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Hall–Littlewood Vertex Operators and Generalized Kostka Polynomials

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A family of vertex operators that generalizes those given by Jing for the Hall–Littlewood symmetric functions is presented. These operators produce symmetric functions related to the Poincaré polynomials referred to as generalized Kostka polynomials in the same way that Jing's operator produces symmetric functions related to Kostka–Foulkes polynomials. These operators are then used to derive commutation relations and new relations involving the generalized Kostka coefficients. Such relations may be interpreted as identities in the $(GL(n)\times\mathbb{C})$-equivariant K-theory of the nullcone. © 2001 Academic Press

1. INTRODUCTION

Kostka–Folkes polynomials may be considered as coefficients of the formal power series representing the character of certain graded $GL(n)$-modules. These $GL(n)$-modules are defined by twisting the coordinate ring of the nullcone by a suitable line bundle [1] and the definition may be generalized by twisting the coordinate ring of any nilpotent conjugacy closure in $gl(n)$ by a suitable vector bundle [13]. The resulting polynomials have been called generalized Kostka polynomials.

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Jing defined a vertex operator that generates the Hall–Littlewood symmetric function $Q[X; q]$ [6], thereby giving an elegant symmetric function recursion for the Kostka–Foulkes polynomials. Garsia used a variant of Jing's vertex operator to derive various new formulas for the Kostka–Foulkes polynomials.

Our point of departure was the observation that the Hall–Littlewood vertex operators can be used to obtain formulas for generalized Kostka polynomials. Our treatment uses Garsia's plethystic type formulas.

One striking fact is that the $\mathbb{Z}[q, q^{-1}]$-linear span of n-fold compositions of components of the Hall–Littlewood vertex operators is isomorphic to $K_{G\times\mathbb{C}^*}(\mathcal{N})$, the $GL(n)\times\mathbb{C}^*$-equivariant K-theory of the nullcone. Under this isomorphism, an n-fold composite operator is sent to the class of the Euler characteristic of a twisted module. This fact has a generalization for all the nilpotent conjugacy class closures in $gl(n)$.

These Grothendieck groups were studied in [7]. We derive many explicit relations among the vertex operators, most of which can be in interpreted as relations in the Grothendieck groups which arise from certain Koszul complexes. This allows for more explicit proofs of some basis theorems for these Grothendieck groups that were proved in [7] using geometric arguments.

There is a particularly well-behaved subfamily of the generalized Kostka polynomials, namely, those that are indexed by a sequence of rectangular partitions. For this subfamily almost all of the formulas for Kostka–Foulkes polynomials have generalizations. There are combinatorial formulas involving Littlewood–Richardson tableaux [10, 11], rigged configurations [8], and inhomogeneous paths with energy function [10, 12].

2. HOPF ALGEBRA OF SYMMETRIC FUNCTIONS AND PLETHYSM

This section contains standard background material on symmetric functions which can be found in [9]. The possible exception to this is the definition of the plethystic notation used here.

Let $\Lambda = \Lambda_{\mathbb{F}}$ be the algebra of symmetric functions over a field $\mathbb{F}$ of characteristic zero. Λ may be defined as the polynomial algebra $\mathbb{F}[p_1, p_2, ...]$ where the p_k are commuting algebraically independent variables. Let $\mathscr{P}$ be the set of partitions. For $\lambda = (\lambda_1, ..., \lambda_k) \in \mathscr{P}$, write $|\lambda| = \sum_{i=1}^k \lambda_i$ and let $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$ denote the power symmetric function. Declaring p_k to have degree k, Λ is endowed with a grading $\Lambda = \bigoplus_{n\geqslant 0} \Lambda^n$ where Λ^n is the homogeneous component of degree n. Λ^n has $\mathbb{F}$-basis $\{p_\lambda \mid \lambda \in \mathscr{P}, |\lambda| = n\}$.

Λ may be realized by symmetric series. Let $X = (x_1, x_2, ...)$ be a countable set of commuting indeterminates. Denote by $\mathbb{F}[[X]]$ the $\mathbb{F}$-algebra of formal power series in the x_i where each x_i has degree 1, and $\mathbb{F}_b[[X]] \subset \mathbb{F}[[X]]$ the subalgebra consisting of series whose monomials are of bounded degree. Let $\Lambda^X \subset \mathbb{F}_b[[X]]$ denote the $\mathbb{F}$-subalgebra consisting of the series that are symmetric in the variables x_i. By the universal mapping property of a polynomial algebra, there is an $\mathbb{F}$-algebra homomorphism $\Lambda \to \Lambda^X$ given by $p_k \mapsto x_1^k + x_2^k + \cdots$. This map is in fact a graded $\mathbb{F}$-algebra isomorphism.

There is a scalar product on Λ for which the power symmetric functions are an orthogonal basis,

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda,$$

where $\delta_{\lambda\mu}$ is the Kronecker delta, $z_\lambda = \prod_i n_i(\lambda)!\, i^{n_i(\lambda)}$, and $n_i(\lambda)$ is the number of parts of size i in the partition λ.

We now define notation for plethystic substitution.

Let $q \in \mathbb{F}$ be a distinguished element that is transcendental over $\mathbb{Q}$. Fix an element $E \in \mathbb{F}_b[[X]]$. Define $p_k[E] \in \mathbb{F}_b[[X]]$ to be the series obtained from E by replacing q by q^k and x_i by x_i^k for all $i \geqslant 1$. By the universal mapping property of a polynomial algebra, there is a unique $\mathbb{F}$-algebra homomorphism $\Lambda \to \mathbb{F}_b[[X]]$ such that $p_k \mapsto p_k[E]$. The image of $P \in \Lambda$ under this map is denoted $P[E]$.

Switching viewpoints and considering $p_k[E]$ for various E, we see directly from the definitions that $E \mapsto p_k[E]$ is a $\mathbb{Q}$-algebra homomorphism $\mathbb{F}_b[[X]] \to \mathbb{F}_b[[X]]$ and $\Lambda^X \to \Lambda^X$ given by $p_k[E \pm E'] = p_k[E] \pm p_k[E']$ and $p_k[EE'] = p_k[E]\, p_k[E']$ for all $E, E' \in \mathbb{F}_b[[X]]$. It is *not* an $\mathbb{F}$-algebra homomorphism since it changes the scalar q to q^k.

