ELSEVIER

Theoretical Computer Science 292 (2003) 679–695

Theoretical Computer Science

www.elsevier.com/locate/tcs

Towards area requirements for drawing hierarchically planar graphs

Xuemin Lin[a,*], Peter Eades[b]

[a] *School of Computer Science and Engineering, The University of New South Wales, Sydney, NSW 2052, Australia*

[b] *Basser Department of Computer Science, The University of Sydney, Sydney, NSW 2006, Australia*

Received 28 September 1999; accepted 22 October 2001

Communicated by G. Ausiello

Abstract

Hierarchical graphs are an important class of graphs for modeling many real applications in software and information visualization. In this paper, we investigate area requirements for drawing hierarchically planar graphs regarding two different drawing standards. Firstly, we show an exponential lower bound for the area needed for *straight-line* drawing of hierarchically planar graphs. The lower bound holds even for *s-t* hierarchical graphs without transitive arcs, in contrast to the results for upward planar drawing. This motivates our investigation of another drawing standard *grid visibility representation*, as a relaxation of straight-line drawing. An application of the existing results from upward drawing can guarantee a quadric drawing area for grid visibility representation but does not necessarily guarantee the minimum drawing area. Motivated by this, we will present a new grid visibility drawing algorithm which is efficient and guarantees the minimum drawing area with respect to a given topological embedding. This implies that the area minimization problem is polynomial time solvable restricted to the class of graphs whose planar embeddings are unique. However, we can show that the problem of area minimization of grid visibility for hierarchically planar graphs is generally NP-hard, even restricted to *s-t* graphs.

Keywords: Graph drawing; Hierarchically planar graph; Straight-line drawing; Visibility representation; Drawing area

* Corresponding author.

E-mail addresses: lxue@cse.unsw.edu.au (X. Lin), peter@cs.usyd.edu.au (P. Eades).

0304-3975/03/$ - see front matter © 2002 Elsevier Science B.V. All rights reserved.
PII: S0304-3975(02)00031-2

1. Introduction

Automatic graph drawing [2,5,13,14,15,16,17,18] plays an important role in many computer-based applications such as CASE tools, software and information visualization, VLSI design, visual data mining, and Internet navigation. Directed acyclic graphs are an important class [2] of graphs to be investigated in this area. The *upward* drawing convention for drawing acyclic directed graphs has received a great deal of attention since last decade; and a number of results for drawing *upward planar graphs* have been published [2,4,6,11].

Consider [7,8,9,19] that directed graphs are not powerful enough to model every real-life application. "Hierarchical" graphs are then introduced, where layering information is added to a directed acyclic graph. Consequently, the "hierarchical" drawing convention is proposed to display the specified layering information.

Due to the additional layering constraint, most problems in hierarchical drawing are inherently different to those in upward drawing. For example, testing for "upward planarity" of directed acyclic graphs is NP-Complete [11], while it can be done in linear time [3,12] for "hierarchical planarity". Therefore, issues, such as, "planar", "straight-line", "convex", and "symmetric" representations have been independently investigated [7,8,12] with respect to "hierarchically planar" graphs.

In this paper, we investigate the problem of area requirements for drawing hierarchically planar graphs with respect to a given *resolution requirement*. In [6], it has been shown that exponential areas are generally necessary for drawing upward planar graphs by the "straight-line" drawing standard. However, only quadric drawing areas are required when "s-t" upward planar graphs are *reduced*, meaning that no "transitive" arcs exist.

In this paper, we show that the results in [6] do not entirely hold for hierarchically planar graphs. Specifically, we show that by the straight-line drawing standard, exponential drawing areas are necessary even for s-t hierarchically planar graphs without transitive arcs. This is the first contribution of the paper.

Secondly, we study the drawing area problem when line segments are allowed to represent vertices. In particular, we study the drawing area problem for "grid visibility representations". An application of the algorithm VISIBILITY_DRAW in [4] gives the quadric area for the grid visibility representation of hierarchically planar graphs. However, this algorithm does not necessarily guarantee the minimal drawing area—an example will be given in Section 4. Motivated by this, we present a new and efficient algorithm for grid visibility representations of hierarchically planar graphs which guarantees the minimum drawing area with respect to a fixed "planar embedding". This is the second contribution of the paper. The third contribution of the paper is to prove that the problem of area minimization is NP-hard for the grid visibility representation if a planar embedding is not fixed. The NP-hardness holds even for s-t hierarchically planar graphs.

The rest of the paper is organized as follows. Section 2 gives the basic terminology and background knowledge, as well as the definitions of problems to be investigated. Section 3 presents the first contribution. Section 4 presents the second and the third contributions. This is followed by the conclusions and remarks.

2. Preliminaries

The basic graph theoretic definitions can be found in [1].

A *hierarchical graph* $H=(V,A,\lambda,k)$ consists of a simple and directed acyclic graph (V,A), a positive integer k, and for each vertex u, an integer $\lambda(u)\in\{1,2,\ldots,k\}$ with the property that if $u\to v\in A$, then $\lambda(u)>\lambda(v)$. For $1\leqslant i\leqslant k$ the set $\{u\colon \lambda(u)=i\}$ of vertices is the *ith layer* of H and is denoted by L_i. An arc $u\to v$ in $H=(V,A,\lambda,k)$ is a *transitive* arc if there exists another directed path from u to v. An arc $u\to v$ is *long* if it spans more than two layers, that is, $\lambda(u)-\lambda(v)\geqslant 2$.

For each vertex u in H, we use A_u to denote the set of arcs incident to u, A_u^+ to denote the set of arcs outgoing from u, and A_u^- to denote the set of arcs incoming to u. A *sink* u of a hierarchical graph H is a vertex that does not have outgoing arcs; that is, $A_u^+=\emptyset$. A *source* of H is a vertex that does not have incoming arcs; that is, $A_u^-=\emptyset$. H is *s-t* if it has only one sink and one source.

A hierarchical graph is *proper* if it has no long arcs. Clearly, adding $\lambda(u)-\lambda(v)-1$ dummy vertices to each long arc $u\to v$ in an improper hierarchical graph H results in a proper hierarchical graph, denoted by H_p. H_p is called the *proper image* of H. Note that $H_P=H$ if H is proper.

To display the specified hierarchical information in a hierarchical graph, the *hierarchical drawing convention* is proposed, where a vertex in each layer L_i is separately allocated on the horizontal line $y=i$ and arcs are represented as curves monotonic in y direction; see Figs. 1(a)–(c). In this paper, we will discuss only hierarchical drawing convention.

