L \) has a right adjoint \( r: L \to M \), defined by the condition that \( h(x) \leq y \) iff \( x \leq r(y) \) for all \( x \in M \) and \( y \in L \), or explicitly as

\[
    r(y) = \bigvee \{ x \in M \mid h(x) \leq y \}.
\]

In particular, if \( h \) is onto then \( r(y) \) is the largest element which \( h \) maps to \( y \). Also, for any \( h \), \( r \) preserves arbitrary meets.

**Lemma 1.8.** Any dense onto homomorphism \( h: M \to L \) whose right adjoint preserves disjoint binary joins preserves connectedness.

**Proof.** Let \( r: L \to M \) be the right adjoint and consider any connected \( c \in M \). If \( h(c) = a \vee b \) and \( a \wedge b = 0 \) then \( rh(c) = r(a) \vee r(b) \) and \( r(a) \wedge r(b) = r(0) = 0 \), the last step
since \( h \) is dense, and hence \( c = (c \wedge r(a)) \vee (c \wedge r(b)) \) because \( c \leq rh(c) \). It follows that \( c = c \wedge r(a) \) or \( c = c \wedge r(b) \) and consequently \( h(c) = a \) or \( h(c) = b \), as desired. \( \square \)

**Lemma 1.9.** In any frame \( L \), if \( x \ll a \vee b \) and \( a \wedge b = 0 \), then \( x \wedge a \ll a \).

**Proof.** As a first step, we have that \( x < a \vee b \) and \( a \wedge b = 0 \) implies \( x \wedge a < a \): take \( y \) such that \( x \wedge y = 0 \) and \( a \vee b \wedge y = e \) and observe that this implies \( (x \wedge a) \wedge (b \vee y) = 0 \) and \( a \vee (b \wedge y) = e \). Now, the same hypothesis implies \( x \wedge b < b \) by symmetry, and then one readily sees that \( x_1 < x_2 < \cdots < x_n < a \vee b \) and \( a \wedge b - 0 \) implies \( x_1 \wedge a < x_2 \wedge a < \cdots < x_n \wedge a < a \). This yields the desired result by the definition of the relation \( \ll \). \( \square \)

**Corollary.** For any frame \( L \), the right adjoint of \( \forall : \beta L \to L \) preserves disjoint binary joins.

**Proof.** It is a familiar fact, but also easily checked directly, that the right adjoint in question is explicitly given as the map \( k : L \to \beta L \) defined by \( k(a) = \{ x \in L \mid x \ll a \} \). Now, for any disjoint \( a, b \in L \), \( x \ll a \vee b \) implies \( x = (x \wedge a) \vee (x \wedge b) \) where \( x \wedge a \ll a \) and \( x \wedge b \ll b \) by Lemma 1.9, and hence \( k(a \vee b) \subseteq k(a) \vee k(b) \), the latter being the join of ideals and hence the join in \( \beta L \). This proves \( k(a \vee b) = k(a) \vee k(b) \), as claimed. \( \square \)

The last result is of particular interest in the case of completely regular \( L \), characterized by the condition that \( \forall : \beta L \to L \) is onto. For such \( L \), Lemma 1.8 and the Corollary of Lemma 1.9, together with Lemma 1.6, prove the following:

**Proposition 1.10.** For any completely regular frame \( L \), \( \forall : \beta L \to L \) preserves and reflects connectedness. \( \square \)

**Remark.** We do not know whether the corresponding result holds for compactifiable frames and their compact regular coreflection, basically because we do not know much about the right adjoint in that situation, except for the special case of normal regular frames which will be discussed later.

2. **Local connectedness of \( \beta L \)**

**Definition 2.1.** A frame \( L \) is called pseudocompact if any sequence \( a_0 \ll a_1 \ll a_2 \ll \cdots \) in \( L \) with \( \forall a_n = e \) terminates, that is, \( a_k = e \) for some \( k \).

