In this paper we develop several new theorems which will allow determination of the deficiency indices of the minimal closed symmetric operator $L_0$ (See Section 17.4 of Ref. [1]) induced in $L^2(a, \infty)$ by the formally self-adjoint real coefficient differential operator

$$\ell(y) = [(ry''')' - py']' + qy.$$ 

For brevity we shall denote this operator by $L_0(r, p, q)$. As is pointed out in Sections 17.4 and 23 of Ref. [1] the deficiency indices of $L_0(r, p, q)$ will be $(m, m)$ where $m$ is an integer $2 \leq m \leq 4$ and $m$ is the dimension of the subspace of solutions to

$$\ell(y) = iy$$

which lie in $L^2(a, \infty)$. In order to summarize our results and to compare them with earlier results we find it convenient to consider the special case where $[a, \infty) = [1, \infty)$, $r(t) = t^\alpha$, $p(t) = \pm t^\beta$, and $q(t) = \pm t^\gamma$ ($\alpha, \beta, \gamma$ real

![Diagram](image-url)
numbers). Naimark's result (Theorem 4, p. 195 of Ref. [1]) requires $\alpha = 0$, $0 < \gamma$, and $\gamma > 4\beta + 4$. The result of Fedorjuk (Theorem 5.1, p. 336 of Ref. [2]) requires $\alpha = 0$, $0 < \gamma$, and $\gamma \geq 2\beta$. In Ref. [3] Everitt has shown that the deficiency indices of $L_0(t^\alpha, \pm t^\beta, t^\gamma)$ are $(2, 2)$ provided $\gamma \geq 0$ and either $\beta \leq 2$ or $\gamma \geq 2(\beta - 2)$. In Ref. [4] he has shown that those of $L_0(t^\alpha, \pm t^\beta, t^\gamma)$ are $(2, 2)$ provided $\beta \leq 2/3$ and that those of $L_0(t^\alpha, \pm t^\beta, -t^\gamma)$ are $(2, 2)$ provided $\beta \leq 2/3$ and $\gamma \leq 4/3$. The theorems which are proved below
DEFICIENCY INDICES

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(2,2) (3,3) FOR -

FIG. 4. Deficiency indices of \( L_0(t^n, \pm t^\theta, -t^\gamma) \) when \( \alpha < 4 \).

together with Theorem 1, p. 192 of Ref. [1] (which allows the case \( \gamma = 0 \) to be deduced from a known case for \( \gamma < 0 \) and conversely) provide the information indicated in Figs. 1–4. Except for the one marked line in Fig. 4 it is not intended that the various boundary lines be included in any region. Applicability of the theorems which follow is summarized in Section III of Ref. [5].

THEOREM 1. Let the definitions and hypotheses of Theorem 2.1 of Ref. [5] hold. It follows that the deficiency indices of \( L_0(r, p, (-1)^l q) \) are (2, 2) for \( l = 1 \) and \( l = 2 \).

Proof. From the theorem cited we have existence of two linearly independent functions \( y_1 \) and \( y_2 \) on \([a, \infty)\) such that

\[
[(ry'_j)' - py'_j]' + (-1)^l qy_j = iy_j
\]

for \( j = 1, 2 \) where \( y_1(t) \to 1 \) and \( y_2(t) \to 1 \) as \( t \to \infty \). It is clear that no nontrivial linear combination of \( y_1 \) and \( y_2 \) can be in \( \mathcal{L}^2(a, \infty) \). Hence the deficiency indices of \( L_0 \) are \((m, m)\) where \( m \leq 2 \). Since by Section 23 of Ref. [1] we know \( m \geq 2 \), the theorem is proved. \( \square \)

THEOREM 2. Let the conditions and definitions stated in Theorem 2.3 of Ref. [5] hold for \( \alpha = i \). It then follows that the deficiency indices of \( L_0(r, p, (-1)^l q) \), are (2, 2) when \( l = 2 \), are (3, 3) when \( l = 1 \) and \( (q^2 r)^{-1/4} \in \mathcal{L}(a, \infty) \) and are (2, 2) when \( l = 1 \) and \( (q^2 r)^{-1/4} \notin \mathcal{L}(a, \infty) \).
Proof. From Theorem 2.3 of Ref. [5] we see that there exist four linearly independent functions $y_1, y_2, y_3,$ and $y_4$ such that

$$[(ry'')' - py']' + (-1)q y_j = iy_j$$

for each $j$ and such that $(q^2r^{1/2}(t)y_j(t) \exp\{-\int_0^t \lambda_j\} \to 1$ as $t \to \infty$ for each $j$. It is easily verified from the definitions of the function $\alpha$ and $\beta$ that

$$q(t) = \kappa_1 \exp \left\{ \int_b^{h(t)} \alpha \right\}$$

and that

$$r(t) = \kappa_2 \exp \left\{ \int_b^{h(t)} \beta \right\}$$

for certain nonzero constants $\kappa_1$ and $\kappa_2$. Remembering $\alpha(s) \to 0$ and $\beta(s) \to 0$ as $s \to \infty$ we have that

$$y_j(t) = d \exp \left\{ \int_b^{h(t)} \mu_j + o(1) \right\} (1 + o(1)). \quad (1)$$