A hierarchical drawing is *planar* if no pair of arcs intersect except, possibly, at common end points. A hierarchical graph is *hierarchically planar* if it has a planar drawing admitting the hierarchical drawing convention.

An *embedding* E_H of a proper hierarchical graph H gives an ordered vertex set $\mathscr{L}_i$ for each layer L_i in H. For a pair of vertices $u,v\in\mathscr{L}_i$, u is on the *left side* of v if $u<v$. An *embedding* of an improper hierarchical graph H means an embedding of the proper image H_p of H, and is also denoted by E_H. Note that for an improper hierarchical graph H, $\mathscr{L}_i$ may contain more vertices than L_i due to additional dummy vertices.

A hierarchical drawing α of H *respects* E_H if for each pair of vertices u,v in a $\mathscr{L}_i$, the x-coordinate value $\alpha(u)$ is smaller than that of $\alpha(v)$ if and only if $u<v$. An embedding E_H is *planar* if a straight-line drawing of H_p respecting E_H is planar.

Various representation standards exist for drawing hierarchically planar graphs. In a *straight-line* drawing α, each vertex v is represented as a point $\alpha(u)$ and each arc $u\to v$ is represented as a line segment connecting $\alpha(u)$ and $\alpha(v)$; see Fig. 1(a). In a *polyline* drawing, each long arc is allowed to be represented as a polygonal chain with bends allocated on some of the k horizontal lines $y=i$ for $1\leqslant i\leqslant k$; see Fig. 1(c). In a *visibility representation* β, each vertex u is represented as a horizontal line segment $\beta(u)$ on $y=\lambda(u)$ and each arc $u\to v$ as a vertical line segment connecting $\beta(u)$ and $\beta(v)$, such that:

- $\beta(u)$ and $\beta(v)$ are disjoint if $u\neq v$, and
- a vertical line segment and a horizontal line segment do not intersect if the corresponding arc and vertex are not incident.

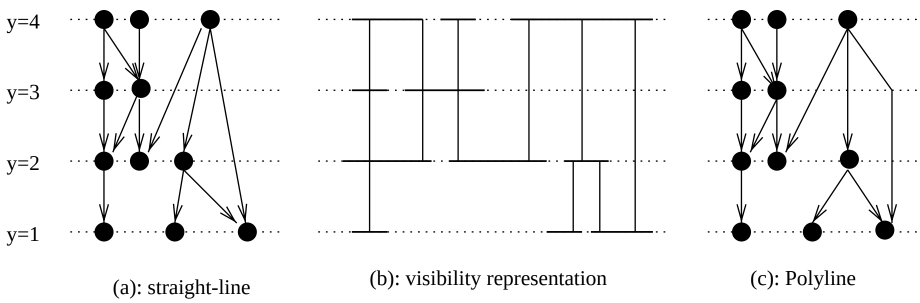


Fig. 1. Various representations of a hierarchical graph.

See Fig. 1(b), for example. Note that in a visibility representation, a line segment used to represent a vertex may be degenerated into a point.

A straight-line drawing is a *grid* drawing if each vertex is at a grid position; and a polyline drawing is a *grid* drawing if vertices and bends are at grid positions. Similarly, in a *grid* visibility representation each horizontal line segment and vertical line segment must use grid points as its ends.

The *area* of a hierarchical drawing is the area of the minimum isothetic rectangle that contains the drawing. The *width* and the *height* of the drawing are width and height, respectively, of this rectangle. Drawing a hierarchically planar graph H consists of two phases: (1) finding a planar embedding E_H, and (2) finding a hierarchical drawing of H respecting E_H.

Note that efficient polynomial time algorithms [3,12] were proposed for phase 1. In this paper, we study the drawing area minimization problem. For a given hierarchical graph H, any hierarchical drawing of H has a fixed height. Consequently, the investigation of the drawing area problem is reduced to that of the drawing width problem.

3. An exponential area lower bound

Below, we first define a class of s-t hierarchically planar graphs $\{H_n\colon n\geqslant 1\}$ where $H_n=(V_n,A_n,\lambda_n,4n-1)$, such that:

1. $|V_n|=10n-6$,
2. H_n has no transitive arcs, and
3. any planar straight-line drawing of H_n requires exponential width with respect to a given vertex resolution requirement.

Specifically, we iteratively define H_n by extending H_{n-1} (for $n\geqslant 2$). The extension follows the same topology.

The graph H_1 consists of four vertices $\{t_1,c_{1,1},c_{2,1},s_1\}$ and three layers $L_3=\{s_1\}$, $L_2=\{c_{1,1},c_{2,1}\}$, and $L_1=\{t_1\}$. Four arcs connect H_1 in a diamond shape (see

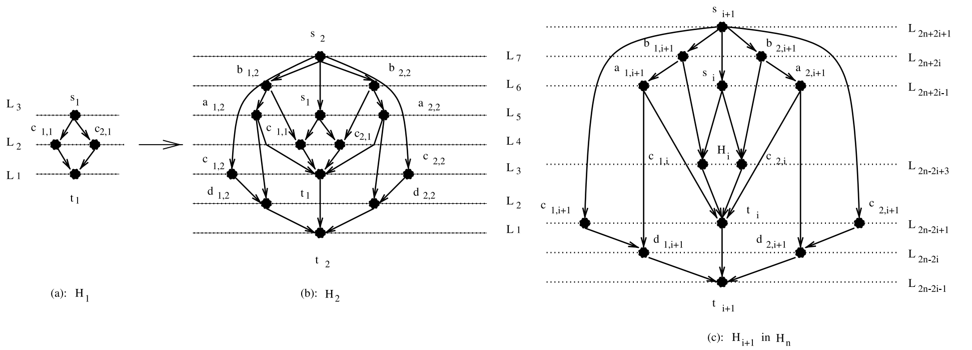


Fig. 2. Construction of H_n.

