INFORMATION AND CONTROL 24, 74–91 (1974)

On the Interconnection Structure of Cellular Networks*

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This paper presents a model which can be used to represent many of the interconnection patterns commonly found in cellular networks. This model is then used to classify cellular networks according to the degree of regularity in their interconnection patterns. Specifically, three classes of cellular networks, corresponding to three forms of interconnection regularity, are defined. A concept of network realization is then developed to detect structural similarities in different networks and is used to compare the computational capabilities of these three classes.

1. INTRODUCTION

The object of this paper is to investigate three different concepts of regularity for the interconnection patterns of cellular networks. A cellular network is first defined as an interconnection of identical finite state machines (cells). The different kinds of interconnection regularity are then defined by means of different constraints on the allowable interconnection patterns. This paper is specifically concerned with the question of whether certain types of constraints result in restrictions in the computational capabilities of the corresponding class of cellular networks.

Early research on cellular networks was directed at the problem of determining the computational capability of a specific network. Typical of this approach is the work of von Neumann (1966), Lee (1963), and Codd (1968). Cole (1966), Smith (1971), Yamada and Amoroso (1969, 1971), and others were interested in the capabilities of a more general class of networks. In particular, their models enable the investigation of different interconnection structures and different cells. Each of their arrays could be embedded in an n-dimensional space.

* This research was supported by the National Science Foundation under Grant No. GJ-750.

Wagner (1966) observed that there exist cellular networks that are intuitively regular but that can not be embedded in n-dimensional spaces. He characterized these as networks whose interconnection structures could be represented by group graphs. A class of finite networks based on this concept of regularity was investigated by Jump (1968).

The cellular network model proposed in this paper includes, as a special case, the interconnection structures based on n-dimensional arrays as well as those based on group graphs. The philosophy of the paper and many of the techniques that are used are closely related to the work of Yamada and Amoroso. Indeed, generalizations of their concepts of structural and behavioral homomorphisms were found to be effective tools for comparing the capabilities of different subclasses of cellular networks.

In the next section, a class of cellular networks is defined and three subclasses, representing three types of interconnection regularity, are identified. In Section 3, the concepts of network simulation and network realization are introduced. These concepts are then used in Section 4 to compare the capabilities of the three subclasses mentioned above.

2. A Cellular Network Model

The essential features of a cellular network can be completely specified in terms of its interconnection structure and its cell structure. The goal of this paper is to investigate the relationships between different classes of cellular networks, as opposed to the properties unique to a specific class. Hence, the interconnection structure is interpreted as a directed graph, and the different classes of networks to be studied are characterized in terms of restrictions on this graph. The cell structure is specified as a function with finite domain. Thus, the following cellular network model will be used.

DEFINITION. A *cellular network* is a quadruple $N = (C, \eta; S, \delta)$, where

(1) C is a countable set of *cells*,

(2) η is the *neighborhood function* which maps C into C^k for some positive integer k,

(3) S is a finite set of *signals*, and

(4) δ is the *cell function* which maps S^k into S.

A cellular network may be viewed as a model for a collection C of cells, each of which has k input terminals and one output terminal. The signals present on any of these terminals are elements of the set S. The signal present on the output terminal of a cell is called the *state* of that cell. All of the cells of a network are identical, and each one realizes the cell function δ. It is assumed that the cell input terminals are ordered so that the k input signals of a cell can be represented by a k-tuple $(x_1, x_2, \ldots, x_k)$, where x_i is the signal on the ith input terminal for $i = 1, 2, \ldots, k$. The network operates in a synchronous mode, and the signal on the output terminal of a cell at time $t + 1$ is given by $\delta(x_1, x_2, \ldots, x_k)$, where x_i is the signal present on the cell's ith input terminal at time t.

The interconnection structure of a cellular network is given by its neighborhood function η. If $\eta(u) = (u_1, u_2, \ldots, u_k)$, then the output terminal of cell u_i is directly connected to the ith input terminal of cell u. In this case, cell u_i is called the ith *neighbor* of cell u, and the set $\{u_1, u_2, \ldots, u_k\}$ is called the *neighborhood* of cell u. The integer k is said to be the *neighborhood index* of the network.

The global behavior of a cellular network is now characterized as follows:

DEFINITION. The *automaton realized by a cellular network* $N = (C, \eta; S, \delta)$ is the ordered pair $A(N) = (S^C, \bar{\delta})$, where

(1) $S^C = \{f \mid f: C \to S\}$ is the set of *configurations* of N, and

(2) $\bar{\delta}: S^C \to S^C$ is the *next configuration function*, defined by

$$\bar{\delta}(f)(u) = \delta \circ f^k \circ \eta(u) \qquad \text{for all } u \text{ in } C.^1$$

A configuration f of a cellular network $N = (C, \eta; S, \delta)$ may be viewed as a specification of the state of N at some time t. Under this interpretation, $f(u)$ corresponds to the state of cell u at time t. The next state of a cell is determined by the current states of its k neighbors and the cell function δ. But the k neighbors of cell u are given by $\eta(u) = (u_1, u_2, \ldots, u_k)$, and $f^k \circ \eta(u) = (f(u_1), f(u_2), \ldots, f(u_k))$ gives their current states under configuration f. Hence

$$\bar{\delta}(f)(u) = \delta \circ f^k \circ \eta(u) = \delta(f(u_1), f(u_2), \ldots, f(u_k))$$

represents the next state of cell u. Therefore, the next configuration function $\bar{\delta}$ maps the configuration f into $\bar{\delta}(f)$ which specifies the state of the network at time $t + 1$. Note that if the set of cells C is finite, then the set of configura-

[1] If f is a function from X to Y and k a positive integer, then f^k denotes the function from X^k to Y^k defined by $f^k(x_1, x_2, \ldots, x_k) = (f(x_1), f(x_2), \ldots, f(x_k))$ for all $(x_1, x_2, \ldots, x_k)$ in X^k. If, in addition, g is a function from Y to Z, then $g \circ f$ denotes the composition of f and g defined by $g \circ f(x) = g(f(x))$ for all x in X.

tions S^C is also finite so that the automaton realized by N is an autonomous finite state machine.