$l = 2$. Note $\mu_1 = (1/\sqrt{2})(1 + i)$, $\mu_2 = (1/\sqrt{2})(-1 + i)$, $\mu_3 = (1/\sqrt{2})(-1 - i)$, and $\mu_4 = (1/\sqrt{2})(1 - i)$. We have

$$|y_j(t)| = |d| \exp \left\{ \int_b^{h(t)} \text{Re} \{\mu_j + o(1)\} \right\} (1 + o(1)). \quad (2)$$

Note that if $c > 0$, $\int_b^{h(t)} [c + o(1)] > (1/2) ch(t)$ for all large $t$, so remembering $h(t) \to \infty$ as $t \to \infty$ we see that $|y_1(t)| \to \infty$ and $|y_4(t)| \to \infty$ as $t \to \infty$. Hence $y_1 \notin \mathcal{L}^2(a, \infty)$ and $y_4 \notin \mathcal{L}^2(a, \infty)$.

In order to establish our conclusion we shall show that no nontrivial linear combination of $y_1$ and $y_4$ is in $\mathcal{L}(a, \infty)$. This problem is compounded by the fact that each of $|y_1|$ and $|y_4|$ has the same rate of growth. Before proceeding to this we need to show that $h' \cdot (1/y_1) \in \mathcal{L}(d_1, \infty)$ (where $d_1 > a$ is such that $1/y_1$ is defined on $[d_1, \infty)$). First, $h' = (q/r)^{1/4}$, so $h'(t)/y_1(t)$ is

$$k \exp \left\{ \int_0^{h(t)} \left[ -(1/\sqrt{2})(1 + i) + o(1) \right] \right\} (1 + o(1))$$

with $k$ a nonzero constant, hence $h'(t)/y_1(t) \to 0$ as $t \to \infty$. Next observe

$$\int_{d_1}^{t} \frac{|H'/y_1|}{H(d_1)} \int_{h(d_1)}^{h(t)} |1/y_1(g(\tau))| \, d\tau.$$
for all large $\tau$. Hence $h' / y_1 \in \mathcal{L}(d_1, \infty)$, and since $h'(t) / y_1(t) \to 0$ as $t \to \infty$, $h' / y_1 \in \mathcal{L}^2(d_1, \infty)$. Returning to the problem at hand, suppose each of $c_1$ and $c_4$ is a complex number not each of which is zero and $c_1 y_1 + c_4 y_4 \in \mathcal{L}^2(a, \infty)$ (we are now looking for a contradiction). Since each of $y_1$ and $y_4$ is not in $\mathcal{L}^2(a, \infty)$, $c_1 \neq 0 \neq c_4$. Hence letting $c = c_1 / c_4$ we have $y_1(c + y_4 / y_1)$ in $\mathcal{L}^2(d_1, \infty)$, hence $h' \cdot (c + y_4 / y_1) = (h' / y_1) \cdot y_1 \cdot (c + y_4 / y_1)$ is in $\mathcal{L}(d_1, \infty)$. Hence $(c + (y_4(g(\cdot)) / y_1(g(\cdot))))$ is in $\mathcal{L}(h(d_1), \infty)$, but from (1) we find that this says

$$\left( c + \exp \left\{ - \int_0^t (\sqrt{2i} + o(1)) \right\} (1 + o(1)) \right) \text{ is in } \mathcal{L}(h(d_1), \infty).$$

But clearly there is an $\varepsilon > 0$ and a subset $E$ of $[b, \infty)$ of infinite measure such that

$$\left| c + \exp \left\{ - \int_0^t (\sqrt{2i} + o(1)) \right\} (1 + o(1)) \right| > \varepsilon$$

for each $t \in E$. Hence we have reached a contradiction.