Fig. 2(a)). To extend H_1 to H_2, ten vertices are added as depicted in Fig. 2(b). Generally, we construct H_{i+1} from H_i by adding the following ten vertices in a way as depicted in Fig. 2(c):

$$V_{i+1} = V_i \cup \{s_{i+1}, t_{i+1}, a_{1,i+1}, a_{2,i+1}, b_{1,i+1}, b_{2,i+1}, c_{1,i+1}, c_{2,i+1}, d_{1,i+1}, d_{2,i+1}\}.$$

Therefore, the layers of H_n can be described below:
$L_1 = \{t_n\}$, $L_{4n-1} = \{s_n\}$, $L_{2n} = \{c_{1,1}, c_{2,1}\}$; and for $1 \leqslant x \leqslant n-1$, $L_{2n+2x} = \{b_{j,x+1}\colon 1 \leqslant j \leqslant 2\}$, $L_{2n+2x-1} = \{s_x\} \cup \{a_{j,x+1}\colon 1 \leqslant j \leqslant 2\}$, $L_{2x+1} = \{t_{n-x}\} \cup \{c_{j,n-x+1}\colon 1 \leqslant j \leqslant 2\}$, and $L_{2x} = \{d_{j,n-x+1}\colon 1 \leqslant j \leqslant 2\}$.
The arc set A_n of H_n consists of:
$\{s_1 \to c_{j,1}, c_{j,1} \to t_1\colon 1 \leqslant j \leqslant 2\}$, $\{s_x \to b_{j,x}, s_x \to c_{j,x}\colon 1 \leqslant j \leqslant 2, 2 \leqslant x \leqslant n\}$, $\{s_x \to s_{x-1}\colon 2 \leqslant x \leqslant n\}$, $\{b_{j,x} \to a_{j,x}, b_{j,x} \to c_{j,x-1}, a_{j,x} \to t_{x-1}, a_{j,x} \to d_{j,x}\colon 1 \leqslant j \leqslant 2, 2 \leqslant x \leqslant n\}$, and $\{c_{j,x} \to d_{j,x}, d_{j,x} \to t_x,\colon 1 \leqslant j \leqslant 2, 2 \leqslant x \leqslant n\}$, $\{t_{x-1} \to t_x\colon 2 \leqslant x \leqslant n\}$.

Based on the structure of H_n, the following two lemmas can be immediately verified by a mathematical induction,

Lemma 1. *For $n \geqslant 1$, H_n is a hierarchically planar graph with no transitive arcs.*

Lemma 2. *For $n \geqslant 1$, the planar embedding E_{H_n} of H_n is unique up to a complete reversal.*

Theorem 1 presents the main result in this section.

Theorem 1. *For each H_n, suppose that α is a hierarchically planar straight-line drawing of H_n, where each pair of vertices in the same layer are at least distance 1 apart. Then α has width at least $\Omega((2n-1)!)$.*

Proof. With respect to α, suppose that for $2 \leqslant i \leqslant n$, the distance between $\alpha(c_{1,i})$ and $\alpha(c_{2,i})$ is l_i.

Lemma 2 tells us that the planar embedding given by Fig. 2(c) is unique to any hierarchically planar drawing of H_n up to a complete reversal.

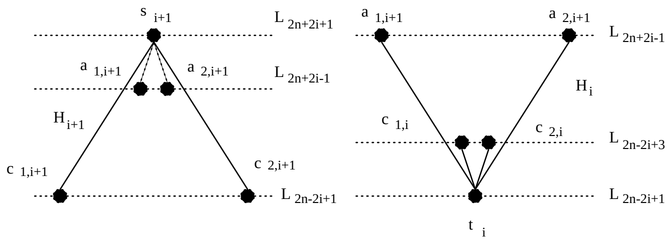


Fig. 3. Relationship among widths.

Without loss of generality, we may assume that α gives the planar embedding as depicted in Fig. 2(c).

Thus, in α the relationship of the vertices orderings between α restricted to H_{i+1} and α restricted to H_i must be as the one illustrated in Fig. 2(c). Consider the two triangles in Fig. 3 with respect to α. Since the drawing α is a planar straight-line drawing, elementary geometry implies $l_{i+1}/l_i \geqslant 2i(2i-1)$. Hence, $l_n \geqslant (2(n-1))!$ Therefore the Theorem holds. □

Note that any hierarchical drawing of H_n has height $4n-2$. Thus each hierarchically planar straight-line drawing of H_n, in which each pair of vertices in the same layer are at least distance 1 apart, has area at least $\Omega(n(2n-2)!)$.

Note that H_n can be drawn upward planar in quadratic area (with respect to the number of vertices in H_n) by the algorithm in [6], but the layering of H_n is not preserved.

4. Visibility representation

In this section, we study the problem of drawing area minimization for visibility representation. As mentioned earlier, the area minimization problem can be reduced to the drawing width minimization problem due to the fixed drawing height for a hierarchical graph. Below we formally present the problem.

Minimum area of grid visibility drawing (MAGVD).

Instance: A hierarchical planar graph H is given.

Question: Find a grid visibility representation of H such that the drawing width is minimized.

Without loss of generality, we assume that in H, there is no *isolated* vertex—a vertex without any incident arcs. In Section 4.2, we will prove that MAGVD is NP-hard. Firstly, however, we show that it is polynomially solvable if the planar embedding is given as part of the input.

4.1. Area minimization for a fixed planar embedding

Di Battista and Tamassia proposed a framework [4], VISIBILITY_DRAW, for drawing s-t upward planar graphs by a grid visibility representation. In fact the algorithm can be immediately applied to hierarchically planar graphs with a fixed planar embedding. Below is the version for hierarchical graphs.

Algorithm VISIBILITY_DRAW

Input: a hierarchically planar graph H and its planar embedding E_H.

Output: a grid visibility representation of H respecting E_H.

Step 1: *Labelling.* Give each arc a an integer $l(a)$.

Step 2: *Drawing.* This step follows immediately Step 1 and draws H based on the output of Step 1. It consists of the following two phases: drawing vertices and drawing arcs of H.

Drawing vertices: For each vertex $u \in H$, let A_u represent the set of arcs in H which are incident to u. Assume $u \in L_i$. Represent u by the horizontal line segment from $(\min_{a\in A_u}\{l(a)\}, i)$ to $(\max_{a\in A_u}\{l(a)\}, i)$.

Drawing arcs: Represent an arc $a = u \to v$ with $u \in L_i$ and $v \in L_j$ by the vertical line segment from $(l(a), i)$ to $(l(a), j)$.

Suppose that the largest x-coordinate value assigned to a grid visibility representation β of H is N, and the smallest is 1. Then the width of β is $N - 1$. Therefore, the key in applying the algorithm VISIBILITY_DRAW to minimizing drawing width is to optimize Step 1—Labelling. Note that in VISIBILITY_DRAW, a *dual graph* technique is adopted while labelling each arc, such that the label of each arc takes the length of the longest path from the source to the node in dual graph which corresponds to the right face of the arc. It is interesting to note that the length of the longest path may be far from the minimum width for a given planar embedding; and thus the labeling technique in [4] does not guarantee the minimality of the drawing width for a fixed planar embedding. This is shown by the following examples.