If η is the neighborhood function and C the cell set of a network N, then for $i = 1, 2,..., k$, η_i will denote the function from C to C defined by $\eta_i = \pi_i \circ \eta$, where π_i is the ith projection from C^k onto C.[2] Hence

$$\eta(u) = (\eta_1(u), \eta_2(u),..., \eta_k(u))$$

for all u in C. The interconnection structure of cellular networks is classified by means of the following directed edge-labeled graph which is constructed from the k functions $\eta_1, \eta_2,..., \eta_k$.

DEFINITION. Let $N = (C, \eta; S, \delta)$ be a cellular network. Then the *connection graph* of N is the ordered pair $\Gamma(N) = (C, E)$, where

$$E = \{(\eta_i(u), u, i) \mid i = 1, 2,..., k \text{ and } u \in C\}.$$

Every triple $(\eta_i(u), u, i)$ in E corresponds to an edge labeled i and directed from node $\eta_i(u)$ to node u. Note that $\eta_i(u)$ is the ith neighbor of u. Hence, an edge labeled i represents a connection from the output terminal of the ith neighbor of a cell to the ith input terminal of that cell.

If the connection graph of a cellular network has ℓ components, then there are ℓ mutually disjoint sets of cells such that no cell in one set is connected to a cell in any other set. Thus, the network consists of ℓ independent cellular networks acting in parallel. These component networks have been called *laminations* [Yamada and Amoroso (1969)]. If there is only one lamination in a network, then that network is said to be *unlaminated.* It has been shown that there is little loss of generality in considering only unlaminated networks [Yamada and Amoroso (1969)].

The three classes of networks studied in this paper can now be introduced.

DEFINITION. A cellular network $N = (C, \eta; S, \delta)$ is said to be *balanced* if it is unlaminated and η_i is a permutation on C for $i = 1, 2,..., k$, where k is the neighborhood index of N.

It can be easily shown that the in-degree and the out-degree of every node in the connection graph of a balanced network is equal to k. Furthermore, there is exactly one edge labeled i directed into and one edge labeled i directed out of every node for $i = 1, 2,..., k$.

[2] π_i is the function defined by $\pi_i(u_1, u_2,..., u_k) = u_i$ for all $(u_1, u_2,..., u_k) \in C^k$ and $1 \leqslant i \leqslant k$.

PROPOSITION 2.1. *Let* $N = (C, \eta; S, \delta)$ *be a balanced network with neighborhood index* k, *and let* u *and* v *be two cells in* C. *Then there are two sequences* $e_1, e_2, \ldots, e_p$ *and* $i_1, i_2, \ldots, i_p$, *where* $e_j \in \{1, -1\}$ *and* $1 \leqslant i_j \leqslant k$ *for* $j = 1, 2, \ldots, p$, *and such that*

$$v = \eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_2}^{e_2} \circ \eta_{i_1}^{e_1}(u).\ ^{3}$$

Proof. Since N is unlaminated, $\Gamma(N)$ has exactly one component. Hence, there is a sequence $x_0, x_1, \ldots, x_p$ of nodes of $\Gamma(N)$ and a sequence $i_1, i_2, \ldots, i_p$ of integers such that $u = x_0$, $v = x_p$, and either (x_{j-1}, x_j, i_j) or (x_j, x_{j-1}, i_j) is an edge of $\Gamma(N)$ for $j = 1, 2, \ldots, p$. If (x_{j-1}, x_j, i_j) is an edge, then, since η_{i_j} is a permutation, $x_j = \eta_{i_j}^{-1}(x_{j-1})$. If (x_j, x_{j-1}, i_j) is an edge, then $x_j = \eta_{i_j}(x_{j-1})$. Hence,

$$v = \eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_2}^{e_2} \circ \eta_{i_1}^{e_1}(u), \qquad \text{with} \quad e_j \in \{1, -1\} \quad \text{for} \quad j = 1, 2, \ldots, p. \quad \blacksquare$$

The second class of networks is obtained by requiring that the connection graph be a group graph.[4]

DEFINITION. A cellular network $N = (G, \eta; S, \delta)$ is *uniform* if N is unlaminated, G is a group with binary operation $\cdot$, and

$$\eta(x) = (\eta_1(\lambda) \cdot x, \eta_2(\lambda) \cdot x, \ldots, \eta_k(\lambda) \cdot x),$$

where $x \in G$, λ is the identity of G, and k is the neighborhood index of N. G is called the *connection group* of the network.

Networks similar to this have been characterized by Wagner (1966). He showed that the networks whose connection graphs are group graphs are exactly those that "look the same when viewed from any node".

It can be easily seen that every uniform network is balanced. Hence, by Proposition 2.1, if x is an arbitrary cell in the uniform network $N = (G, \eta; S, \delta)$, then x can be expressed in the form

$$x = \eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_2}^{e_2} \circ \eta_{i_1}^{e_1}(\lambda),$$

where λ is the identity of G. This observation plus the definition of uniform networks can be used to prove the following.

[3] If $e = 1$, then η_i^e denotes the function η_i, and, if $e = -1$, then η_i^e denotes η_i^{-1}, the inverse of η_i.