**Remark.** Recall that a real-valued continuous function on a frame \( L \) is a homomorphism to \( L \) from the frame \( L(\mathbb{R}) \) of real numbers. Such \( \varphi : L(\mathbb{R}) \to L \) is called bounded whenever \( \varphi(U) = 0 \) for some \( U = (-\infty, -n) \vee (n, \infty) \) in \( L(\mathbb{R}) \), \( n \) some natural number, and \( L \) itself is usually called pseudocompact if all \( \varphi : L(\mathbb{R}) \to L \) are bounded. Moreover, one can prove that a space \( X \) is pseudocompact if \( \partial X \) is a pseudocompact
frame. On the other hand, Gilmour [8] has recently shown how to express the pseudocompactness of a frame $L$ in terms of a 'cover condition': $L$ is pseudocompact iff any sequence $a_0 \ll a_1 \ll a_2 \ll \cdots$ in $L$ such that $\bigvee a_n = e$ terminates. We have adopted the latter condition as our definition of pseudocompactness since this eliminates the reference to the reals from our context and hence falls in line with the definition of complete regularity for frames and our approach to $\beta L$.

The following technical lemma is related to Lemma 4.5 of [7].

**Lemma 2.2.** In any locally connected pseudocompact frame $L$, if $a \ll b$, then only finitely many components of $b$ meet $a$.

**Proof.** Suppose there are components $c_n (n \in \omega)$ of $b$, different for different $n$, such that all $a \wedge c_n \neq 0$, and put $c = \bigvee c_n$. Then $a \wedge c \ll c$ by Lemma 1.9, and hence we have $u_k (k \in \omega)$ such that $a \wedge c \ll u_0 \ll u_1 \ll \cdots \ll c$. Further, by the familiar fact that $a \wedge c \ll u_0$ implies $u_0 \ll (a \wedge c)^*$, there exist $u_k (k \in \omega)$ for which $u_0 \ll u_0 \ll u \ll \cdots \ll (a \wedge c)^*$. Now consider the sequence

$$w_0 = u_0, \quad w_1 = u_1 \vee (u_0 \wedge c_0), \quad \ldots, \quad w_n = u_n \vee (u_{n-1} \wedge c_1) \vee \cdots \vee (u_0 \wedge c_{n-1}), \ldots$$

Here, $w_n \ll w_{n+1}$ for each $n$, by the usual properties of $\ll$ relative to $\vee$, and since $u_k \wedge c_n \ll u_{k+1} \wedge c_n$ for all $k$ and $n$ by Lemma 1.9,

$$\bigvee w_n \geq u_0 \vee (\bigvee u_1 \wedge c_1) \geq u_0 \vee u = e.$$ 

Hence, by pseudocompactness, $w_n = e$ for some $n$ and thus, a fortiori, $u_n \vee c_0 \vee \cdots \vee c_{n-1} = e$. Now, since

$$(a \wedge c_k) \wedge c_n \leq (a \wedge c_k) \wedge (a \wedge c)^* = 0$$

for any $k$ and $n$, and $(a \wedge c_k) \wedge c_i = 0$ for all $i < n$ and $k \geq n$, it follows that $a \wedge c_k = 0$ for all $k \geq n$, a contradiction. \qed

**Remark.** It may be worth emphasizing that the above argument is choice-free. If $(z_{nk})_{n, k=0, 1, \ldots}^T$ is any sequence exhibiting the fact that $x \ll y$ for some $x$ and $y$, then, for any $n$ and $k$, one has an explicitly described subsequence of $(z_{nk})$ which shows that $z_{nk} \ll z_{nk+1}$, and hence $x \ll z_{11} \ll z_{23} \ll z_{37} \ll \cdots \ll y$. This, together with the explicit nature of the proof of Lemma 1.9 and of the argument that $x \ll y$ implies $y^* \ll x^*$, shows that the above $u_k$ and $v_k$ can be directly described in terms of the given hypothesis that $a \ll b$.

We are now ready to prove our main result.

**Proposition 2.3.** For any completely regular frame $L$, $\beta L$ is locally connected iff $L$ is locally connected and pseudocompact.
Proof. (⇒) By Proposition 1.10, $L$ is locally connected. For pseudocompactness, we first show that, whenever $a << b$ in $L$, only finitely many components of $b$ meet $a$. To begin with, note that $k(c)$ is a component of $k(b)$ for each component $c$ of $b$: $k(c)$ is connected by Proposition 1.10, and, for the corresponding component $J \subseteq k(c)$ of $k(b)$, $b \geq \bigvee J \geq c$ shows that $\bigvee J = c$, again by that proposition, therefore $J \subseteq k(c)$ and hence equality. Next, there is the familiar fact that $a << b$ in $L$ implies $k(a) \prec k(b)$ in $\beta L$ [4], and by compactness $k(a)$ is therefore covered by finitely many components of $k(b)$. Hence $k(a)$ meets only finitely many of the $k(c)$—but then $a$ meets only finitely many of the $c$.