By abuse of notation the variable X will also be used to represent the formal sum $x_1 + x_2 + x_3 + \cdots$. By definition

$$p_k[X] = p_k[x_1 + x_2 + x_3 + \cdots] = x_1^k + x_2^k + x_3^k + \cdots.$$

EXAMPLE 1. $e_2 = p_1^2/2 - p_2/2 \in \Lambda$. For the expression $E = x_1 + x_2$ we have

$$e_2[x_1 + x_2] = (x_1 + x_2)^2/2 - (x_1^2 + x_2^2)/2 = x_1 x_2.$$

If $E = 1/(1-q)$ then $p_1[1/(1-q)] = 1/(1-q)$ and $p_2[1/(1-q)] = 1/(1-q^2)$, so that

$$e_2\left[\frac{1}{1-q}\right] = \frac{1}{2(1-q)^2} - \frac{1}{2(1-q^2)} = \frac{q}{(1-q)(1-q^2)}.$$

Recall that each $E \in \mathbb{F}_b[[X]]$ defines an $\mathbb{F}$-algebra homomorphism $\Lambda \to \mathbb{F}_b[[X]]$ such that $P \mapsto P[E]$.

(1) If $E = X$ then $P \mapsto P[X]$ yields the above isomorphism $\Lambda \cong \Lambda^X$.

(2) If $E = 0$ then the $\mathbb{F}$-algebra homomorphism ε: $\Lambda \to \mathbb{F}$ given by $P \mapsto P[0]$ is the *counit* for Λ; it selects the constant term.

(3) If $E = x_1 + x_2 + \cdots + x_m$ then $P \mapsto P[x_1 + x_2 + \cdots + x_m]$ is the $\mathbb{F}$-algebra epimorphism $\Lambda \to \mathbb{F}[x_1, ..., x_m]^{S_m}$ onto the $\mathbb{F}$-algebra of symmetric polynomials in x_1 through x_m where S_m is the symmetric group on m letters.

(4) If $E = X/(q-1)$ then the map $P[X] \mapsto P[X/(q-1)]$ is an $\mathbb{F}$-algebra isomorphism $\Lambda \to \Lambda$ with inverse map $P[X] \mapsto P[X(q-1)]$. In particular if $\{f_\lambda\}$ is a basis of Λ then so are $\{f_\lambda[X(q-1)]\}$ and $\{f_\lambda[X/(q-1)]\}$.

We now discuss the Hopf algebra structure on Λ. Let $Y = y_1 + y_2 + \cdots$ where the y_i are another countable set of commuting indeterminates. Let $\Lambda^{X;\,Y}$ denote the $\mathbb{F}$-subalgebra of $\mathbb{F}_b[[X, Y]]$ consisting of the series that are symmetric in both the x_i and the y_j. Then there is an isomorphism $\Lambda \otimes_{\mathbb{F}} \Lambda \to \Lambda^{X;\,Y}$ given by $f \otimes g \mapsto f[X]\, g[Y]$ for $f, g \in \Lambda$. Letting $E = X + Y$, the map $P \mapsto P[X+Y]$ defines an $\mathbb{F}$-algebra homomorphism Δ: $\Lambda \to \Lambda \otimes \Lambda$. Δ is the comultiplication map for Λ. The maps Δ and ε give Λ the structure of a coassociative cocommutative Hopf algebra.

There is a scalar product on $\Lambda \otimes \Lambda$ defined by

$$\langle f_1[X]\, g_1[Y], f_2[X]\, g_2[Y] \rangle_{\Lambda^X \otimes \Lambda^Y} = \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle$$

for all $f_i, g_i \in \Lambda$ and this relation is then extended by linearity. With respect to this scalar product on $\Lambda \otimes \Lambda$, the map Δ is adjoint to the multiplication map $\Lambda \otimes \Lambda \to \Lambda$ given by $f \otimes g \mapsto fg$. That is,

$$\langle f, gh \rangle = \langle f[X+Y], g[X]\, h[Y] \rangle \tag{1}$$

for all $f, g, h \in \Lambda$.

To deal with the Cauchy kernel it is necessary to work in a completion of Λ. Let $\hat{\Lambda} = \mathbb{F}[[p_1, p_2, ...]]$, the $\mathbb{F}$-algebra of formal power series in the p_i. The symmetric realization $\hat{\Lambda}^X$ of $\hat{\Lambda}$ is given by the $\mathbb{F}$-subalgebra of symmetric series in $\mathbb{F}[[X]]$. Given any element $E \in \mathbb{F}_b[[X]]$ with no constant term and $P \in \hat{\Lambda}$, the plethysm $P[E] \in \mathbb{F}[[X]]$ may be defined by substitution as before.

Define the Cauchy kernel $\Omega \in \hat{\Lambda}$ by

$$\Omega = \exp\left(\sum_{r \geqslant 1} p_r/r \right).$$

The following formulas are consequences of the definitions,

$$\begin{aligned}
\Omega[zX] &= \prod_i \frac{1}{1-zx_i} = \sum_{k\geqslant 0} z^k h_k[X] \\
\Omega[-zX] &= \prod_i 1-zx_i = \sum_{k\geqslant 0} (-z)^k\, e_k[X] \\
\Omega[X+Y] &= \Omega[X]\,\Omega[Y] \\
\Omega[X-Y] &= \Omega[X]\,\Omega[-Y] = \Omega[X]/\Omega[Y] \\
\Omega[XY] &= \sum_{\lambda\in\mathscr{P}} s_\lambda[X]\, s_\lambda[Y],
\end{aligned}$$

where h_k, e_k, and s_λ are the homogeneous, elementary, and Schur symmetric functions, respectively.

Two bases $\{a_\lambda\}$ and $\{b_\lambda\}$ of Λ are dual with respect to the scalar product if and only if $\sum_{\lambda\in\mathscr{P}} a_\lambda[X]\, b_\lambda[Y] = \Omega[XY]$.

For $f\in\Lambda$, let $f^\perp$ be the linear operator on Λ that is adjoint to multiplication by f with respect to the scalar product,

$$\langle f^\perp(g), h\rangle = \langle g, fh\rangle \tag{2}$$

for all g, $h\in\Lambda$. This operator is usually referred to as “f-perp” or “skewing by f.”