[4] An introduction to group graphs can be found in Grossman and Magnus (1964).

PROPOSITION 2.2. *Let* $N = (G, \eta; S, \delta)$ *be a uniform network,* λ *the identity of* G, *and*

$$x = \eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_2}^{e_2} \circ \eta_{i_1}^{e_1}(\lambda)$$

an arbitrary element of G. *Then*

$$x = \eta_{i_p}^{e_p}(\lambda) \cdot \cdots \cdot \eta_{i_2}^{e_2}(\lambda) \cdot \eta_{i_1}^{e_1}(\lambda),$$

and

$$x^{-1} = \eta_{i_1}^{-e_1} \circ \eta_{i_2}^{-e_2} \circ \cdots \circ \eta_{i_p}^{-e_p}(\lambda).$$

Also, if

$$y = \eta_{j_q}^{d_q} \circ \cdots \circ \eta_{j_2}^{d_2} \circ \eta_{j_1}^{d_1}(\lambda),$$

then

$$x \cdot y = \eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_1}^{e_1} \circ \eta_{j_q}^{d_q} \circ \cdots \circ \eta_{j_1}^{d_1}(\lambda).$$

A finitely generated Abelian group can be decomposed into a direct sum of a finite number of cyclic groups. If such a group is used as the connection group of a uniform network, then this decomposition can be used to establish a coordinate system for the cell set of the network. This motivates the final classification.

DEFINITION. A cellular network is called an *array* if it is uniform and its connection group is Abelian.

The interconnection patterns of most of the cellular networks in the literature are derived from group graphs. Furthermore, the most common groups are finitely generated and Abelian. Hence, most of these earlier networks can be modeled as arrays. Every array is a uniform network and every uniform network is balanced. On the other hand, there exist balanced networks that are not uniform and uniform networks that are not arrays. Hence, the cellular network model of this paper can be used to represent a larger class of interconnection structures than earlier models.

3. NETWORK REALIZATIONS

The possibility of one cellular network simulating the behavior of another is considered in this section. The global behavior of a cellular network has been characterized in terms of the automaton realized by that network.

Therefore, the following generalization of sequential machine homomorphisms will be used to formalize the concept of behavioral simulation.

DEFINITION. Let $N_1 = (C_1, \eta; S_1, \delta_1)$ and $N_2 = (C_2, \tau; S_2, \delta_2)$ be two cellular networks and $A(N_1) = (C_1^{S_1}, \bar{\delta}_1)$ and $A(N_2) = (C_2^{S_2}, \bar{\delta}_2)$ the respective automatons realized by N_1 and N_2. Then a *behavioral homomorphism* from N_1 to N_2 is a (partial) function β from $C_1^{S_1}$ to $C_2^{S_2}$ such that, for all f in the domain of β, $\bar{\delta}_1(f)$ is also in the domain of β and $\bar{\delta}_2 \circ \beta(f) = \beta \circ \bar{\delta}_1(f)$. If β is onto $C_2^{S_2}$, then N_1 is said to *simulate* the behavior of N_2.

Each element in the domain of a behavioral homomorphism β from N_1 to N_2 is a function whose domain is the entire cell set of N_1. Moreover, the condition that β must satisfy is stated in terms of the next configuration functions $\bar{\delta}_1$ and $\bar{\delta}_2$. As a result, β is most naturally viewed as a global transformation of a subset of the state set of $A(N_1)$. Furthermore, the existence of the behavioral homomorphism β does not necessarily imply the existence of structural similarities in the networks N_1 and N_2. In order to investigate these similarities, the following two structure preserving operations are introduced.

DEFINITION. Let $N_1 = (C_1, \eta; S_1, \delta_1)$ and $N_2 = (C_2, \tau; S_2, \delta_2)$ be two cellular networks with neighborhood index equal to k. Then:

(1) A function ϕ from C_1 to C_2 is *neighborhood preserving* if $\phi^k \circ \eta(x) = \tau \circ \phi(x)$ for all x in C_1;

(2) a function ψ from S_1 to S_2 is said to be a *cell homomorphism* if ψ is onto S_2 and $\psi \circ \delta_1(u_1, u_2, \ldots, u_k) = \delta_2 \circ \psi^k(u_1, u_2, \ldots, u_k)$ for all $(u_1, u_2, \ldots, u_k)$ in $S_1{}^k$.

The pair (ϕ, ψ) is called a *network homomorphism* from N_1 to N_2. If ϕ is onto C_2, then the network homomorphism is *onto* N_2, and N_1 is said to *realize*, or to be a *realization* of, N_2.

The existence of a neighborhood preserving function from N_1 to N_2 implies certain similarities between the connection graphs of N_1 and N_2. A cell homomorphism establishes a close correlation between the behavior of cells in N_1 and cells in N_2. It will be shown in Theorem 3.1 that network N_1 simulates the behavior of network N_2 if N_1 realizes N_2. Hence, network homomorphisms provide a means of comparing the behavior of cellular networks that is explicitly based on the structure of the networks.

As an example of the definitions in this section, consider the two cellular networks $N_1 = (C_1, \eta; \{0, 1\}, \delta_1)$ and $N_2 = (C_2, \tau; \{0, 1\}, \delta_2)$, where δ_1 is the Boolean OR function of two variables, δ_2 is the Boolean AND function of two variables, and C_1, η, C_2, and τ are given by the connection graphs $\Gamma(N_1)$

and $\Gamma(N_2)$ in Fig. 1. N_1 is a uniform network since $\Gamma(N_1)$ is the graph of the dihedral group of 12 elements. N_2 is a balanced network, but it is not uniform. Neither network is an array. Now, let ϕ and ψ be the functions defined in Table 1. Then it can be easily verified that the pair (ϕ, ψ) is a network homomorphism from N_1 onto N_2. Hence, the uniform network N_1 is a realization of the nonuniform network N_2.