Now take any sequence $a_0 << a_1 << \cdots$ in $L$ such that $\bigvee a_n = e$ and define

$$u = (a_2^* \land a_3^*) \lor (a_5^* \land a_9^*) \lor (a_7^* \land a_4^*) \lor \cdots$$

and

$$v = (a_1^* \land a_4^*) \lor (a_5^* \land a_{10}^*) \lor (a_9^* \land a_{10}^*) \lor \cdots.$$  

We claim that $u << v$. Clearly, by familiar properties of $\ll$ [4],

$$a_2^* \land a_3^* \ll a_1^* \land a_4^*, \quad a_5^* \land a_9^* \ll a_7^* \land a_{10}^*, \quad \cdots.$$  

Then, consider any $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$ such that

$$a_2^* \land a_3 \leq x_0 < y_0 \leq a_1^* \land a_4, \quad a_5^* \land a_9 \leq x_1 < y_1 \leq a_7^* \land a_{10}, \quad \cdots.$$  

Putting $x = \bigvee x_n$ and $y = \bigvee y_n$, we want to prove that $x < y$. For this, let

$$z = (x_0^* \land x_3^*) \lor (x_5^* \land x_9^*) \lor (x_7^* \land x_{10}) \lor \cdots.$$  

Then

$$x_0 \land z = 0 \quad \text{since} \quad x_0 \leq a_4,$$

$$x_1 \land z = 0 \quad \text{since} \quad x_1 \leq a_7^* \leq a_5^* \quad \text{and} \quad x_1 \leq a_{10},$$

$$x_2 \land z = 0 \quad \text{since} \quad x_2 \leq a_{13}^* \leq a_{11}^* \leq a_5^* \quad \text{and} \quad x_2 \leq a_{16},$$

$$\cdots$$

so that, in all, $x \land z = 0$. On the other hand

$$y_0 \lor z \geq y_0 \lor (x_0^* \land x_3^*) = a_5 \quad \text{since} \quad y_0 \lor x_0^* = e \quad \text{and} \quad y_0 \leq a_4,$$

$$y_1 \lor z \geq y_1 \lor (a_4^* \land x_4^* \land a_{11}) = (y_1 \lor a_4^*) \land a_{11} \quad \text{since} \quad y_1 \lor x_1^* = e \quad \text{and} \quad y_1 \leq a_{11},$$

$$y_2 \lor z \geq y_2 \lor (a_{10}^* \land x_7^* \land a_{17}) = (y_2 \lor a_{10}^*) \land a_{17} \quad \text{since} \quad y_2 \lor x_2^* = e \quad \text{and} \quad y_2 \leq a_{17},$$

$$\cdots$$

and hence

$$y_0 \lor y_1 \lor z \geq a_5 \lor ((y_1 \lor a_{11}^*) \land a_{11}) = a_{11},$$

$$y_0 \lor y_1 \lor y_2 \lor z \geq a_{13}^* \lor ((y_2 \lor a_{17}^*) \land a_{17}) = a_{17},$$

$$\cdots$$

showing that $y \lor z = e$ since $\bigvee a_n = e$ and $a_0 \leq a_1 \leq \cdots$. This proves $x < y$, as desired. It follows now that the information given by
pieces together so that we obtain \( u \ll v \).

Now, by the first part of this proof, \( u \) is covered by finitely many components of \( v \), and since the elements in the definition of \( v \) are pairwise disjoint it follows that there exists a \( k \) for which

\[
(a_1^* \land a_2) \lor \cdots \lor (a_{6k+1}^* \land a_{6k+4}) = 0.
\]

Hence \( a_{6n+2} \land a_{6n+3} = 0 \), or \( a_{6n+3} = a_{6n+2}^* \), for all \( k > n \), and by shifting the indices appropriately we obtain the analogous relations

\[
a_{6n+4} = a_{6n+3}^*, \quad a_{6n+5} = a_{6n+4}^*, \quad \cdots
\]

for all sufficiently large \( n \) so that, finally, \( a_{l+1} = a_l^* \) for all sufficiently large \( l \). For these \( l \), we then have

\[
a_{l+2} = a_{l+1}^* = (a_l^*)^* = a_l^* = a_{l+1},
\]

showing that the sequence does indeed terminate.