The skewing operators and the map Δ (or equivalently the plethystic map $P\mapsto P[X+Y]$) can be expressed in terms of each other. For any f, g, $h\in\Lambda$,

$$\langle f^\perp(g), h\rangle = \langle g[X+Y], f[Y]\, h[X]\rangle \tag{3}$$

using (2), (1), and the commutativity of multiplication in Λ. Let $\{a_\lambda\}$ and $\{b_\lambda\}$ be dual bases of Λ. Then for every $P\in\Lambda$,

$$P[X+Y] = \sum_\lambda (b_\lambda^\perp(P))[X]\, a_\lambda[Y]. \tag{4}$$

3. DEFINITION OF THE OPERATOR

Using plethystic substitution we define a family of linear operators on symmetric functions. Define the formal Laurent series $H(Z^k)$ in an ordered set of variables $Z^k = (z_1, z_2, ..., z_k)$ with coefficients given by operators on Λ, which acts on $P\in\Lambda$ by

$$H(Z^k)\, P[X] = P[X-(1-q)\, Z^*]\, \Omega[ZX]\, R(Z^k), \tag{5}$$

where $Z^* = \sum_{i=1}^k z_i^{-1}$, $Z = \sum_{i=1}^k z_i$, and $R(Z^k) = \prod_{1 \leqslant i < j \leqslant k} 1 - z_j/z_i$. For $v \in \mathbb{Z}^k$, define the operator H_v^q by

$$H_v^q P[X] = H(Z^k)\, P[X]|_{z^v}. \tag{6}$$

Remark 2. (1) If $k=1$ this is Garsia's version [2, 3] of Jing's Hall–Littlewood vertex operator [6].

(2) At $q=0$ this formula reduces to plethystic notation for a repeated application of the Schur function vertex operator that is due to Bernstein [3; 9, p. 96, No. 29(d)]. So $H^0_{(v_1)} H^0_{(v_2)} \cdots H^0_{(v_k)} = H^0_v$ for any $v \in \mathbb{Z}^k$. If v is a partition $\lambda = (\lambda_1, ..., \lambda_k)$ with at most k parts then $H^0_\lambda 1 = s_\lambda[X]$.

(3) When $q=1$ and λ is a partition, this formula reduces to multiplication by the Schur function s_λ: $\Omega[ZX]\, R(Z^k)|_{z^\lambda} = s_\lambda[X]$.

(4) $H^q_\lambda 1 = s_\lambda$ for any partition $\lambda = (\lambda_1, ..., \lambda_k)$ with at most k parts. If $v_k < 0$ then $H^q_v 1 = 0$.

This operator possesses the same shifted skew symmetry in its subscript that Schur functions have. Let $\mathbb{Z}^k_{\geqslant}$ denote the set of dominant integral weights in $\mathbb{Z}^k$, that is, the weakly decreasing sequences in $\mathbb{Z}^k$. Let $\mathscr{P}^k \subset \mathbb{Z}^k_{\geqslant}$ be the set of partitions of length at most k, which are always regarded as having k parts, some of which may be zero.

Proposition 3. *Let* $v \in \mathbb{Z}^k$. *Then*

$$H^q_v = -H^q_{(v_1, v_2, ..., v_{i+1}-1, v_i+1, ..., v_k)}.$$

In particular if $v_{i+1} = v_i + 1$ *then* $H^q_v = 0$.

Proof. Let $s_i Z^k$ be the sequence Z^k with z_i and z_{i+1} exchanged. Then $H(s_i Z^k) = ((1 - z_i/z_{i+1})/(1 - z_{i+1}/z_i))\, H(Z^k) = -(z_i/(z_{i+1}))\, H(Z^k)$. The result follows by taking the coefficient of z^v on both sides of this equation. If $v_{i+1} = v_i + 1$ then $H^q_v = -H^q_v$ and hence must be zero. ∎

The following corollary shows that for every $v \in \mathbb{Z}^k$, H^q_γ is either zero or up to sign equal to H^q_ν for some $\nu \in \mathbb{Z}^k_{\geqslant}$.

Corollary 4. *Let* $v \in \mathbb{Z}^k$, $\sigma \in S_k$, *and* $\rho = \rho^{(k)} = (k-1, k-2, ..., 0)$. *Then*

$$H^q_v = \operatorname{sign}(\sigma)\, H^q_{\sigma(v+\rho)-\rho}. \tag{7}$$

In particular, if $v + \rho$ *has two equal parts then* $H^q_v = 0$. *Otherwise there is a unique* $\sigma \in S_k$ *such that* $\sigma(v+\rho)$ *is strictly decreasing, so that* $\sigma(v+\rho) - \rho \in \mathbb{Z}^k_{\geqslant}$.

4. ANOTHER FORMULA FOR THE VERTEX OPERATOR

We derive another formula for the operator H^q_ν, which will be used to prove a strong linear independence property of the operators. To do this, it is convenient to introduce some notation for the irreducible rational characters of $GL(k)$. The polynomial representation ring of $GL(k)$ is isomorphic to the $\mathbb{F}$-algebra $\mathbb{F}[z_1, z_2, ..., z_l]^{S_k}$ of symmetric polynomials in the variables z_i. The rational representation ring $R(GL(k))$ is isomorphic to the $\mathbb{F}$-algebra of symmetric Laurent polynomials in the variables $z_1, z_2, ...,$ z_k or equivalently the localization $\mathbb{F}[z_1, ..., z_k]^{S_k}[\det[Z]^{-1}]$ of the algebra of symmetric polynomials obtained by inverting the character $\det[Z] = z_1 z_2 \cdots z_k$ of the determinant module. $R(GL(k))$ has a distinguished basis $\{s_\lambda[Z] \mid \lambda \in \mathbb{Z}^k_{\geqslant}\}$ where $s_\lambda[Z]$ is the character of the irreducible finite-dimensional $GL(n)$-module V^λ of highest weight λ. Explicitly, if m is an integer such that $m \geqslant -\lambda_k$ then $s_\lambda[Z] = \det[Z]^{-m} s_{\lambda+(m^k)}[Z]$ where $m^k = (m, m, ..., m) \in \mathbb{Z}^k$, $\lambda + (m^k)$ is the vector sum in $\mathbb{Z}^k$ (which is a partition) and $s_{\lambda+(m^k)}[Z]$ is the Schur polynomial. For $v \in \mathbb{Z}^k$, define its dual weight $v^* = (-v_k, -v_{k-1}, ..., -v_1)$. If $\lambda \in \mathbb{Z}^k_{\geqslant}$ then so is λ^*; it satisfies $(V^\lambda)^* \cong V^{\lambda^*}$ where V^* is the contragredient dual of V.