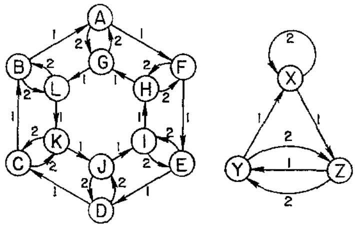

FIG. 1. Interconnection structures for N_1 and N_2.

TABLE 1

Network Homomorphism from N_1 onto N_2

(a) Neighborhood-preserving function ϕ

Cell in N_1	A	B	C	D	E	F	G	H	I	J	K	L
Image under ϕ	X	Y	Z	X	Y	Z	X	Y	Z	X	Y	Z

(b) Cell homomorphism ψ

Signal in N_1	0	1
Image under ψ	1	0

The relationship between network homomorphisms and behavioral homomorphisms is now considered. Given a network homomorphism (ϕ, ψ) from N_1 onto N_2, a behavioral homomorphism β from N_1 onto N_2 will be constructed from the two functions ϕ and ψ. It is first necessary to construct a subset of $C_1^{S_1}$, the state set of $A(N_1)$, which can be used as the domain of β.

DEFINITION. Let (ϕ, ψ) be a network homomorphism from $N_1 = (C_1, \eta; S_1, \delta_1)$ onto $N_2 = (C_2, \tau; S_2, \delta_2)$, and let f be a configuration of N_1.

Then f is said to be *consistent* with (ϕ, ψ) if

$$\psi \circ f(x) = \psi \circ f(y)$$

for all x and y in C_1 such that $\phi(x) = \phi(y)$.

LEMMA 3.1. *Let (ϕ, ψ) be a network homomorphism from N_1 to N_2, and let $F(\phi, \psi)$ denote the set of all configurations of N_1 that are consistent with (ϕ, ψ). Then $F(\phi, \psi)$ is closed under $\bar{\delta}_1$, the next configuration function of $A(N_1)$.*

Proof. Let f be an element of $F(\phi, \psi)$. It must be shown that $\bar{\delta}_1(f)$ is also in $F(\phi, \psi)$. To this end, let x and y be two cells in C_1 such that $\phi(x) = \phi(y)$. Then

$$\begin{aligned}\psi(\bar{\delta}_1(f)(x)) &= \psi \circ \delta_1 \circ f^k \circ \eta(x), \text{ by the definition of } \bar{\delta}_1, \\ &= \delta_2 \circ \psi^k \circ f^k \circ \eta(x), \text{ since } \psi \text{ is a cell homomorphism.}\end{aligned}$$

Similarly, $\psi(\bar{\delta}_1(f)(y)) = \delta_2 \circ \psi^k \circ f^k \circ \eta(y)$. But

$$\phi^k \circ \eta(x) = \tau \circ \phi(x) = \tau \circ \phi(y) = \phi^k \circ \eta(y),$$

since ϕ is neighborhood preserving and $\phi(x) = \phi(y)$. It therefore follows that

$$\psi^k \circ f^k \circ \eta(x) = \psi^k \circ f^k \circ \eta(y),$$

since f is consistent with (ϕ, ψ). Hence

$$\begin{aligned}\psi(\bar{\delta}_1(f)(x)) &= \delta_2 \circ \psi^k \circ f^k \circ \eta(x) \\ &= \delta_2 \circ \psi^k \circ f^k \circ \eta(y) \\ &= \psi(\bar{\delta}_1(f)(y)),\end{aligned}$$

so that $\bar{\delta}_1(f)$ is consistent with (ϕ, ψ). ▮

THEOREM 3.1. *If the cellular network $N_1 = (C_1, \eta; S_1, \delta_1)$ realizes the network $N_2 = (C_2, \tau; S_2, \delta_2)$, then N_1 simulates N_2.*

Proof. Let (ϕ, ψ) be a network homomorphism from N_1 onto N_2. The function β from $F(\phi, \psi)$ onto $C_2^{S_2}$ is defined as follows. Let $f \in F(\phi, \psi)$. Then $\beta(f)$ is the function from C_2 to S_2 defined by

$$(\beta(f) \circ \phi)(x) = \psi \circ f(x)$$

for all x in C_1. Thus $\beta(f) \circ \phi = \psi \circ f$. Since ϕ is onto C_2, the domain of $\beta(f)$ is C_2. $\beta(f)$ is a well-defined function since f is consistent with (ϕ, ψ). In order to show that β is a behavioral homomorphism, let y be an element of C_2. Then $y = \phi(x)$ for some x in C_1 so that

$$\begin{aligned}
\bar{\delta}_2(\beta(f))(y) &= \bar{\delta}_2(\beta(f))(\phi(x)) \\
&= \delta_2 \circ \beta(f)^k \circ \tau \circ \phi(x), \text{ by the definition of } \bar{\delta}_2, \\
&= \delta_2 \circ \beta(f)^k \circ \phi^k \circ \eta(x), \text{ since } \phi \text{ is neighborhood preserving}, \\
&= \delta_2 \circ \psi^k \circ f^k \circ \eta(x), \text{ by the definition of } \beta, \\
&= \psi \circ \delta_1 \circ f^k \circ \eta(x), \text{ since } \psi \text{ is a cell homomorphism}, \\
&= \psi \circ \bar{\delta}_1(f)(x), \text{ by the definition of } \bar{\delta}_1, \\
&= \beta(\bar{\delta}_1(f)) \circ \phi(x), \text{ by the definition of } \beta, \\
&= \beta(\bar{\delta}_1(f))(y).
\end{aligned}$$

Hence, $\bar{\delta}_2 \circ \beta = \beta \circ \bar{\delta}_1$, so that β is a behavioral homomorphism. To show that β is onto $C_2^{S_2}$, let f' be any configuration of N_2. Define a configuration f of N_1 by setting $f(x)$ equal to any $u \in S_1$ such that $\psi(u) = f'(\phi(x))$. Since ψ and ϕ are both onto, this can always be done. $\beta(f)$ is clearly equal to f' and consistent with (ϕ, ψ). ∎

Due to the following proposition, neighborhood-preserving functions are a particularly convenient tool for comparing the behavior of balanced networks.