A hierarchical graph H_1 and its dual graph are illustrated in Fig. 4(a), where the dual graph is depicted by rectangles and dotted arcs. An application of the algorithm VISIBILITY_DRAW produces the grid visibility representation of H_1 with width 3 as shown in Fig. 4(b). However, the minimum width of a grid visibility representation of H is 2 as shown in Fig. 4(c). Actually, the drawing in Fig. 4(c) is output by our algorithm.

We can generalize the example in Fig. 4 to the graph H_2 as shown in Fig. 5, where H_1 in Fig. 4 is duplicated n times in H_2. It can be immediately verified that the length of the longest path from the source to the sink of the dual graph of H_2 is $4n$. Consequently, the width of the grid visibility representation of H_2 produced by the algorithm VISIBILITY_DRAW is $4n - 1$. However, it is easy to show that the minimum width of a grid visibility representation of H_2 is $3n - 1$.

Next, we provide a new algorithm OPTIMAL_LABELLING to Step 1, which guarantees the minimum drawing area for a hierarchically planar graph with a fixed planar

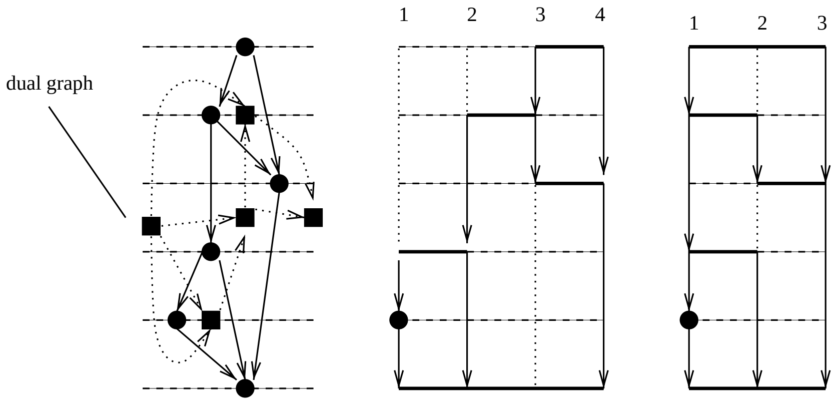


Fig. 4. Counter example 1.

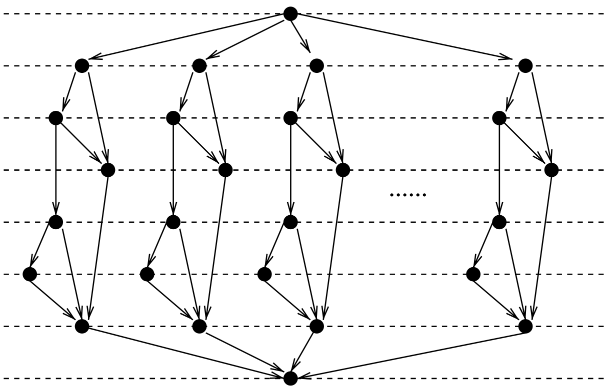


Fig. 5. Counter example 2.

embedding. The basic idea is immediate—labelling each arc with the minimal possible integer.

To describe OPTIMAL_LABELLING, the following notation is needed. For two different arcs $a_1 = u_1 \rightarrow v_1$, $a_2 = u_2 \rightarrow v_2 \in H$, a_1 is on the *left* side of a_2 with respect to E_H if and only if in E_H there are a vertex u on a_1 and a vertex v on a_2 such that u and v are in the same layer and u is on the left side of v. Note that E_H is a planar embedding of H_p, and thus u and v could be dummy vertices on the long arcs. By adding the restriction that such two vertices u and v are always taken from the highest possible layer, the all possible cases are then limited to four which are depicted in Figs. 6(a)–(d), where dotted lines indicate possible extensions to long arcs. Note that Fig. 6(c) also includes the horizontal flip of case Fig. 6(d).

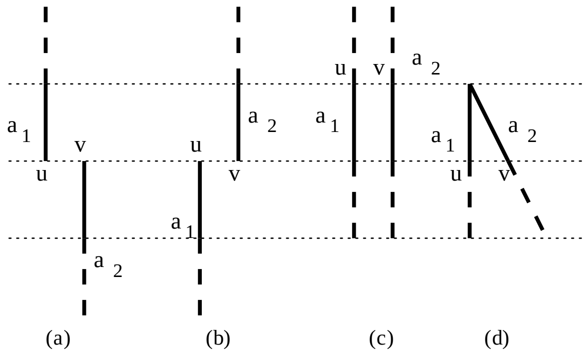


Fig. 6. The four possible cases where a_1 is on the left side of a_2.

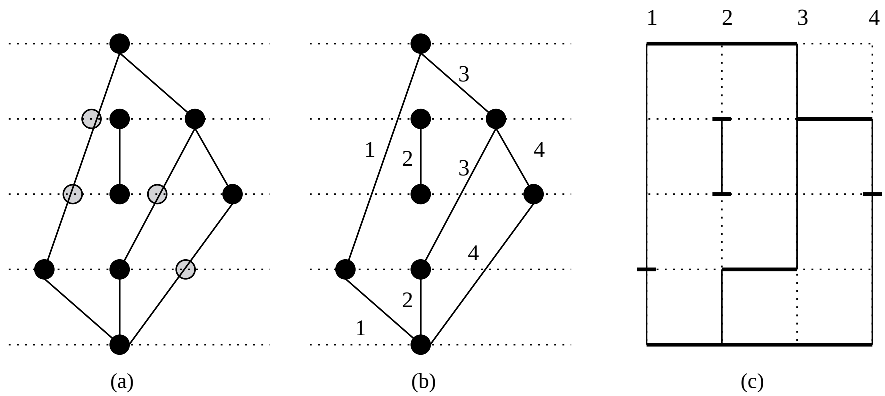


Fig. 7. OPTIMAL_LABELLING.

An arc a in H is *left-most* with respect to E_H if there is no arc in H that is on the left side of a.

The algorithm OPTIMAL_LABELLING iteratively finds the left-most arcs (with respect to E_H) in H to label. In each iteration i:

S1: OPTIMAL_LABELLING scans the hierarchical graph H from the top layer to the bottom layer to label the left-most arcs in the current H with the integer i. Go to S2.