Denote by $\bar{c}^\lambda_{\mu\nu}$ the tensor product multiplicities,

$$s_\mu[Z]\, s_\nu[Z] = \sum_{\lambda \in \mathbb{Z}^k_{\geqslant}} \bar{c}^\lambda_{\mu\nu} s_\lambda[Z]$$

for $\lambda, \mu, \nu \in \mathbb{Z}^k_{\geqslant}$. It is well known that

$$\bar{c}^\lambda_{\mu\nu} = \dim(V^{\lambda^*} \otimes V^\mu \otimes V^\nu)^{GL(n)}, \tag{8}$$

where V^G denotes the submodule of V fixed by G. Moreover for any finite-dimensional G-module,

$$\dim((V^*)^G) = \dim(V^G). \tag{9}$$

Let

$$\bar{H}(Z)\, P[X] = P[X - (1-q)\, Z^*]\, \Omega[ZX]. \tag{10}$$

Then by (5) and (6),

$$\bar{H}(Z) = \sum_{\nu \in \mathbb{Z}^k_{\geqslant}} s_\nu[Z]\, H^q_\nu. \tag{11}$$

Proposition 5. *For $\nu \in \mathbb{Z}^k_{\geqslant}$,*

$$H^q_\nu = \sum_{\lambda, \mu \in \mathscr{P}^k} \bar{c}^\lambda_{\mu\nu} s_\lambda[X]\, s_\mu[X(q-1)]^\perp. \tag{12}$$

Proof. Let Y be another set of variables and let $\bar{H}(Z)$ act on the X variables. We have

$$\bar{H}(Z)\,\Omega[XY] = \Omega[XY]\,\Omega[(q-1)\,Z^*Y]\,\Omega[XZ]$$
$$= \Omega[XY] \sum_{\nu \in \mathbb{Z}^k_{\geqslant}} \sum_{\lambda, \mu \in \mathscr{P}^k} s_\nu[Z]\,\bar{c}^{\nu}_{\mu^*\lambda} s_\mu[(q-1)\,Y]\,s_\lambda[X].$$

The tensor product multiplicity can be rewritten using (8) and (9) so that $\bar{c}^{\nu}_{\mu^*\lambda} = \bar{c}^{\lambda}_{\mu\nu}$. Clearly we have for fixed $\gamma \in \mathscr{P}^k$ and $\nu \in \mathbb{Z}^k_{\geqslant}$

$$H^q_\nu s_\gamma[X] = \bar{H}(Z)\,\Omega[XY]|_{s_\nu[Z]\,s_\gamma[Y]}$$
$$= \sum_{\lambda, \mu \in \mathscr{P}^k} \bar{c}^{\lambda}_{\mu\nu} \langle s_\mu[(q-1)\,Y]^\perp\, s_\gamma[Y], \Omega[XY]\,s_\lambda[X]\rangle_Y$$
$$= \sum_{\lambda, \mu \in \mathscr{P}^k} \bar{c}^{\lambda}_{\mu\nu} s_\lambda[X]\,s_\mu[(q-1)\,X]^\perp\, s_\gamma[X].$$

It is seen that (12) holds on the Schur basis and hence on Λ. ∎

5. LINEAR INDEPENDENCE

The full strength of the following result is required later, when we work with operators on Λ that are infinite linear combinations of the H^q_ν.

PROPOSITION 6. *Let $\{c_\nu \in \mathbb{F} \mid \nu \in \mathbb{Z}^k_{\geqslant}, |\nu| = d\}$ be an arbitrary collection of scalars. Then the map $F = \sum_{\nu \in \mathbb{Z}^k_{\geqslant}\ |\nu| = d} c_\nu H^q_\nu$ is a well-defined linear operator on Λ. Moreover this operator is 0 if and only if $c_\nu = 0$ for all $\nu \in \mathbb{Z}^k_{\geqslant}$ with $|\nu| = d$.*

Proof. Any infinite linear combination of operators of the form $s_\lambda[X]\,s_\mu[X(q-1)]^\perp$ with $|\lambda| = |\mu| + d$ is well-defined because its action on any given element of the basis $\{s_\lambda[X/(q-1)]\}$ has only finitely many non-zero summands.

The operator $F = \sum_{\nu \in \mathbb{Z}^k_{\geqslant}\ |\nu| = d} c_\nu H^q_\nu$ is well-defined because for fixed partitions λ and μ, $s_\lambda[X]\,s_\mu[X(q-1)]^\perp$ appears in the formula given in Proposition 5 for only finitely many H^q_ν, namely for those such that $\bar{c}^{\lambda}_{\mu\nu} \neq 0$.

For $\gamma \in \mathbb{Z}^k_{\geqslant}$, set $\alpha(\gamma) \in \mathscr{P}^k$ to be the partition with $\alpha(\gamma)_i = \max(\gamma_i, 0)$ and $\beta(\gamma) \in \mathscr{P}^k$ be the partition defined by $\beta(\gamma)_i = -\min(\gamma_{k+1-i}, 0)$.

By Proposition 5, for $\tau \in \mathscr{P}^k$,

$$H^q_\gamma(s_\tau[X/(q-1)]) = \sum_{\lambda, \mu \in \mathscr{P}^k} \bar{c}^{\lambda}_{\mu\gamma} s_\lambda[X]\,s_\mu[X(q-1)]^\perp\,(s_\tau[X/(q-1)]).$$

The Littlewood Richardson rule implies that the coefficient $\bar{c}^{\lambda}_{u\gamma}=0$ unless $\mu \supseteq \beta(\gamma)$. Since $\{s_\mu[X(q-1)]\}$ and $\{s_\tau[X/(q-1)]\}$ are dual bases, we calculate that $H^q_\gamma(s_\tau[X/(q-1)])=0$ if $|\tau| \leqslant |\beta(\gamma)|$ and $\tau \neq \beta(\gamma)$, and $H^q_\gamma(s_{\beta(\gamma)}[X/(q-1)])=s_{\alpha(\gamma)}[X]$.

It follows now that $F=0$ if and only if $c_\gamma=0$ for all γ. For if there exist non-zero coefficients, then γ is chosen such that c_γ is non-zero and $|\beta(\gamma)|$ is a minimum. We see then that

$$F(s_{\beta(\gamma)}[X/(q-1)]) = \sum_{\beta(\nu)=\beta(\gamma)} c_\nu s_{\alpha(\nu)}[X] \neq 0. \quad \blacksquare \tag{14}$$

6. CONNECTION WITH GENERALIZED KOSTKA POLYNOMIALS

Next it is shown that the operators H^q_ν have the same relation to the generalized Kostka polynomials of [13] that the components of Garsia's modified Hall–Littlewood vertex operator has to the Kostka–Foulkes polynomials.

Let us recall the definition of the generalized Kostka polynomials. Let $\eta=(\eta_1, \eta_2, ..., \eta_t)$ be a sequence of positive integers summing to n. Let

$$\text{Roots}_\eta = \{(i, j) \mid 1 \leqslant i \leqslant \eta_1 + \cdots + \eta_k < j \leqslant n \text{ for some } k\}$$

be the set of strictly upper block triangular positions in an $n \times n$ matrix with diagonal block sizes given by η. Let Z be the sequence of variables z_1 through z_n. Define the formal power series $B_\eta[Z; q]$ by

$$B_\eta[Z; q] = \prod_{(i, j) \in \text{Roots}_\eta} \frac{1}{1-qz_i/z_j}.$$

Let J be the antisymmetrizer $J=\sum_{\sigma \in S_n} \text{sign}(\sigma)\,\sigma$. Define the linear operator π: $\mathbb{F}[z_1^{\pm}, ..., z_n^{\pm}] \to R(GL(n))$ given by $\pi(f)=J(z^\rho)^{-1} J(z^\rho f)$.