Proposition 3.1. *Let ϕ be a neighborhood preserving function from a balanced network $N_1 = (C_1, \eta; S_1, \delta_1)$ to a balanced network $N_2 = (C_2, \tau; S_2, \delta_2)$. Then ϕ is uniquely determined by its value at a single cell of N_1.*

Proof. Let k be the neighborhood index of N_1 and N_2, and assume that the value of ϕ is known for some x in C_1. Let y be an arbitrary cell in C_1. Since N_1 is unlaminated, y can be expressed as

$$y = \eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_2}^{e_2} \circ \eta_{i_1}^{e_1}(x),$$

where $e_j = \pm 1$ and $1 \leqslant i_j \leqslant k$ for $j = 1, 2, ..., p$. Then it can be easily shown, by induction on p, that

$$\phi(y) = \tau_{i_p}^{e_p} \circ \cdots \circ \tau_{i_2}^{e_2} \circ \tau_{i_1}^{e_1} \circ \phi(x).$$

Hence, the value of ϕ at y is uniquely determined by its value at x. ∎

The concepts of behavioral and network homomorphisms are similar to the behavioral and structural homomorphisms studied by Yamada and Amoroso (1969). The major differences are that the behavioral homomorphism used in this paper may be a partial function and neighborhood-preserving functions are not necessarily one-to-one. Theorem 3.1 above is derived from Theorem 4.1 in Yamada and Amoroso (1969) by modifying the proof to apply to the more general interconnection structures used in cellular networks.

4. An Investigation of Regular Interconnection Structures

In Section 2, three subclasses of cellular networks were defined by placing increasingly restrictive constraints on network connection graphs. Thus, the class of all arrays was obtained as a proper subclass of the class of all uniform networks which was seen to be a proper subclass of the class of all balanced networks. In this section, these three classes are investigated to determine whether restricting the class of cellular networks in this way results in corresponding restrictions in the possible behavior that can be realized. The network homomorphisms introduced in Section 3 will be used for this purpose. In particular, one class of networks will be said to be as powerful as another class if every network in this second class can be realized by some network in the first class.

It is first shown, in Theorem 4.1, that the class of uniform networks is as powerful as the class of balanced networks.

Theorem 4.1. *Let $N = (C, \eta; S, \delta)$ be a balanced cellular network. Then there exists a uniform network which realizes N.*

Proof. Let N be a balanced network with neighborhood index k. Then the functions η_i are one-to-one and onto for $i = 1, 2,..., k$. Hence, the set $X = \{\eta_i \mid 1 \leqslant i \leqslant k\}$ is a subset of the group of all permutations on C. Let G_m denote the subgroup of this group generated by X, and let μ be the function from G_m to $G_m{}^k$ defined by

$$\mu(g) = (\eta_1 \circ g, \eta_2 \circ g, ..., \eta_k \circ g)$$

for all g in G_m. Since $\mu_i(\lambda) = \eta_i$ and the group operation of G_m is functional composition, the cellular network $N_m = (G_m, \mu; S, \delta)$ is uniform.

To show that the uniform network N_m is a realization of the balanced network N, let b be an element of C. Then define ϕ_m as the function from G_m to C, where

$$\phi_m(\gamma) = \gamma(b)$$

for all γ in G_m. In other words, the image under ϕ_m of the permutation γ is the cell obtained by applying γ to the distinguished cell b. Now

$$\begin{aligned}
\phi_m{}^k \circ \mu(\gamma) &= \phi_m{}^k(\eta_1 \circ \gamma, \eta_2 \circ \gamma,..., \eta_k \circ \gamma), \text{ by the definition of } \mu,\\
&= (\phi_m(\eta_1 \circ \gamma), \phi_m(\eta_2 \circ \gamma),..., \phi_m(\eta_k \circ \gamma))\\
&= (\eta_1 \circ \gamma(b), \eta_2 \circ \gamma(b),..., \eta_k \circ \gamma(b)),\\
&\qquad \text{by the definition of } \phi_m,\\
&= (\pi_1 \circ \eta \circ \gamma(b), \pi_2 \circ \eta \circ \gamma(b),..., \pi_k \circ \eta \circ \gamma(b)),\\
&\qquad \text{by the definition of } \eta_1, \eta_2,..., \eta_k,\\
&= \eta \circ \gamma(b)\\
&= \eta \circ \phi_m(\gamma), \text{ by the definition of } \phi_m.
\end{aligned}$$

Hence, $\phi_m{}^k \circ \mu = \eta \circ \phi_m$, so that ϕ_m is a neighborhood preserving function. Since N is unlaminated, ϕ_m is onto.

Let ψ be the identity function on S. Then (ϕ_m, ψ) is a network homomorphism from N_m onto N. Hence, there is a uniform realization of N. ∎

Due to the following proposition, the network N_m, defined in the proof of Theorem 4.1, will be called the *minimal uniform realization* of N.