$\ell = 1$. Noting that the $\mathcal{P}$ of Lemma 2.2 and Theorem 2.3 of Ref. [5] is given in this case by $\mathcal{P}(z, s) = z^4 - \gamma(s) z^2 - 1 + \delta(s)$, where $-i\delta(s)$ is positive, we have (from the proof of Lemma 2.2) that $\lambda_4(s) = -\lambda_4(s)$ for all large $s$ and that $\Re \lambda_4(s) \neq 0$ for all $s$. Hence either $\Re \lambda_2(s) > 0$ for all $s$ or $\Re \lambda_4(s) > 0$ for all $s$. Moreover from the proof of Lemma 2.2 (equation 2.3) we see that for all large $s$

$$\lambda_j(s) = \sum_{m=0}^{\infty} a_j(m, 0) \gamma^m(s) + \sum_{m=0}^{\infty} a_j(m, n) \gamma^m(s) \delta^n(s).$$

Since the $\eta_j$ given by

$$\eta_j(s) = \sum_{m=0}^{\infty} a_j(m, 0) \gamma^m(s)$$

satisfy $\eta_j^4(s) - \gamma(s) \eta_j^2(s) - 1 = 0$ and since $\gamma(s)$ is real we see (since $\mu_j = \lambda_j(\infty) = a_j(0, 0)$, implying $a_2(0, 0) = i$ and $a_4(0, 0) = -i$, $a_3(0, 0) = -a_3(0, 0) = 1$) for large $s$, $\eta_2(s) = -\eta_4(s) = \eta_4(s)$. Hence $\Re \eta_2(s) = \Re \eta_4(s) = 0$ for all large $s$. Thus for $j = 2, 4$,

$$\Re \lambda_j(s) = \Re \eta_j(s) + \Re \left\{ \delta(s) \sum_{m=0}^{\infty} a_j(m, n) \gamma^m(s) \delta^{n-1}(s) \right\}$$

is in $L(b, \infty)$ when $\delta \in \mathcal{P}(b, \infty)$. 
\[ l = 1 \text{ and } (q^3r)^{-1/4} \in \mathcal{L}(a, \infty). \] A change of variables shows that 
\((q^3r)^{-1/4} \in \mathcal{L}(a, \infty)\) if and only if \(\delta \in \mathcal{L}(0, \infty)\). In case \(l = 1, \mu_1 = 1, u_2 = i, \mu_3 = -1, \mu_4 = -i\), and from the above remarks we see that

\[ \exp \left\{ \int_{b}^{h(t)} \text{Re} \lambda_j \right\} \]

is bounded when \(j = 2\) and \(j = 4\). So since \(\lambda_3(t) = -1 + o(1)\), from

\[ y_j(t) = (q^3r)^{-1/8} \exp \left\{ \int_{b}^{h(t)} \lambda_j \right\} (1 + o(1)) \]

we then see that when \((q^3r)^{-1/4} \in \mathcal{L}(a, \infty)\) and \(l = 1\), each \(|y_2|^2, |y_3|^2, \) and \(|y_4|^2\) is the product of a function in \(\mathcal{L}(a, \infty)\) with a bounded function. Hence each of \(y_2, y_3,\) and \(y_4\) is in \(\mathcal{L}^2(a, \infty)\). From (1), noting that here \(\mu_1 = 1\), we see \(|y_4(t)| \to \infty\) as \(t \to \infty\) so that \(y_4 \notin \mathcal{L}^2(a, \infty)\). Thus there exist three and no more than three linearly independent solutions of \([ryy' - py']' + qy = iy\) which are in \(\mathcal{L}^2(a, \infty)\) and our conclusion of the theorem for this case is proved.

\[ l = 1 \text{ and } (q^3r)^{-1/4} \notin \mathcal{L}(a, \infty). \] In this case, for the same reason we have just noted, \(y_3 \notin \mathcal{L}^2(a, \infty)\). Letting \(j_0\) be the one of 2 and 4 so that \(\text{Re} \lambda_j = 0\) for all \(s\) (see the remarks under the section for \(l = 1\)) we have

\[ |y_{j_0}(t)|^2 = (q^3r)^{-1/4} \exp \left( \int_{b}^{h(t)} 2 \text{Re} \lambda_0 \right)^2 (1 + o(1)), \]

and since \(\exp\left\{ \int_{b}^{h(t)} 2 \text{Re} \lambda_j \right\}(1 + o(1))\) is bounded below by a positive number for all large \(t\) it follows that \(|y_{j_0}(\cdot)|^2\) is not in \(\mathcal{L}(a, \infty)\) when \((q^3r)^{-1/4}\) is not in \(\mathcal{L}(a, \infty)\). If \(c_1y_1 + c_{j_0}y_{j_0}\) is a nontrivial linear combination of \(y_1\) and \(y_{j_0}\) and \(c_1 = 0\) clearly \(c_1y_1 + c_{j_0}y_{j_0} \notin \mathcal{L}^2(a, \infty)\); on the other hand if \(c_1 \neq 0,\)
\(c_1y_1 + c_{j_0}y_{j_0} = y_4(c_1 + o(1))\) is clearly not in \(\mathcal{L}^2(a, \infty)\). This completes the proof of the theorem. \[ \blacksquare \]

**Theorem 3.** Let the conditions and definitions in Theorem 2-5 of Ref. [5] be satisfied for \(\sigma = i\). It follows that the deficiency indices of \(L_0(r, p, q)\) are \((2, 2)\).