Let $\gamma \in \mathbb{Z}^n$. Define the generating series

$$\mathbb{H}_{\eta,\gamma}[Z; q] = \pi(z^\gamma B_\eta[Z; q]).$$

Since $\{s_\lambda[Z] \mid \lambda \in \mathbb{Z}^n_{\geqslant}\}$ is a basis of $R(GL(n))$, define $K_{\lambda\gamma\eta}(q) \in \mathbb{Z}[[q]]$ by

$$\mathbb{H}_{\eta,\gamma}[Z; q] = \sum_{\lambda \in \mathbb{Z}^n_{\geqslant}} s_\lambda[Z]\, K_{\lambda\gamma\eta}(q).$$

It is shown in [13] that $K_{\lambda\gamma\eta}(q) \in \mathbb{Z}[q]$.

By developing the product of geometric series in $B_\eta[Z;q]$ and using the shifted skew symmetry of the index for irreducible characters

$$s_v[Z] = \operatorname{sign}(\sigma)\, s_{\sigma(v+\rho)-\rho}[Z]$$

for $v \in \mathbb{Z}^n$ and $\sigma \in S_n$, one obtains the following expression [13],

$$K_{\lambda\gamma\eta}(q) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \sum_{\substack{m:\, \mathrm{Roots}_\eta \to \mathbb{N} \\ \mathrm{wt}(m) = \sigma(\lambda+\rho) - (\gamma+\rho)}} q^{|m|}, \tag{15}$$

where ε_1 through ε_n is the standard basis of $\mathbb{Z}^n$, $|m| = \sum_{(i,j) \in \mathrm{Roots}_\eta} m(i,j)$, and $\mathrm{wt}(m) = \sum_{(i,j) \in \mathrm{Roots}_\eta} m(i,j)(\varepsilon_i - \varepsilon_j)$.

Given η and γ, write $\gamma = (\gamma^{(1)}, \gamma^{(2)}, ..., \gamma^{(t)})$ with $\gamma^{(i)} \in \mathbb{Z}^{\eta_i}$ and assume that $\gamma^{(i)} \in \mathbb{Z}^{\eta_i}_{\geqslant}$.

PROPOSITION 7.

$$H^q_{\gamma^{(1)}} H^q_{\gamma^{(2)}} \cdots H^q_{\gamma^{(t)}} = \sum_{\lambda \in \mathbb{Z}^n_{\geqslant}} K_{\lambda,\gamma,\eta}(q)\, H^q_\lambda. \tag{16}$$

Proof. Let Z^n be the entire set of n variables z_1 through z_n. Break this set into t collections of successive variables such that the ith collection $Z^{(i)}$ has size η_i for $1 \leqslant i \leqslant t$. We have the relation $H(U^k)\, H(V^\ell) = \Omega[qU^*V]\, H(U^k, V^\ell)$, which is verified by showing it holds when evaluated on an arbitrary $P \in \Lambda$,

$$\begin{aligned} H(U^k)\, H(V^\ell)\, P[X] &= P[X - (1-q)(U^* + V^*)]\, \Omega[UX] \\ &\quad \times \Omega[V(X - (1-q)\, U^*)]\, R(U^k)\, R(V^\ell) \\ &= P[X - (1-q)(U^* + V^*)]\, \Omega[(U+V)\, X] \\ &\quad \times \Omega[qU^*V]\, R(U^k, V^\ell). \end{aligned}$$

It follows that

$$H(Z^{(1)})\, H(Z^{(2)}) \cdots H(Z^{(t)}) = H(Z^n) \prod_{1 \leqslant i < j \leqslant t} \Omega[q(Z^{(i)})^*\, Z^{(j)}]. \tag{17}$$

Observe that $H^q_{\gamma^{(1)}} H^q_{\gamma^{(2)}} \cdots H^q_{\gamma^{(t)}}$ is the coefficient of z^γ on the left hand side of (17). The coefficient of z^γ on the right hand side is $\sum_m q^{|m|}\, H^q_{\gamma + \mathrm{wt}(m)}$ where m runs over the functions m: $\mathrm{Roots}_\eta \to \mathbb{N}$. By Proposition 6 the operators H^q_λ are independent for $\lambda \in \mathbb{Z}^n_{\geqslant}$. Hence we may take the coefficient of H^q_λ on both sides. Using Corollary 4 the right hand side becomes precisely the expression (15) of $K_{\lambda\gamma\eta}(q)$. ∎

If γ is such that $\gamma^{(i)} \in \mathscr{P}^{\eta_i}$, define the symmetric function

$$\mathscr{H}_{(\gamma^{(1)}, \ldots, \gamma^{(t)})}[X; q] = H^q_{\gamma^{(1)}} H^q_{\gamma^{(2)}} \cdots H^q_{\gamma^{(t)}} 1. \tag{18}$$

Then

$$\mathscr{H}_{(\gamma^{(1)}, \ldots, \gamma^{(t)})}[X; q] = \sum_{\lambda \in \mathscr{P}^n} K_{\lambda\gamma\eta}(q)\, s_\lambda[X] \tag{19}$$

which is a finite sum, unlike the expansion (16) which is an infinite sum. At $q=1$ this is the expansion of the product of Schur functions $s_{\gamma^{(1)}} \cdots s_{\gamma^{(t)}}$ in the Schur basis.

We have shown that the coefficients of the expansions of $\mathbb{H}_{\eta,\gamma}[Z; q]$ in terms of irreducible characters, and $H^q_{\gamma^{(1)}} \cdots H^q_{\gamma^{(t)}}$ in terms of the H^q_λ for $\lambda \in \mathbb{Z}^n_{\geqslant}$, are the same. Consequently, at least on the level of characters, the questions of [7] regarding a certain Grothendieck ring of graded modules, can be translated into questions regarding the span of the above operators.

7. SPACES OF VERTEX OPERATORS AND COMMUTATION RELATIONS

By manipulating the order of the variables we may derive several sorts of explicit commutation relations. These are applied to prove basis theorems for spaces of our operators. Let $\mathscr{G}(k, n)$ denote the $\mathbb{Z}[q]$-span of operators of the form $H^q_\mu H^q_\nu$ where $\mu \in \mathbb{Z}^k$ and $\nu \in \mathbb{Z}^n$, and $\mathscr{G}_{\geqslant}(k, n)$ the $\mathbb{Z}[q]$-span of such operators with the concatenated weight dominant, that is, $(\mu, \nu) \in \mathbb{Z}^{k+n}_{\geqslant}$.