PROPOSITION 4.1. *Let $N' = (G, \tau; S', \delta')$ be a uniform realization of the balanced network $N = (C, \eta; S, \delta)$. Then N' is also a realization of $N_m = (G_m, \mu; S, \delta)$, the minimal uniform realization of N.*

Proof. Let (ϕ', ψ') be a network homomorphism from N' onto N, and let ϕ_m be the neighborhood-preserving function from N_m onto N defined by $\phi_m(\gamma) = \gamma(\phi'(\lambda))$ for all γ in G_m, and let λ be the identity of G. Since N' is unlaminated and uniform, any element x in G can be expressed as $x = \tau_{i_p}^{e_p} \circ \cdots \circ \tau_{i_2}^{e_2} \circ \tau_{i_1}^{e_1}(\lambda)$. Define a function ϕ from G onto G_m as follows.

(1) $\phi(\lambda) = \Lambda$, the identity of G_m,

(2) $$\phi(\tau_{i_p}^{e_p} \circ \cdots \circ \tau_{i_2}^{e_2} \circ \tau_{i_1}^{e_1}(\lambda)) = \mu_{i_p}^{e_p} \circ \cdots \circ \mu_{i_2}^{e_2} \circ \mu_{i_1}^{e_1}(\Lambda)$$

$$= \eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_2}^{e_2} \circ \eta_{i_1}^{e_1},$$

by the definition of μ.

It must be shown that ϕ is well defined. To this end, assume that x is an element of G such that

$$x = \tau_{i_p}^{e_p} \circ \cdots \circ \tau_{i_2}^{e_2} \circ \tau_{i_1}^{e_1}(\lambda) = \tau_{j_q}^{d_q} \circ \cdots \circ \tau_{j_2}^{d_2} \circ \tau_{j_1}^{d_1}(\lambda).$$

Let u be a cell in C. Then, since ϕ' is onto, there is a cell y in G such that $\phi'(y) = u$. Now

$$\tau_{i_p}^{e_p} \circ \cdots \circ \tau_{i_2}^{e_2} \circ \tau_{i_1}^{e_1}(y) = \tau_{j_q}^{d_q} \circ \cdots \circ \tau_{j_2}^{d_2} \circ \tau_{j_1}^{d_1}(y), \text{ since } N' \text{ is uniform.}$$

Hence

$$\phi'(\tau_{i_p}^{e_p} \circ \cdots \circ \tau_{i_2}^{e_2} \circ \tau_{i_1}^{e_1}(y)) = \phi'(\tau_{j_q}^{d_q} \circ \cdots \circ \tau_{j_2}^{d_2} \circ \tau_{j_1}^{d_1}(y)),$$

so that

$$\eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_2}^{e_2} \circ \eta_{i_1}^{e_1}(\phi'(y)) = \eta_{j_q}^{d_q} \circ \cdots \circ \eta_{j_2}^{d_2} \circ \eta_{j_1}^{d_1}(\phi'(y))$$

or

$$\eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_2}^{e_2} \circ \eta_{i_1}^{e_1}(u) = \eta_{j_q}^{d_q} \circ \cdots \circ \eta_{j_2}^{d_2} \circ \eta_{j_1}^{d_1}(u), \qquad \text{for all } u \text{ in } C.$$

Thus,

$$\eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_2}^{e_2} \circ \eta_{i_1}^{e_1} = \eta_{j_q}^{d_q} \circ \cdots \circ \eta_{j_2}^{d_2} \circ \eta_{j_1}^{d_1},$$

so that ϕ is well-defined. ϕ is clearly neighborhood preserving. Therefore, (ϕ, ψ') is a network homomorphism form N' onto N_m. ∎

COROLLARY 4.1. *If* $N = (G, \eta; S, \delta)$ *is a uniform network, then* N_m, *the minimal uniform realization of* N, *is network isomorphic to* N.

Proof. Let I_G and I_S be the identity functions on G and S, respectively. Then (I_G, I_S) is a network isomorphism from N to N. Hence, there is a network homomorphism (ϕ, I_S) from N onto N_m, by Proposition 4.1, and one from N_m onto N, by Theorem 4.1. In particular, let (ϕ_m, I_S) be the network homomorphism from N_m onto N, where $\phi_m(g) = g(\lambda)$ for all g in the connection group of N_m and λ is the identity of G. Then it can be easily shown that $(I_G, I_S) = (\phi_m \circ \phi, I_S \circ I_S)$, so that ϕ_m is one-to-one. Hence, (ϕ_m, I_S) is a network isomorphism. ∎

It is now shown that if only finite networks are to be realized, then a converse of Theorem 4.1 can be obtained.

PROPOSITION 4.2. *If ϕ is a neighborhood preserving function from a uniform network $N' = (C', \eta'; S', \delta')$ onto a finite network $N = (C, \eta; S, \delta)$, then N is balanced.*

Proof. It is first shown that η_i is onto. To this end, let $x \in C$. Since ϕ is onto, there is a cell y in C' such that $\phi(y) = x$. Since N' is uniform, η_i' is onto. Hence, $\eta_i'(z) = y$ for some cell z in C'. But $x = \phi(y) = \phi(\eta_i'(z)) = \eta_i(\phi(z))$, since ϕ is neighborhood preserving. Hence, η_i is onto. Since C is a finite set, η_i is also one-to-one. But this is sufficient for N to be balanced. ▌

COROLLARY 4.2. *Let N be a finite cellular network. Then there is a uniform realization of N iff N is balanced.*