S2: OPTIMAL_LABELLING deletes all arcs labelled in this iteration; and deletes the isolated vertices resulted after arcs deletion in H. Go to $(i+1)$th iteration.

The algorithm terminates if all arcs in H are labelled.

For instance, Fig. 7(b) shows the result after applying OPTIMAL_LABELLING to the graph with respect to the planar embedding depicted in Fig. 7(a). Fig. 7(c) illustrates the result after applying Step 2 in VISIBILITY_DRAW to the output (Fig. 7(b)) of OPTIMAL_LABELLING.

It can be immediately verified that the drawing, produced by applying a combination of OPTIMAL_LABELLING and Step 2 in VISIBILITY_DRAW, respects the given planar embedding E_H; that is.

Lemma 3. *The combination of OPTIMAL_LABELLING and Step* 2 *in VISIBILITY_DRAW gives a grid visibility representation of H respecting a given planar embedding E_H.*

Below, we show that the labelling algorithm actually gives the minimum drawing width for a fixed planar embedding.

Theorem 2. *Respecting a given planar embedding E_H of a hierarchically planar graph H, the grid visibility representation of H, produced by the combination of OPTIMAL_LABELLING and Step* 2 *in VISIBILITY_DRAW, has the minimum drawing width.*

Proof. Any grid visibility representation β' of H that respects E_H *induces* a labelling $l_{\beta'}$ of the arc set of H by assigning the abscissa of the vertical line, representing an arc, as the label of this arc. Suppose that the maximal label in $l_{\beta'}$ is N'. The drawing width of β' is $N' - 1$. Note that β' respects E_H. It immediately implies that for each pair of arcs a and a', $l_{\beta'}(a) \leqslant l_{\beta'}(a') - 1$ if a is on the left side of a'.

Applying mathematic induction and based on the above fact, we can immediately verify that every arc has been assigned the minimum label, by the algorithm OPTIMAL_LABELLING, for all possible visibility representations respecting E_H. Therefore, the theorem holds. □

Note that a hierarchically planar graph with n nodes has $O(n)$ arcs; and the number of labels produced by our algorithm OPTIMAL_LABELLING are no more than the number of arcs. Further, a hierarchical graph with n nodes spans at most n layers. Therefore, the drawing given by our algorithm, a combination of our algorithm OPTIMAL_LABELLING with Step 2 in VISIBILITY_DRAW, occupies area $O(n^2)$.

Suppose that vertices in each layer L_i in H are stored from left to right according to their ordering given by E_H, as well as the vertices in $\mathscr{L}_i$ do. Assume that for each vertex u, arcs in A_u^+ are also stored from left to right according their ordering. To execute OPTIMAL_LABELLING efficiently, S1 and S2 can be integrated together in each iteration. In each iteration i, start with the leftmost vertex u in the top layer of the remaining H, and search down along the leftmost arc $a = u \to v$ in A_u^+ to see if a is the leftmost arc in the current H:

Case 1: If a is also the leftmost arc in the current H, then label a with i and delete a from H. Consequently, delete any resultant isolated vertex from H. Continue the iteration from the layer one level below the layer of v if A_v^+ is empty; otherwise continue the iteration from the layer of v.

Case 2: If a is not the leftmost arc in the current H, then in the remaining H there must be a vertex w such that the leftmost arc $b = u_1 \to v_1$ of A_w^+ is on the left side of a and u_1 is the leftmost vertex in the layer of u_1. Choose such a vertex u_1 that its layer number is maximized. Then continue the iteration i from the layer of u_1.

Clearly, the computation involved in the above two cases is proportional to a scan of the first vertices in the layers spanned by a. Consequently, for a $H=(V,A,\lambda,k)$ each iteration takes $O(k)$ time. Note that the number of iteration must be less than the number of arcs, because each iteration labels at least one arc. Therefore, the algorithm OPTIMAL_LABELLING runs in time $O(k|A|)$. As H is planar, $|A|=O(|V|)$; and thus the algorithm runs in $O(k|V|)$.

4.2. The complexity of MAGVD

In this section, we prove the NP-hardness of MAGVD. In fact, we are able to show a bit stronger result; that is, MAGVD is NP-hard even restricted to s-t hierarchically planar graphs. Clearly, we need only to prove the NP-completeness of the corresponding decision problem.

Decision Problem for MAGVD (DPMAGVD)
Instance: An s-t hierarchically planar graph H, and an integer K.
Question: Is there a grid visibility representation of H such that its width is not greater than K?

It is well known [10] that the 3-PARTITION problem is NP-complete. In our proof, we will transform 3-PARTITION to a special case of DPMAGVD.

3-PARTITION
Instance: A finite set S of $3n$ elements, an integer B, and an integer weight $s(e)$ for each element $e\in S$ are given such that each $s(e)$ satisfies $B/4<s(e)<B/2$ and $\sum_{e\in S} s(a)=nB$.
Question: Can S be partitioned into n disjoint sets $S_1,S_2,\ldots,S_n$ such that for $1\leqslant i\leqslant n$, $\sum_{e\in S_i} s(e)=B$?

Below, we transform an instance I_{3P} of 3-PARTITION to an instance $D_{I_{3P}}=(H_{I_{3P}},K_{I_{3P}})$ of DPMAGVD by applying a scaling technique. Without loss of generality, we can assume that $n\geqslant 2$. The s-t hierarchically planar graph $H_{I_{3P}}$ has eight layers, and is constructed as follows.

1. The top layer L_8 and the bottom layer L_1, respectively, contain only the source u_0 and the sink v_0; see Fig. 8(a).
2. Each element $e\in S$ corresponds to a graph G_e that has vertices on layers L_1, L_5, L_6, L_7, L_8. Here, G_e has only one vertex on L_7 called "sub-source" of G_e, and has only one vertex on L_5 called "sub-sink" of G_e. Besides, G_e has $6n^2\times s(e)+1$ vertices on L_6 connecting to the sub-sink and sub-source. Further, there is an arc from the source v_0 to the sub-source, called "source" arc of G_e. Similarly, G_e has an arc from the sub-sink to the sink v_0, called "sink" arc. Fig. 8(b) shows the topology of G_e.