The following are vertex operator analogues of results proven in [7] for Grothendieck groups. Our proofs have the advantage of being explicit and working over $\mathscr{E}[q]$ instead of $\mathbb{Z}[q, q^{-1}]$. Moreover our relations among vertex operators can be lifted to relations in the Grothendieck groups.

Theorem 8. $\mathscr{G}(k, n) = \mathscr{G}_{\geqslant}(k, n)$.

Theorem 9. *If* $k > n$ *then* $\mathscr{G}(k, n) \subset \mathscr{G}(k-1, n+1)$ *and* $\mathscr{G}(n, k) \subset \mathscr{G}(n+1, k-1)$.

Theorem 10. $\mathscr{G}(k, n) = \mathscr{G}(n, k)$.

Consider the following generating function equation that follows easily from manipulating the operators on an arbitrary symmetric function.

$$H(U^k, V^\ell)\, H(W^m, Z^n)\, \Omega[-q(U^*W+V^*W+V^*Z)] = H(U^k, W^m)\, H(V^\ell, Z^n)\, \Omega[-q(U^*V+W^*V+W^*Z)](-1)^{\ell m} \times \prod_{i=1}^{m} w_i^\ell \prod_{i=1}^{\ell} v_i^{-m}. \tag{20}$$

Setting $\ell=m=1$ in this formula gives enough relations to prove Theorem 8. For brevity of notation, define $|_\beta^\alpha=|\alpha|-|\beta|$. If $\mu, \nu \in \mathbb{Z}_\geqslant^n$ then we will say that ν/μ is a vertical strip, and denote this by $\nu/\mu \in \mathscr{V}$, provided that $\nu_i-\mu_i \in \{0, 1\}$ for all $1 \leqslant i \leqslant n$.

Lemma 11. *For all $a, b \in \mathbb{Z}$ and $\mu \in \mathbb{Z}_\geqslant^k$ and $\nu \in \mathbb{Z}_\geqslant^n$,*

$$\sum_{\alpha/\mu \in \mathscr{V}} \sum_{\nu/\beta \in \mathscr{V}} (-q)^{|_\mu^\alpha+|_\beta^\nu} (H^q_{(\alpha, a+|_\beta^\nu)} H^q_{(b-|_\mu^\alpha, \beta)} - qH^q_{(\alpha, a+|_\beta^\nu+1)} H^q_{(b-|_\mu^\alpha-1, \beta)}) = \sum_{\alpha/\mu \in \mathscr{V}} \sum_{\nu/\beta \in \mathscr{V}} (-q)^{|_\mu^\alpha+|_\beta^\nu} \times (qH^q_{(\alpha, b+|_\beta^\nu)} H^q_{(a-|_\mu^\alpha, \beta)} - H^q_{(\alpha, b+|_\beta^\nu-1)} H^q_{(a-|_\mu^\alpha+1, \beta)}). \tag{21}$$

Proof. From Eq. (20),

$$H(U^k, v)\, H(w, Z^n)\, \Omega[-q(U^*w+v^*w+v^*Z)] = H(U^k, w)\, H(v, Z^n)\, \Omega[-q(U^*v+w^*v+w^*Z)](-w/v).$$

The desired identity is obtained by taking the coefficient of $u^\mu v^a w^b z^\nu$. ∎

Note that if $b=a+1$, then (21) reduces significantly to the identity

$$\sum_{\alpha/\mu \in \mathscr{V}} \sum_{\nu/\beta \in \mathscr{V}} (-q)^{|_\mu^\alpha+|_\beta^\nu} H^q_{(\alpha, a+|_\beta^\nu)} H^q_{(a+1-|_\mu^\alpha, \beta)} = \sum_{\alpha/\mu \in \mathscr{V}} \sum_{\nu/\beta \in \mathscr{V}} q(-q)^{|_\mu^\alpha+|_\beta^\nu} H^q_{(\alpha, a+|_\beta^\nu+1)} H^q_{(a-|_\mu^\alpha, \beta)}. \tag{22}$$

Relations (21) and (22) suffice to prove Theorem 8. Before giving the proof an example is helpful.

Example 12. We wish to show that $H^q_{(22)}H^q_{(41)}$ is a linear combination of $H^q_\alpha H^q_\beta$ with $\alpha, \beta \in \mathbb{Z}^2$ such that (α, β) is dominant.

$$\begin{aligned} &H^q_{(22)}H^q_{(41)} - qH^q_{(23)}H^q_{(40)} - qH^q_{(32)}H^q_{(31)} + q^2H^q_{(33)}H^q_{(30)} \\ &\quad - qH^q_{(23)}H^q_{(31)} + q^2H^q_{(24)}H^q_{(30)} + q^2H^q_{(33)}H^q_{(21)} - q^3H^q_{(34)}H^q_{(20)} \\ &= qH^q_{(24)}H^q_{(21)} - q^2H^q_{(25)}H^q_{(20)} - q^2H^q_{(34)}H^q_{(11)} + q^3H^q_{(35)}H^q_{(10)} \\ &\quad - H^q_{(23)}H^q_{(31)} + qH^q_{(24)}H^q_{(30)} + qH^q_{(33)}H^q_{(21)} - q^2H^q_{(34)}H^q_{(20)}. \end{aligned}$$

By Corollary 4 many terms vanish and others cancel with each other. The terms $H^q_{(23)}H^q_{(40)}$, $H^q_{(23)}H^q_{(31)}$, $H^q_{(34)}H^q_{(20)}$, and $H^q_{(34)}H^q_{(11)}$ are all zero. The terms $q^2H^q_{(24)}H^q_{(30)}$ and $q^2H^q_{(33)}H^q_{(30)}$ cancel and so do $qH^q_{(24)}H^q_{(21)}$ and $qH^q_{(33)}H^q_{(21)}$. When this relation is reduced and $H^q_{(22)}H^q_{(41)}$ is expressed alone on the left hand side of the equation we have

$$H^q_{(22)}H^q_{(41)} = qH^q_{(32)}H^q_{(31)} - q^2H^q_{(33)}H^q_{(21)} + q^2H^q_{(43)}H^q_{(20)} - q^3H^q_{(44)}H^q_{(10)} - qH^q_{(33)}H^q_{(30)}.$$

On the right hand side of this equation, only $H^q_{(32)}H^q_{(31)}$ does not have the property that the concatenated indexing weights are dominant. We apply relation (22) to this operator:

$$H^q_{(32)}H^q_{(31)} - qH^q_{(33)}H^q_{(30)} - qH^q_{(42)}H^q_{(21)} + q^2H^q_{(43)}H^q_{(20)} = qH^q_{(33)}H^q_{(21)} - q^2H^q_{(34)}H^q_{(20)} - q^2H^q_{(43)}H^q_{(11)} + q^3H^q_{(44)}H^q_{(10)}.$$

Therefore

$$H^q_{(22)}H^q_{(41)} = (q^2-q)\,H^q_{(33)}H^q_{(30)} + q^2H^q_{(42)}H^q_{(21)} - q^3H^q_{(43)}H^q_{(11)} + (q^2-q^3)\,H^q_{(43)}H^q_{(20)} + (q^4-q^3)\,H^q_{(44)}H^q_{(10)}.$$

We now give the proof of Theorem 8.