To see that Proposition 4.1 may not hold if the realized network N is infinite, let $C' = \{..., -2, -1, 0, 1, 2,...\}$, $C = \{0, 1, 2,...\}$, and $k = 1$. Define η' and η as follows.

$$\eta'(x) = x - 1 \qquad \text{for all } x \text{ in } C',$$

$$\eta(x) = \begin{cases} x - 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Let ϕ be the function from C' onto C given by

$$\phi(x) = \begin{cases} x, & \text{if } x \geqslant 0, \\ 0, & \text{if } x \leqslant 0. \end{cases}$$

It can be easily verified that ϕ is a neighborhood-preserving function from the uniform network $N' = (C', \eta'; S, \delta)$ onto the network $N = (C, \eta; S, \delta)$, where S and δ are arbitrary. But N is not balanced since $\eta(0) = \eta(1) = 0$, and, therefore, $(\eta(0), 0, 1)$ and $(\eta(1), 1, 1)$ are two different edges directed out of cell 0 and labeled 1.

In order to determine whether or not balanced networks and uniform networks can be realized by arrays, several properties of neighborhood-preserving functions are now developed.

PROPOSITION 4.3. *If there is a neighborhood-preserving function from a uniform network $N' = (G', \tau; S', \delta')$ onto a uniform network $N = (G, \eta; S, \delta)$, then there is a group homomorphism from the connection group G' onto the connection group G.*

Proof. Let ϕ be a neighborhood-preserving function from N' to N, and let ν be the function from G' to G defined by $\nu(g) = \phi(g) \cdot \phi(\lambda')^{-1}$, where g is in G' and λ' is the identity of G'. Let

$$x = \tau_{i_p}^{e_p} \circ \cdots \circ \tau_{i_2}^{e_2} \circ \tau_{i_1}^{e_1}(\lambda') \qquad \text{and} \qquad y = \tau_{j_q}^{d_q} \circ \cdots \circ \tau_{j_2}^{d_2} \circ \tau_{j_1}^{d_1}(\lambda')$$

be two arbitrary cells in G'. Then

$$\begin{aligned}\nu(xy) &= \phi(\tau_{i_p}^{e_p} \circ \cdots \circ \tau_{i_1}^{e_1}(\lambda') \cdot \tau_{j_q}^{d_q} \circ \cdots \circ \tau_{j_1}^{d_1}(\lambda')) \cdot \phi(\lambda')^{-1} \\ &= \phi(\tau_{i_p}^{e_p} \circ \cdots \circ \tau_{i_1}^{e_1} \circ \tau_{j_q}^{d_q} \circ \cdots \circ \tau_{j_1}^{d_1}(\lambda')) \cdot \phi(\lambda')^{-1} \\ &= (\eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_1}^{e_1} \circ \eta_{j_q}^{d_q} \circ \cdots \circ \eta_{j_1}^{d_1}(\phi(\lambda'))) \cdot \phi(\lambda')^{-1};\end{aligned}$$

since ϕ is neighborhood preserving,

$$\begin{aligned}&= \eta_{i_p}^{e_p}(\lambda) \cdot \cdots \cdot \eta_{i_1}^{e_1}(\lambda) \cdot \eta_{j_q}^{d_q}(\lambda) \cdot \cdots \cdot \eta_{j_1}^{d_1}(\lambda) \cdot \phi(\lambda') \cdot \phi(\lambda')^{-1} \\ &= \eta_{i_p}^{e_p}(\lambda) \cdot \cdots \cdot \eta_{i_1}^{e_1}(\lambda) \cdot \eta_{j_q}^{d_q}(\lambda) \cdot \cdots \cdot \eta_{j_1}^{d_1}(\lambda),\end{aligned}$$

where λ is the identity of G. On the other hand,

$$\begin{aligned}\nu(x) \cdot \nu(y) &= \phi(\tau_{i_p}^{e_p} \circ \cdots \circ \tau_{i_1}^{e_1}(\lambda')) \cdot \phi(\lambda')^{-1} \cdot \phi(\tau_{j_q}^{d_q} \circ \cdots \circ \tau_{j_1}^{d_1}(\lambda')) \cdot \phi(\lambda)^{-1} \\ &= \eta_{i_p}^{e_p}(\lambda) \cdot \cdots \cdot \eta_{i_1}^{e_1}(\lambda) \cdot \eta_{j_q}^{d_q}(\lambda) \cdot \cdots \cdot \eta_{j_1}^{d_1}(\lambda).\end{aligned}$$

Hence, $\nu(xy) = \nu(x) \cdot \nu(y)$ so that ν is a group homomorphism from G to G'. ν is clearly onto since ϕ is onto. ▌

COROLLARY 4.3. *Let ϕ be a neighborhood-preserving function from a uniform network N' to a uniform network N. Then the equivalence relation induced by ϕ is a congruence relation for the connection group of N'.*

Proof. Let ν be the homomorphism from the connection group of N' to the connection group of N, defined in the proof of Proposition 4.3. Let x and y be any two elements in the connection group of N' such that $\phi(x) = \phi(y)$. Then $\nu(x) = \phi(x) \cdot \phi(\lambda')^{-1} = \phi(y) \cdot \phi(\lambda')^{-1} = \nu(y)$. Hence, the equivalence relation induced by ϕ is equal to the one induced by the homomorphism ν. ▌

Let ϕ be a neighborhood-preserving function from a uniform network N' to a balanced network N. Then the *kernel* of ϕ is the set of cells of N' which have that same image under ϕ as the identity of the connection group of N'. In the case that N is also uniform, then the kernel of ϕ is the same as the kernel of the congruence relation induced by ϕ.