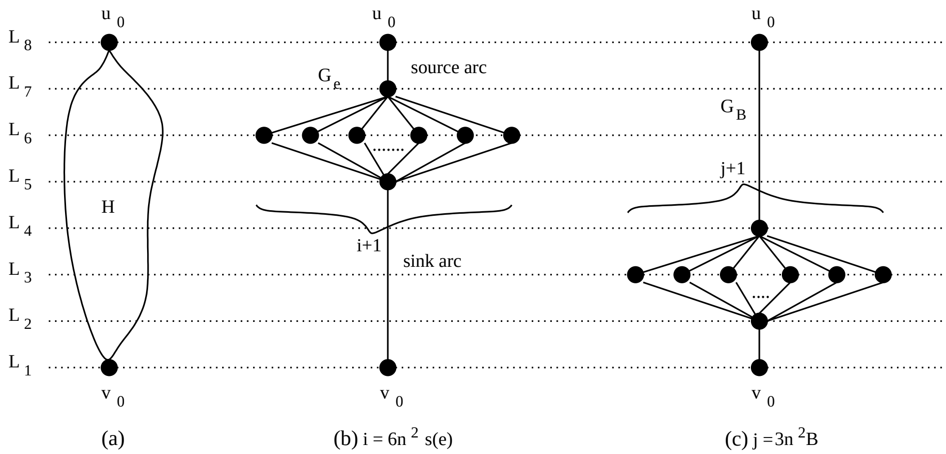


Fig. 8. Transforming I_{3P}.

3. In $H_{I_{3P}}$, we also duplicate $2n$ times a graph G_B that takes a similar topology to that of G_e, and is depicted in Fig. 8(c).
4. We assign $K_{I_{3P}}$ as $6n^3B + (5n-1)$.

Note that a planar embedding of G_e is not unique, neither is a planar embedding of G_B. However, any planar embedding of G_e is able to lead to a grid visibility representation, produced by our algorithm, with the minimum width; this is also true for G_B. More specifically, the following lemma can be immediately verified based on the structures of G_e and G_B.

Lemma 4. *The minimum drawing width of a grid visibility representation of G_e is $6n^2 \times s(e)$, and the minimum drawing width of G_B is $3n^2 \times B$.*

Below, we prove that $K_{I_{3P}}$ is a lower bound of the drawing width for a grid visibility representation of $H_{I_{3P}}$.

Theorem 3. *Let wid_β denote the width of a grid visibility representation β of $H_{I_{3P}}$. Then $wid_\beta \geqslant K_{I_{3P}}$.*

Proof. Clearly, in β each pair of G_Bs cannot have an intersection apart from v_0 and u_0; and each G_B takes a width at least $3n^2 \times B$. Note that there are $2n$ G_Bs, and each pair must be 1 distance apart. Further, there are totally $3n$ sink arcs in G_es passing the layer L_3. From these, it immediately follows that $wid_\beta \geqslant 2n \times (3n^2B) + 5n - 1$. $\square$

We need the following notation. In a planar embedding of $H_{I_{3P}}$, the $2n$ G_Bs can be ordered from left to right because there is no intersection between two G_Bs apart from the source u_0 and the sink v_0. We call the middle space between the ith G_B and the $(i+1)$th G_B (for $1 \leqslant i \leqslant 2n-1$) "$i$th bucket", denoted by Q_i. We denote the left space

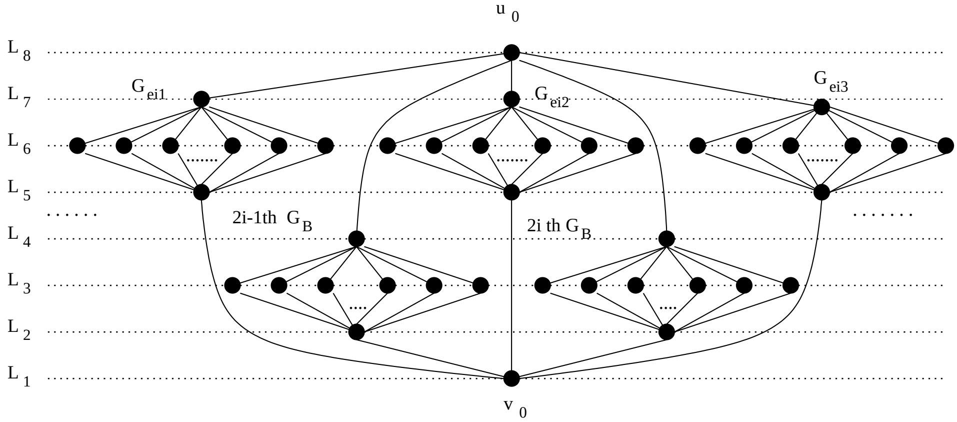


Fig. 9. Planar embedding for $H_{I_{3P}}$.

of the 1st G_B by Q_0, and denote the right space of the last ($2n$th) G_B by Q_{2n}. Q_0 and Q_{2n} are also called buckets.

Next, we show that 3-PARTITION has a solution for I_{3P} if and only if DPMAGVD has a solution for $H_{I_{3P}}$. First, we show "if" part.

Theorem 4. *If* 3-*PARTITION has a solution to* I_{3P}, *then DPMAGVD has a solution for* $H_{I_{3P}}$.

Proof. Suppose that I_{3P} has a solution. Then, there are $S_1, S_2, \dots, S_n$ forming a disjoint partition for S; and each S_i has the total weight B. Clearly, each S_i (for $1 \leqslant i \leqslant n$) must contain exactly three elements, because $B/4 < B < B/2$.

For each S_i, suppose that e_{i1}, e_{i2}, e_{i3} are the three elements in S_i. We can construct a planar embedding, such that $G_{e_{i1}}, G_{e_{i2}}, G_{e_{i3}}$, the $(2i-1)$th G_B, and the $2i$th G_B are neighbouring to each other as depicted in Fig. 9.

Applying our algorithm in the previous section, the planar embedding can be drawn with the width $6n^3B + 5n - 1$. For illustration, the drawing of the subgraph in Fig. 9 is depicted in Fig. 10. $\square$

To show the "only if" part, we first prove that if DPMAGVD has a solution then any drawing with width less than or equal to $K_{I_{3P}}$ must *induce* (respect) a planar embedding with a similar topology to that in the proof of Theorem 4. This can be proved step by step as follows.

Lemma 5. *Suppose that for a grid visibility representation* β *of* $H_{I_{3P}}$, $wid_\beta \leqslant K_{I_{3P}}$, *and* E *is the planar embedding respected by* β. *Then, both the buckets* Q_0 *and* Q_{2n} *in* E, *respectively, contains one and only sink arc from a* G_e.