Proof. It is enough to show that if $(\mu, a) \in \mathbb{Z}^k_{\geqslant}$ and $(b, v) \in \mathbb{Z}^n_{\geqslant}$ then

$$H^q_{(\mu, a)}H^q_{(b, v)} \in \mathscr{G}_{\geqslant}(k, n).$$

If $b = a+1$, then in the relation (22) the only term that is not indexed by weights such that $(\alpha, \beta) \in \mathbb{Z}^{k+n}_{\geqslant}$, is $H^q_{(\mu, a)}H^q_{(a+1, v)}$, and hence it is in $\mathscr{G}_{\geqslant}(k, n)$.

Otherwise assume that $b > a+1$. All of the terms of Eq. (21) are of the form $H^q_{(\alpha, a+|^v_\beta)}H^q_{(b-|^\alpha_\mu, \beta)}$, $H^q_{(\alpha, a+|^v_\beta+1)}H^q_{(b-|^\alpha_\mu-1, \beta)}$, $H^q_{(\alpha, b+|^v_\beta)}H^q_{(a-|^\alpha_\mu, \beta)}$, or $H^q_{(\alpha, b+|^v_\beta-1)}H^q_{(a-|^\alpha_\mu+1, \beta)}$. Let $H^q_{(\theta, c)}H^q_{(d, \gamma)}$ be one of these terms after it has been rewritten using Corollary 4 so that $(\theta, c) \in \mathbb{Z}^k_{\geqslant}$ and $(d, \gamma) \in \mathbb{Z}^n_{\geqslant}$.

Consider $H^q_{(\theta, c)}H^q_{(d, \gamma)}$ of the first form since the others follow from essentially the same remark. We verify that $d-c < b-a$ (unless $H^q_{(\theta, c)}H^q_{(d, \gamma)} = H^q_{(\mu, a)}H^q_{(a+1, v)}$) and hence by induction the theorem is true since if $d-c \leqslant 0$ then $(\theta, c, d, \gamma) \in \mathbb{Z}^{k+n}_{\geqslant}$. By definition,

$$c = \min_{1 \leqslant i \leqslant k-1} \{a + |^v_\beta, \alpha_i + (k-i)\}$$

$$d = \max_{1 \leqslant i \leqslant n-1} \{b - |^\alpha_\mu, \beta_i - i\}.$$

We note that

$$c \geqslant \min_{1 \leqslant i \leqslant k-1} \{a + |^{\nu}_{\beta}, \mu_i + (k-i)\} \geqslant \min\{a + |^{\nu}_{\beta}, a+1\}$$

and hence $c \geqslant a$ with equality if and only if $|^{\nu}_{\beta} = 0$. Similarly

$$d \leqslant \max_{1 \leqslant i \leqslant n-1} \{b - |^{\alpha}_{\mu}, \nu_i - i\} \leqslant \max\{b - |^{\alpha}_{\mu}, b - i\}$$

and so $d \leqslant b$ with equality if and only if $|^{\alpha}_{\mu} = 0$. ∎

The generalization of this statement, that $H^q_{\gamma^{(1)}} H^q_{\gamma^{(2)}} \cdots H^q_{\gamma^{(t)}}$ with $\gamma^{(i)} \in \mathbb{Z}^{\eta_i}$ is in the $\mathbb{Z}[q]$ span of the operators $H^q_{\alpha^{(1)}} H^q_{\alpha^{(2)}} \cdots H^q_{\alpha^{(t)}}$ with $\alpha^{(i)} \in \mathbb{Z}^{\eta_i}$ and $(\alpha^{(1)}, \alpha^{(2)}, ..., \alpha^{(t)})$ a dominant weight, is conjectured to be true [7], but does not seem to follow easily from these relations.

By setting $\ell = 1$ and $m = 0$ in (20), we immediately obtain another relation among the operators which gives us another basis theorem.

Lemma 13. *For all* $a \in \mathbb{Z}$, $\mu \in \mathbb{Z}^k_{\geqslant}$, *and* $\nu \in \mathbb{Z}^n_{\geqslant}$,

$$\sum_{\substack{\beta \in \mathbb{Z}^n_{\geqslant} \\ \nu/\beta \in \mathscr{V}}} (-q)^{|^{\nu}_{\beta}} H^q_{(\mu, a + |^{\nu}_{\beta})} H^q_{\beta} = \sum_{\substack{\alpha \in \mathbb{Z}^k_{\geqslant} \\ \alpha/\mu \in \mathscr{V}}} (-q)^{|^{\alpha}_{\mu}} H^q_{\alpha} H^q_{(a - |^{\alpha}_{\mu}, \nu)}. \tag{23}$$

Proof. From Eq. (20),

$$H(U^k, v)\, H(Z^n)\, \Omega[-qv^*Z] = H(U^k)\, H(v, Z^n)\, \Omega[-qvU^*] \tag{24}$$

The stated identity is obtained by taking the coefficient of $u^{\mu} v^{a} z^{\nu}$. ∎

Before proving Theorem 9 we give an example.

Example 14. We wish to write the composition of operators $H^q_{(53)} H^q_{(2)}$ as a linear combination of operators $H^q_{\alpha} H^q_{\beta}$ with $\alpha \in \mathbb{Z}^1$ and $\beta \in \mathbb{Z}^2$,

$$H^q_{(53)} H^q_{(2)} - q H^q_{(54)} H^q_{(1)} = H^q_{(5)} H^q_{(32)} - q H^q_{(6)} H^q_{(22)}$$
$$H^q_{(54)} H^q_{(1)} - q H^q_{(55)} H^q_{(0)} = H^q_{(5)} H^q_{(41)} - q H^q_{(6)} H^q_{(31)}$$
$$H^q_{(55)} H^q_{(0)} = H^q_{(5)} H^q_{(50)} - q H^q_{(6)} H^q_{(40)}.$$

Making repeated substitutions, we have the relation

$$H^q_{(53)} H^q_{(2)} = q^2 H^q_{(5)} H^q_{(50)} - q^3 H^q_{(6)} H^q_{(40)} + q H^q_{(5)} H^q_{(41)} - q^2 H^q_{(6)} H^q_{(31)} + H^q_{(5)} H^q_{(32)} - q H^q_{(6)} H^q_{(22)}.$$

We now give the proof of Theorem 9.