THEOREM 4.2. *Let $N' = (G, \tau; S', \delta')$ be a uniform network and $N = (C, \eta; S, \delta)$ a balanced network. Then the kernel of any neighborhood-preserving function from G onto C is a subgroup of G. Moreover, N is uniform iff this subgroup is normal.*

Proof. Let ϕ be a neighborhood-preserving function from G onto C, and let x and y be any two cells in the kernel of ϕ. Then x, y, and xy^{-1} have the forms

$$x = \tau_{i_p}^{e_p} \circ \cdots \circ \tau_{i_2}^{e_2} \circ \tau_{i_1}^{e_1}(\lambda),$$

$$y = \tau_{j_q}^{d_q} \circ \cdots \circ \tau_{j_2}^{d_2} \circ \tau_{j_1}^{d_1}(\lambda),$$

and

$$xy^{-1} = \tau_{i_p}^{e_p} \circ \cdots \circ \tau_{i_1}^{e_1} \circ \tau_{j_1}^{-d_1} \circ \cdots \circ \tau_{j_q}^{-d_q}(\lambda),$$

where λ is the identity of G. Since x and y are in the kernel of ϕ,

$$\phi(\lambda) = \phi(x) = \eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_2}^{e_2} \circ \eta_{i_1}^{e_1}(\phi(\lambda))$$

$$= \phi(y) = \eta_{j_q}^{d_q} \circ \cdots \circ \eta_{j_2}^{d_2} \circ \eta_{j_1}^{d_1}(\phi(\lambda)).$$

Hence,

$$\eta_{j_1}^{-d_1} \circ \eta_{j_2}^{-d_2} \circ \cdots \circ \eta_{j_q}^{-d_q}(\phi(\lambda)) = \phi(\lambda),$$

so that

$$\phi(xy^{-1}) = \eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_1}^{e_1} \circ \eta_{j_1}^{-d_1} \circ \cdots \circ \eta_{j_q}^{-d_q}(\phi(\lambda))$$

$$= \eta_{i_p}^{e_p} \circ \cdots \circ \eta_{i_1}^{e_1}(\phi(\lambda))$$

$$= \phi(\lambda).$$

Therefore, xy^{-1} is in the kernel of ϕ, proving that it is a subgroup of G.

If N is uniform, then there is a group homomorphism ν from G onto C, and the kernel of ν is the same as the kernel of ϕ. Hence, the kernel of ϕ is normal. Conversely, assume that the kernel of ϕ is a normal subgroup of G. Let H be the factor group of G with respect to this normal subgroup. Let ζ be the function from H to H^k defined by $\zeta([g]) = ([\tau_1(g)], [\tau_2(g)], \ldots, [\tau_k(g)])$, where $g \in G$, $[g]$ denotes the coset containing g, and k is the neighborhood index of N and N'. It can be easily shown that the uniform network $(H, \zeta; S, \delta)$ is network isomorphic to N. Therefore, if the kernel of ϕ is normal, then N is uniform. ▮

Finally, it is shown that no array can be used to realize a balanced network unless that network is also an array.

LEMMA 4.1. *Let N be a balanced cellular network and N_m its minimal uniform realization. Then N is an array iff N_m is an array.*

Proof. If N is uniform, then, by Corollary 4.1, there is a network isomorphism between N and N_m. Hence, N is an array iff N_m is an array. If N is not uniform, then due to Theorem 4.2, there is a subgroup of the connection group of N_m that is not normal. Hence, this connection group is not Abelian, and N_m is not an array. ∎

THEOREM 4.3. *Let N be a balanced cellular network. Then there is an array realization of N iff N is an array.*

Proof. Let N' be an array realization of the balanced network N, and let N_m be the minimal uniform realization of N. By Proposition 4.1, N' realizes N_m. This together with Proposition 4.3 implies that there is a group homomorphism from the connection group of N', which is Abelian, onto the connection group of N_m. Hence, the connection group of N_m must also be Abelian, and N_m is an array. Due to Lemma 4.1, it follows that N is also an array.

Every cellular network is a realization of itself. Therefore, there is an array realization of N iff N is an array. ∎

From Theorem 4.3, it follows that the only balanced networks that can be realized by arrays are arrays. Hence, the class of arrays is less powerful than the class of balanced networks and the class of uniform networks.

5. SUMMARY

The cellular network model introduced in this paper has been used to classify and investigate regularities in the interconnection structure of networks composed of identical logic modules. Three subclasses of cellular networks, corresponding to three types of interconnection regularity, were defined. The most regular networks, called arrays, can be embedded in an n-dimensional coordinate space. Uniform networks are characterized by the fact that every cell is connected to its neighbors "in the same way". Finally, in a balanced array, the output of each cell is connected to k other cells, where k is the neighborhood index of the network. Moreover, the connections from the output of each cell can be ordered as being connected to the first input terminal of some cell, the second input terminal of another cell, etc. Balanced networks may be considered to be less regular than uniform networks, which may be thought of as being less regular than arrays.

The simulation and realization of one cellular network by another were defined in terms of behavioral homomorphism and network homomorphism, respectively. Network homomorphisms are directly based on the structure of the network. Moreover, it was shown that if a network N_1 realizes a network N_2, then N_1 also simulates N_2. Hence, network homomorphisms, rather than behavioral homomorphisms, were used to compare the behavior of the three classes of networks.

It was shown that any balanced network can be realized by some uniform network. Arrays, on the other hand, can not be used to realize balanced or uniform networks that are not arrays. Moreover, since there is a uniform realization of a given finite network iff that network is balanced, arrays are the only finite networks that can be realized by arrays.

RECEIVED: November 21, 1972

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