(\( \Leftarrow \)) For any \( J \in \beta L \) and \( a \in J \), take \( b \in J \) such that \( a \ll b \). Then \( a \land c \ll c \) for each component \( c \) of \( b \) by Lemma 1.9 so that \( a \land c \in k(c) \) where \( k(c) \) is connected, as noted before, and \( k(c) \subseteq J \) trivially. Now, \( a \land c = 0 \) for all but finitely many \( c_1, \ldots, c_n \) of these \( c \) by Lemma 2.2, and therefore

\[
a = (a \land c_1) \lor \cdots \lor (a \land c_n) \in k(c_1) \lor \cdots \lor k(c_n) \subseteq J,
\]

showing that \( J \) is the join of all connected \( k(x) \subseteq J \). This proves that \( \beta L \) is locally connected. \( \square \)

Remark. The above proof also shows: a locally connected frame \( L \) is pseudocompact iff, for any \( a \ll b \), only finitely many components of \( b \) meet \( a \).

3. Normal regular frames

For these frames, we consider the universal compactification \( \nabla : \mathcal{R}L \rightarrow L \) mentioned earlier, where \( \mathcal{R}L \) is the frame of all regular ideals of \( L \). As in the case of \( \beta L \), the right adjoint \( r : L \rightarrow \mathcal{R}L \) has a convenient explicit description: for any \( a \in L \), \( r(a) = \{ x \in L \mid x \prec a \} \). That this is an ideal follows immediately from the general properties of the relation \( \prec \); its regularity, on the other hand, is a specific consequence of normality: if \( x \land y = 0 \) and \( a \lor y = e \), take \( u \) and \( v \) in \( L \) such that \( u \lor y = e = a \lor u \) and \( u \land v = 0 \), and note that \( x \prec u \prec a \). Finally, it is quite obvious that \( \nabla J \leq a \) iff \( J \subseteq r(a) \) for any regular ideal \( J \) and any \( a \in L \).

Lemma 3.1. The map \( r : L \rightarrow \mathcal{R}L \) is a lattice homomorphism.
Proof. Since $r$ preserves zero and unit as well as arbitrary meets it is sufficient to show that it preserves all binary joints. For this, first note that $x < a \lor b$ implies that $x < c \lor b$ for some $c < a$: if $y$ is such that $x \land y = 0$ and $a \lor b \lor y = e$, take $u$ and $v$ such that $u \lor b \lor y = e = a \lor v$ and $u \lor y = 0$ to obtain $x < u \lor b$ and $u < a$. It follows now that $x < a \lor b$ implies $x \leq c \lor d$ for suitable $c < a$ and $d < b$, showing that $r(a \lor b) \subseteq r(a) \lor r(b)$, giving the non-trivial part of the desired identity $r(a \lor b) = r(a) \lor r(b)$. □

An immediate consequence of this result, by Lemmas 1.6 and 1.8, is the following counterpart of Proposition 1.10:

**Proposition 3.2.** For any normal regular frame $L$, $\vee : \mathcal{R}L \rightarrow L$ preserves and reflects connectedness. □

The following introduces a property which is of particular significance in the present context.

**Definition 3.3.** A regular frame $L$ is said to have property $WS$ if every finite cover of $L$ has a finite refinement consisting of connected elements.

**Remark.** This notion evolved in topology in the following way: Motivated by a similar concept used by Wilder [14], a space $X$ is said to have property S in [12] if every finite cover of $X$ has a refinement consisting of connected sets. Later on, Henriksen and Isbell [9] show that, for regular $X$, this holds iff every finite open cover of $X$ has a finite refinement consisting of connected open sets, that is, iff the frame $\mathcal{D}X$ has the above defined property WS. We prefer this terminology to the historically perhaps more natural 'property S' because (i) it seems appropriate to emphasize that this concept originates with Wilder [14], and (ii) we want to reserve the latter term for a property of uniform frames, to be studied in a later paper, which will be the direct frame counterpart of the property S considered by Sierpinski for metric spaces (see, for instance, [13]) and later by Collins [6] for uniform spaces.