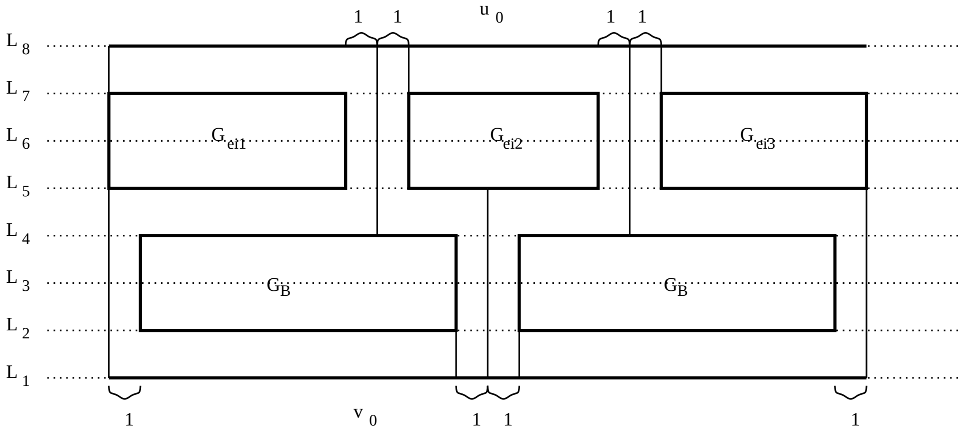


Fig. 10. From 3-PARTITION to DPMAGVD.

Proof. Without loss of generality, we need only to prove that Q_0 has this property, while the proof for Q_{2n} is similar. If Q_0 does not have the property, then there are two cases:

Case 1: Q_0 does not contain any sink arc.

Case 2: Q_0 contains at least two sink arcs.

For case 1, the minimum width of the first (leftmost) G_{e_1} is $6n^2s(e_1)$ according to Lemma 4. The minimum width of the first G_B is $3n^2B$. Therefore, $wid_\beta \geqslant 3n^2B + (6n^2\sum_{e\in S} s(e) - 6n^2s(e_1))$. This implies $wid_\beta \geqslant 6n^3B + 3n^2(B - 2s(e_1))$. Note that $B - 2s(e_1) \geqslant 1$. Therefore, $wid_\beta > K_{I_{3P}}$ since $n \geqslant 2$; and it is contradictory.

For case 2, $wid_\beta \geqslant 6n^2s(e_1) + 2n * 3n^2B > K_{I_{3P}}$; and it is contradictory. □

Lemma 6. *Suppose that for a grid visibility representation β of $H_{I_{3P}}$, $wid_\beta \leqslant K_{I_{3P}}$, and E is the planar embedding respected by β. Then, each bucket Q_i $(1 \leqslant i \leqslant 2n-1)$ in E must contain at least one sink arc but cannot contain more than two sink arcs.*

Proof. Suppose that there is a bucket Q_i that does not contain any sink arc. Let G_{e_1} be the closest space left to the ith G_B, and G_{e_2} be the closest space right to the $(i+1)$th G_B. Then,

$$wid_\beta \geqslant 2 \times 3n^2B + \left(6n^2\sum_{e\in S} s(e) - 6n^2s(e_1) - 6n^2s(e_2)\right).$$

Note that $s(e_1) + s(e_2) < B$. Thus, $wid_\beta > K_{I_{3P}}$. This is contradictory. This means that each bucket must contain at least one sink arc.

Suppose that there is bucket Q_i in E that contains at least three sink arcs, and the corresponding subgraphs are $G_{e_1}, G_{e_2}, \ldots, G_{e_j}$ according to their order in E where

$j \geqslant 3$. Clearly,

$$wid_\beta \geqslant 6n^2 s(e_2) + 2n \times 3n^2 B > K_{I_{3P}}.$$

This is contradictory. Thus the lemma holds. □

Lemma 7. *Suppose that for a grid visibility representation β of $H_{I_{3P}}$, $wid_\beta \leqslant K_{I_{3P}}$, and E is the planar embedding respected by β. Then, Q_1 and Q_{2n-1} in E must, respectively, contain exactly one sink arc.*

Proof. Without loss of generality, we need only to prove that Q_1 has the property, as the proof for Q_{2n-1} will be similar. If Q_1 does not have the property, then Q_1 must contain two sink arcs by applying Lemma 6. Let G_{e_1} and G_{e_2} be the two subgraphs whose sink links are on the left and right, respectively, of the leftmost G_B. Then, we have:

$$wid_\beta \geqslant 6n^2(s(e_1) + s(e_2)) + (2n-1)3n^2 B.$$

Note that $2(s(e_1) + s(e_2)) \geqslant B + 1$. Thus, we have $wid_\beta > K_{I_{3P}}$. It is contradictory. □

Applying a similar technique to the proof of Lemma 7, we can immediately prove the following lemma.

Lemma 8. *Suppose that for a grid visibility representation β of $H_{I_{3P}}$, $wid_\beta \leqslant K_{I_{3P}}$, and E is the planar embedding respected by β. Then, for each pair of Q_i and Q_{i+1} in E for $2 \leqslant i \leqslant 2n-2$, one of them must contain exactly one sink arc.*

Now we are able to prove the "only if" part.

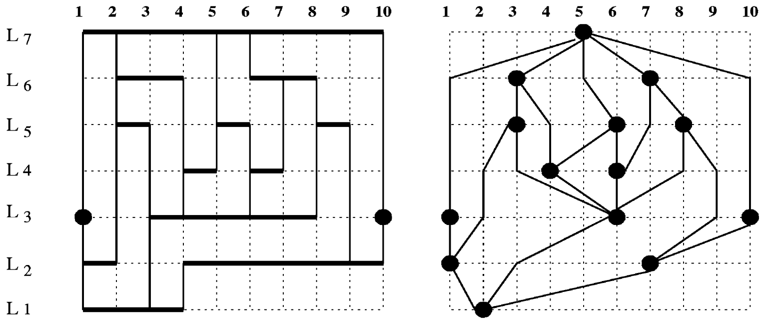
Theorem 5. *If DPMAGVD has a solution for $H_{I_{3P}}$, then 3-PARTITION has a solution for I_{3P}.*

Proof. Let β be a grid visibility representation of $H_{I_{3P}}$ with $wid_\beta \leqslant K_{I_{3P}}$, and E is the planar embedding of $H_{I_{3P}}$ respected by β. Applying the Lemmas 5–8, we can immediately conclude that for $1 \leqslant i \leqslant n-1$, Q_{2i} contains exactly two sink arcs, while each other Q_i contains exactly one sink arc. Consequently, we can divide S in I_{3P} into n disjoint sets $\{S_i\colon 1 \leqslant i \leqslant n\}$ such that for $1 \leqslant i \leqslant n$, $S_i = \{e_{i1}, e_{i2}, e_{i3}\}$ where the sink arc of $G_{e_{i1}}$ is the right (if $i > 1$) one contained in Q_{2i-2}, the sink arc of $G_{e_{i2}}$ is contained in Q_{2i-1}, and the sink arc of $G_{e_{i3}}$ is the left ($i < n$) one contained in Q_{2i}; see Fig. 9 for example.