**Proof.** From Theorem 2.5 of Ref. [5] we have the existence of four linearly independent solutions \(y_j, j = 1, \ldots, 4\) of \([ryy' - py']' + qy = iy\) so that

\[ y_j(t) = r^{-1/8} \exp \left( \int_{b}^{h(t)} (\mu_j + o(1)) \right)^2 (1 + o(1)), \]
where \( \mu_j \) is the fourth root of \( i \) in the \( j \)-th quadrant. As in Theorem 2 we may take the \( r^{1/8} \) "upstairs" so that
\[
y_j(t) = d \exp \left\{ \int_0^{h(t)} (\mu_j + o(1)) \right\} (1 + o(1)),
\]
d a nonzero constant. Since \( 0 < \text{Re} \mu_4 < \text{Re} \mu_1 \) it is easily verified that no nontrivial linear combination of \( y_4 \) and \( y_1 \) is in \( \mathcal{L}^2(a, \infty) \).

**Theorem 4.** Let the definitions and conditions stated in Theorem 2.7 of Ref. [5] hold for \( \sigma = i \). It follows that the deficiency indices for \( L_0(r, p, (-1)^q) \) are \((2, 2)\).

The proof is completely analogous to that of Theorem 1.

**Theorem 5.** Let the conditions and definitions stated in Theorem 2.8 of Ref. [5] hold for \( \sigma = i \). It then follows that the deficiency indices of \( L_0(r, (-1)^p, q) \) are \((3, 3)\) when \( \ell' = 1 \), and are \((2, 2)\) when \( \ell' = 2 \).

**Proof.** \( \ell = 1 \). In this case Theorem 2.8 of Ref. [5] gives the existence of four linearly independent solutions \( y_j, j = 1, \ldots, 4 \) of \([r y'' + pr']' + qy = iy\) such that
\[
y_1(t) = (r/p^3)^{1/4}(t) \cdot \exp(ih(t))(1 + o(1)),
y_2(t) = (r/p^3)^{1/4}(t) \exp(-ih(t))(1 + o(1)),
y_3(t) = h^\nu(t)(r/p^3)^{1/2}(t) o(1), \quad \text{and} \quad y_4(t) = 1 + o(1).
\]
It is clear that \( y_4 \notin \mathcal{L}^2(a, \infty) \). A change of variables shows \( I^\nu \cdot \delta \in \mathcal{L}(0, \infty) \) implies that \( h^\nu \cdot (r/p^3)^{1/2} \) is in \( \mathcal{L}(a, \infty) \). The further requirement that \( h^\nu \cdot (r/p^3)^{1/2} \) be expressible as the product of a monotone function and a bounded function which is bounded below by a positive number then ensures that \( h^\nu \cdot (r/p^3)^{1/2} \in \mathcal{L}^2(a, \infty) \). Hence \( y_4 \in \mathcal{L}^2(a, \infty) \). Also \( h^\nu \cdot (r/p^3)^{1/2} \) in \( \mathcal{L}^2(a, \infty) \) clearly implies \( (r/p^3)^{1/2} \in \mathcal{L}^2(a, \infty) \). Thus each of \( y_1 \) and \( y_2 \) is in \( \mathcal{L}^2(a, \infty) \).

\( \ell = 2 \). In this case Theorem 2.8 gives a solution \( y_1 \) such that \( y_1(t) = (r/p^3)^{1/4} \exp(h(t))(1 + o(1)) \) and a solution \( y_4 \) such that \( y_4(t) = 1 + o(1) \) of \([r y'' - py']' + qy = iy\) Noting \( y_1(t) = d \exp(\int_0^{h(t)} (1 + o(1)))(1 + o(1)) \) for some \( d \neq 0 \) we see that \( |y_1(t)| \to \infty \) as \( t \to \infty \). Hence no nontrivial linear combination of \( y_1 \) and \( y_4 \) is in \( \mathcal{L}^2(a, \infty) \).

It is of interest to note that deficiency indices \((4, 4)\) did not occur in any of the above theorems.
Our results (see Fig. 2) together with those of Everitt cited earlier show that the deficiency indices of $L_0(l, t^\beta, 1)$ are $(2, 2)$ for any real $\beta$ and of $L(1, -t^\beta, 1)$ are $(2, 2)$ provided $\beta \leq 2/3$ while they are $(3, 3)$ provided $\beta > 2$.

REFERENCES