The remarkable strength of property WS is shown by the following lemma:

**Lemma 3.4.** Any regular frame $L$ with property WS is locally connected and pseudocompact.

**Proof.** For local connectedness, consider any $x < a$ in $L$. Then, $\{a, x^*\}$ is a cover of $L$, and if $C \subseteq L$ is a finite refinement consisting of connected elements provided by WS then, for any $c \in C$, $c \leq a$ whenever $x \land c \neq 0$ and therefore

$$x = \bigvee \{x \land c \mid c \in C\} \leq \bigvee \{c \in C \mid x \land c \neq 0\} \leq a.$$ 

The regularity of $L$ now immediately implies that $a$ is a join of connected elements.
To obtain pseudocompactness, consider any sequence $a_0 << a_1 << \cdots$ in $L$ such that $\bigvee a_n = e$. Then, define

$$u = a_1 \lor (a_2 \land a_1^*) \lor (a_3 \land a_2^*) \lor \cdots \quad \text{and} \quad v = (a_2 \land a_0^*) \lor (a_4 \land a_2^*) \lor \cdots$$

and note that $uv = e$ since $a_n \leq uv$ for each $n$ and $\bigvee a_n = e$. Now, if $C$ is a finite refinement of the cover $\{u, v\}$ obtained from WS then each $c \in C$ is below one of the elements

$$a_1, a_3 \land a_1^*, a_5 \land a_3^*, \ldots \quad \text{or} \quad a_2 \land a_0^*, a_4 \land a_2^*, \ldots$$

since $c \leq u$ or $x \leq v$, and the elements in either sequence are disjoint. It follows that $a_1 \lor (a_2 \land a_0^*) \lor \cdots \lor (a_n \land a_{n-2}^*) = e$ for sufficiently large $n$, and then $a_{n+2} = e$. □

The following result characterizes property WS in terms of $\mathfrak{R}L$ for the frames under consideration here.

**Proposition 3.5.** For any normal regular frame $L$, $\mathfrak{R}L$ is locally connected iff $L$ has property WS.

**Proof.** (⇒) If $a_1 \lor \cdots \lor a_n = e$ in $L$ then $r(a_1) \lor \cdots \lor r(a_n) = L$ in $\mathfrak{R}L$ by Lemma 3.1. Hence, by the compactness and local connectedness of $\mathfrak{R}L$, the cover $\{r(a_1), \ldots, r(a_n)\}$ of $\mathfrak{R}L$ is refined by a finite cover $C$ of connected elements, and then $\{\bigvee J \mid J \in C\}$ is a finite cover of $L$, refining $\{a_1, \ldots, a_n\}$ since $J \subseteq r(a_i)$ implies $\bigvee J \leq a_i$, and consisting of connected elements by Proposition 3.2.

($\Leftarrow$) For any $J \in \mathfrak{R}L$ and $a \in J$, take $b \in J$ such that $a < b$ and let $C \subseteq L$ be a refinement of the corresponding cover $\{b, a^*\}$ provided by WS. Then, $\bigvee C = e$ implies $\bigvee \{r(c) \mid c \in C\} = L$ by Lemma 3.1, and hence there exist $x_c \in r(c)$ for each $c \in C$ such that $\bigvee \{x_c \mid c \in C\} = e$. Now, for any $c \in C$, $a \land x_c \neq 0$ implies $a \land c \neq 0$, hence $c \leq a^*$ and therefore $c \leq b$. This means that $\bigvee \{r(c) \mid a \land x_c \neq 0\}$ is contained in $J$, and since $a$ is the join of the $a \land x_c \neq 0$ it belongs to $\bigvee \{r(c) \mid a \land x_c \neq 0\}$. It follows that $J$ is the join of all $r(x) \subseteq J$ with connected $x$, and since these $r(x)$ are connected by Proposition 3.2, this proves that $\mathfrak{R}L$ is locally connected. □

Alternatively, the property WS can be characterized as follows, for the frames presently under consideration, albeit under the assumption of an appropriate choice principle:

**Proposition 3.6.** Assuming CDC, a normal regular frame has property WS iff it is locally connected and pseudocompact.

**Proof.** As noted earlier, CDC here implies complete regularity and the identity $\mathfrak{RL} = \beta L$. Therefore, if $L$ is locally connected and pseudocompact then $\mathfrak{RL}$ is locally connected by Proposition 2.3, and hence $L$ has property WS by Proposition 3.5. The converse follows from Lemma 3.4. □
**Remark.** There are other ways of obtaining the ‘if’ part of this result, without explicit use of Propositions 2.3 and 3.5, but we do not know any argument avoiding CDC. On the other hand, even with the full Axiom of Choice, we do not know whether, say for completely regular $L$, property WS is strictly stronger than local connectedness plus pseudocompactness. In view of the Remark after Proposition 2.3, this is equivalent to the question whether, for a completely regular locally connected frame $L$, property WS is strictly stronger than the condition that, for any $a \ll b$ in $L$, only finitely many components of $b$ meet $a$.

The proof of Proposition 3.5 makes crucial use of Lemma 3.1. It may therefore be of interest to note that this lemma essentially characterizes the situation in question:

**Proposition 3.7.** If $h: M \to L$ is a compactification whose adjoint $q: L \to M$ is a lattice homomorphism then $L$ is normal regular and $h: M \to L$ is isomorphic to $\forall : \mathcal{RL} 	o L$.

**Proof.** If $a \vee b = e$ in $L$ then $q(a) \vee q(b) = e$ in $M$, and by the normality of compact regular frames there exist $s$ and $t$ in $M$ for which $q(a) \vee t = e = s \vee q(b)$ and $s \wedge t = 0$. Then $u = h(s)$ and $v = h(t)$ satisfy the conditions $a \wedge u = a \wedge v = e$, $u \wedge v = 0$, showing that $L$ is normal. Being regular anyway, we have that $L$ is normal regular as claimed.

It follows that $\forall : \mathcal{RL} \to L$ is the universal compactification of $L$, and thus there exists a homomorphism $\bar{h}: M \to \mathcal{RL}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{RL} & \xrightarrow{\forall} & L \\
\downarrow \bar{h} & & \downarrow h \\
M & & \\
\end{array}
$$

We claim that $\bar{h}$ is an isomorphism. Clearly, $\bar{h}$ is dense since $h$ is, and by a familiar property of compact regular frames this makes it one-one. Thus, it remains to show $\bar{h}$ is onto. This can be achieved by proving that all $r(a) = \{x \in L \mid x < a\}$, $a \in L$, belong to the image of $\bar{h}$ since they generate $\mathcal{RL}$: each regular ideal $J$ is the join of all $r(a)$, $a \in J$. Indeed, we claim that $\bar{h}q(a) = r(a)$ for all $a \in L$. Given any $x \in \bar{h}q(a)$, there exist $y \in q(a)$ such that $x < y$, and since $y \leq \forall \bar{h}q(a) = \bar{h}q(a) = a$, this shows $x < a$. Hence $\bar{h}q(a) \subseteq r(a)$. For the reverse inclusion, take any $x < a$ and then $y$ such that $x \wedge y = 0$ and $a \vee y = e$. Since $q$ and therefore $\bar{h}q$ is a lattice homomorphism, this implies $\bar{h}q(x) \wedge \bar{h}q(y) = 0$ and $\bar{h}q(a) \vee \bar{h}q(y) = L$. The latter means that $s \vee t = e$ for suitable $s \in \bar{h}q(a)$ and $t \in \bar{h}q(y)$. Now, $t \leq y$ and hence $x \wedge t = 0$; therefore $x = x \wedge s \leq s$ which shows $x \in \bar{h}q(a)$. It follows that $r(a) \subseteq \bar{h}q(a)$, and hence we have the desired equality $\bar{h}q(a) = r(a)$. $\square$
Remark. Following Alexandroff [1], Wallace [12] defines, for any subset \( Z \) of a completely regular Hausdorff space \( X \),
\[
Z^\circ = \beta X \setminus \text{cl}_{\beta X}(X \setminus Z).
\]
He then proves the following for a normal Hausdorff space \( X \):

**Lemma A.** For any open \( U \) and \( V \) in \( X \),
\[
\begin{align*}
(i) \quad & (U \cap V)^\circ = U^\circ \cap V^\circ, \\
(ii) \quad & (U \cup V)^\circ = U^\circ \cup V^\circ, \\
(iii) \quad & W \subseteq (W \cap X)^\circ \text{ for any open } W \text{ in } \beta X. 
\end{align*}
\]

**Lemma B.** If \( U \) is open in \( \beta X \), then \( U \) is connected if \( U \cap X \) is connected. If \( V \) is open in \( X \), then \( V \) is connected iff \( V^\circ \) is connected.