Further, if there exists an S_i such that $s(e_{i1}) + s(e_{i2}) + s(e_{i3}) \geqslant B + 1$, then for any grid visibility representation β of E,

$$wid_\beta \geqslant 6n^2 B + 6n^2 + (2n-2) \times 3n^2 B > K_{I_{3P}}.$$

It is contradictory. Therefore, the total weight of each S_i is B. □



(a) (b)

Fig. 11. Apply the algorithm GRID_DRAW.

Note that Theorems 4 and 5 do not necessarily imply that there is a polynomial time transformation, with respect to the input size n of 3-PARTITION, between an instance of I_{3P} and $D_{I_{3P}}$, because B may be arbitrarily larger. However, it has been shown [10] that 3-PARTITION is strongly NP-complete, that is, it is NP-complete even if B is bounded by a polynomial of n. This, together with Theorems 4 and 5, proves that a reduction from 3-PARTITION to DPMAGVD can be found. Consequently [10]:

Theorem 6. *MAGVD is NP-hard even restricted to s-t hierarchically planar graphs.*

5. Conclusions and remarks

In this paper, we have shown an exponential area lower bound for planar straight-line drawings of hierarchically planar graphs without transitive arcs in contrast to the result [6] for upward planar drawing. An efficient algorithm has been presented for producing a grid visibility representation with the minimal drawing area with respect to a fixed planar embedding. Further, we proved that the drawing area minimization problem for grid visibility representation is generally NP-hard; the result holds even for s-t hierarchically planar graphs. Note that a modification of our proof construction by adding two "walls" horizontally and vertically, to fix the minimum height and the minimum width, may immediately lead to the NP-hardness of the corresponding problem for s-t upward planar graphs.

We should note that if the algorithm GRID_DRAW [4] is applied to the output of our algorithm in Section 3, then a grid polyline drawing is obtained, which guarantees the following properties:

- each long arc is represented by a polyline with at most two bends;
- the drawing area is $O(n^2)$.

Fig. 11(b) shows the result after applying the algorithm GRID_DRAW to the drawing in Fig. 11(a).

For a possible future study, we are interested in investigating:

- whether or not similar results in Section 4.1 exist for upward planar graphs;
- a good approximation algorithm for solving MAGVD; and
- symmetric drawing issues.

Acknowledgements

The authors are very grateful to the anonymous referees for a careful review and many valuable comments for improving the presentation of the paper.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Application, The Macmillan Press LTD., New York, 1977.

[2] G. Di Battista, P.D. Eades, R. Tamassia, I. Tollis, Algorithms for automatic graph drawing: an annotated bibliography Comput. Geometry: Theory Appl. 4 (1994) 235–282.

[3] G. Di Battista, E. Nardelli, Hierarchies and planarity theory, IEEE Trans. Systems Man Cybernet. 18 (6) (1988) 1035–1046.

[4] G. Di Battista, R. Tamassia, Algorithms for plane representations of acyclic diagraphs, Theoret. Comput. Sci. 61 (1988) 175–198.

[5] G. Di Battista, R. Tamassia, On-line planarity testing, SIAM J. Comput. 25(5) (1996).

[6] G. Di Battista, R. Tamassia, I.G. Tollis, Area requirement and symmetry display of planar upward drawings, Discrete Comput. Geometry 7 (1992) 381–401.

[7] P. Eades, Q. Feng, X. Lin, Straight-line drawing algorithms for hierarchical graphs and clustered graphs, Algorithmic, to appear (extended abstract, Proc. Graph Drawing'96, Lectures Notes in Computer Science, Vol. 1190, Springer, Berlin, 1997, pp. 113–128.)

[8] P. Eades, X. Lin, R. Tamassia, An algorithm for drawing a hierarchical graph, Int. J. Comput. Geometry Appl. 6 (2) (1996) 145–155.

[9] Q. Feng, Algorithms for drawing clustered graphs, Ph.D. Thesis, Department of Computer Science and Software Engineering, University of Newcastle, 1997.

[10] M.R. Garey, D.S. Johnson, Computers and Intractability—a Guide to the Theory of NP-Completeness, Freeman, New York, 1979.

[11] A. Garg, R. Tamassia, On the computational complexity of upward and rectilinear planarity testing, Proc. Graph Drawing'94, Lecture Notes in Computer Science, Springer, Berlin, 1995, pp. 286–297.

[12] L.S. Heath, S.V. Pemmaraju, Recognizing leveled-planar dags in linear time, Proc. Graph Drawing'95, Lecture Notes in Computer Science, Vol. 1027, Springer, Berlin, 1995.

[13] M. Juenger, S. Leipert, P. Mutzel, Pitfalls of using PQ-trees in automatic graph drawing, Proc. Graph Drawing 1997, Lecture Notes in Computer Science, Vol. 1353, Springer, Berlin, 1998, pp. 193–204.

[14] X. Lin, Analysis of algorithms for drawing graphs, Ph.D. Thesis, University of Queensland Press, St. Lucia, 1992.

[15] K. Sugiyama, K. Misue, Graph drawing by the magnetic spring model, J. Visual Languages Comput. 6 (3) (1995) 217–232.

[16] K. Sugiyama, S. Tagawa, M. Toda, Methods for visual understanding of hierarchical system structures, IEEE Trans. Systems Man Cybernet. SMC-11 (2) (1981) 109–125.

[17] R. Tamassia, On embedding a graph in the grid with the minimum number of bends, SIAM J. Comput. 16 (3) (1987) 421–444.

[18] R. Tamassia, I.G. Tollis, Tessellation representations of planar graphs, Proc. 27th Annual Allerton Conf., 1989.

[19] A. Yamaguchi, A. Sugimoto, An approximation algorithm for the two-layered graph drawing problem, Proc. COCOON'99, Lecture Notes in Computer Science, Vol. 1627, Springer, Berlin, 1999, pp. 81–91.