**Lemma C.** \( X \) has property S iff \( \beta X \) has property S.

It is obvious that, for any open \( U \) in \( X \), \( U^\circ \) is the largest open \( W \) in \( \beta X \) such that \( W \cap X = U \), and hence the frame homomorphism \( \mathcal{O}(\beta X) \rightarrow \mathcal{O}X \) by \( W \mapsto W \cap X \) has the map \( U \mapsto U^\circ \) as its right adjoint. Using the fact that \( \mathcal{O}(\beta X) \cong \mathcal{F}(\mathcal{O}X) \) we can therefore see that Lemma A is essentially our Lemma 3.1 and Lemma B is a version of Proposition 3.2. Finally, up to the result of Henriksen and Isbell [9] concerning property S which was mentioned earlier, Lemma C is Proposition 3.5.

4. Concluding remarks

It may be worthwhile to compare the proof of our main result with the classical one for spaces. Both directions of the latter, naturally, use the points of the spaces involved but some arguments appear to be more deeply point-dependent than others.

The implication \((\Rightarrow)\), which was essentially obtained in [2], was proved by means of the filters on the space \( X \) which appear as the traces of the neighbourhood filters of the points of \( \beta X - X \). In particular, the local connectedness of \( X \) was obtained by means of the property of such filters \( \mathcal{R} \) from an earlier paper that \( U \cup V \in \mathcal{R} \) implies \( U \in \mathcal{R} \) or \( V \in \mathcal{R} \), for any disjoint open \( U \) and \( V \). This does find its expression in our present context: it is the precise counterpart of the corollary of Lemma 1.9, as one sees from the fact that the largest open set of \( \beta X \) intersecting \( X \) in \( U \) consists of \( U \) together with all those points of \( \beta X - X \) whose associated trace filter contains \( U \). Thus, the way we obtain the local connectedness of \( L \) from that of \( \beta L \) has certain features in common with the original proof. It might be considered the point-free essence of the latter.

Somewhat similar comments could be applied to the original argument which establishes the pseudocompactness of \( X \), except here a more specifically point-based feature enters: the property that the trace filters have bases of connected open sets.
Still, this seems to be faintly related to the step in the corresponding proof here which shows that, for any $a << b$, only finitely many components of $b$ meet $a$.

By way of contrast, the proof of Henriksen–Isbell [9] that $\beta X$ is locally connected for locally connected pseudocompact $X$ employs a totally point-dependent consideration: it initially proves connectedness im kleinen (at every point, every neighbourhood contains a connected—but not necessarily open—neighbourhood) and then uses the fact that this implies local connectedness. Very clearly, this is totally different from the proof given here, quite apart from various other features of [9], such as the use of uniformities, which have no counterpart here. Moreover, [9] actually proves more: it shows that any Tychonoff extension $Y$ of $X$, containing $X$ densely, is locally connected for locally connected pseudocompact $X$.

We should point out that the latter result has so far eluded us. What seems to be needed here is that

\[(*) \quad \text{any regular subframe of a locally connected compact regular frame is locally connected.}\]

Assume this and consider any dense onto $M \to L$ for completely regular $L$ and $M$, $L$ locally connected and pseudocompact. Then, the completely regular compactification $\beta M \to M \to L$ factors through an embedding $\beta M \to \beta L$, thus $\beta M$ is locally connected by Proposition 2.3 and $(*)$, and hence $M$ is locally connected, again using Proposition 2.3.

What of $(*)$? If one assumes the Boolean Ultrafilter Theorem, this indeed holds because then the frames involved are spatial, and one can apply the familiar result that any quotient of a locally connected space is locally connected. It would seem strange if $(*)$ cannot be proved without this assumption, but for the time being we do not know how to do this.

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