Proof. We prove that if $k>n$ then $\mathcal{G}(k,n)\subset\mathcal{G}(k-1,n+1)$ as the other part is proven in the same manner. It is enough to show that if $k>n$, $(\mu,a)\in\mathbb{Z}^k_{\geqslant}$ and $\nu\in\mathbb{Z}^n_{\geqslant}$ then

$$H^q_{(\mu,a)}H^q_\nu\in\mathcal{G}_{\geqslant}(k-1,n+1).$$

We use (23) with $k-1$ and n. The proof proceeds by induction on μ_1-a. If $\mu_1-a=0$, then there is exactly one term on the left hand side of Eq. (23), namely $H^q_{(\mu,a)}H^q_\nu$. The right hand side is a $\mathbb{Z}[q]$-linear combination of $H^q_\gamma H^q_\rho$ with $\gamma\in\mathbb{Z}^{k-1}$ and $\rho\in\mathbb{Z}^{n+1}$.

If $\mu_1-a>0$, then the left hand side is a linear combination of operators of the form $H^q_{(\gamma,c)}H^q_\beta$ with $\gamma_1=\mu_1$ and

$$\begin{aligned}c&=\min_{1\leqslant i\leqslant k-1}\{a+|^\nu_\beta,\mu_i+(k-i)\}\\&\geqslant\min_{1\leqslant i\leqslant k-1}\{a+|^\nu_\beta,a+(k-i)\}.\end{aligned}$$

It follows that $c\geqslant a$ with equality if and only if $\nu=\beta$. The right hand side of this equation is again an element of $\mathcal{G}(k-1,n+1)$. ∎

Finally we prove Theorem 10.

Proof. We shall prove $\mathcal{G}(k+\ell,k)\subseteq\mathcal{G}(k,k+\ell)$ for $k,\ell>0$, the other inclusion being similar. Consider (20) with $m=0$ and $n=k$ so $U=U^k$, $V=V^\ell$, $Z=Z^k$:

$$H(U,V)\,H(Z)\,\Omega[-qV^*Z]=H(U)\,H(V,Z)\,\Omega[-qU^*V].\qquad(25)$$

Take the coefficient of $u^\alpha v^\beta z^\gamma$ in this equation, where $(\alpha,\beta)\in\mathbb{Z}^{k+\ell}_{\geqslant}$ and $\gamma\in\mathbb{Z}^k_{\geqslant}$. The entire right hand side consists of terms in $\mathcal{G}(k,k+\ell)$. Expanding the expression $\Omega[-qV^*Z]=\prod(1-qz_i/v_j)$, the left hand side is in the set

$$H^q_{\alpha,\beta}H^q_\gamma+\mathbb{Z}[q]\sum_{\beta',\gamma'}H^q_{\alpha,\beta'}H^q_{\gamma'},$$

where $\beta'\in\mathbb{Z}^\ell$ such that $\beta'_j\leqslant\beta_j+k$ for all $1\leqslant j\leqslant\ell$ and $|\beta'|>|\beta|$. Rewriting a typical term $H^q_{\alpha,\beta'}$ by Corollary 4, one obtains either 0 or up to sign $H^q_{\alpha',\beta''}$, say. This means $\alpha'_1+(k+\ell-1)$ is the largest part of the weight $(\alpha,\beta')+\rho^{(k+\ell)}$. Now $\alpha_1\geqslant\beta_j$ for all $1\leqslant j\leqslant\ell$ by the dominance of (α,β), so

$$\begin{aligned}\alpha_1+k+\ell-1&\geqslant\beta_j+k+\ell-1\geqslant\beta'_j+\ell-1\\&\geqslant\beta'_j+\ell-j.\end{aligned}$$

It follows that $\alpha'_1 = \alpha_1$, $(\alpha', \beta'') \in \mathbb{Z}_{\geqslant}^{k+\ell}$ and $|\alpha'| + |\beta''| > |\alpha| + |\beta|$. There are only finitely many elements of $\mathbb{Z}_{\geqslant}^{k+\ell}$ with these properties, so the terms $H^q_{(\alpha', \beta'')} H^q_{\gamma'}$ can again be rewritten in the same way, and the process terminates. ∎

8. RECOVERING IDENTITIES VIA COMMUTATION RELATIONS

The commutation relations in the previous section may be used to recover some identities among the operators H^q_ν that correspond to known identities among generalized Kostka polynomials.

PROPOSITION 15. *For $a \in \mathbb{Z}$ and positive integers k and n,*

$$H^q_{(a^n)} H^q_{(a^k)} = H^q_{(a^k)} H^q_{(a^n)}. \tag{26}$$

Proof. In the proof of Theorem 10, take $\alpha = (a^k)$, $\beta = (a^\ell)$, and $\gamma = (a^k)$. The proof shows there is only one term on either side of the identity coming from (25), which agrees with (26) when $n = k + \ell$. ∎

PROPOSITION 16.

$$H^q_{(a^k)} H^q_{((a+1)^k)} = q^k H^q_{((a+1)^k)} H^q_{(a^k)}. \tag{27}$$

Proof. When Lemma 13 is applied with $\mu = (a^{k-1})$, a, and $\nu = ((a+1)^k)$, there are no surviving terms on the right and exactly two on the left, indexed by $\beta = \nu = ((a+1)^k)$ and $\beta = (a^k)$. Applying Corollary 4 to the term with $\beta = (a^k)$ the desired identity is obtained. ∎

PROPOSITION 17. *For all $a \in \mathbb{Z}$ and $k \geqslant 1$,*

$$H^q_{(a^k)} H^q_{(a^k)} = H^q_{(a^{k+1})} H^q_{(a^{k-1})} + q^k H^q_{((a+1)^k)} H^q_{((a-1)^k)}. \tag{28}$$

Proof. Apply Lemma 13 with $n = k-1$, $\mu = (a^k)$, and $\nu = (a^{k-1})$. On the left side one has the single nonvanishing term $H^q_{(a^{k+1})} H^q_{(a^{k-1})}$ corresponding to the summand $\beta = \nu = (a^{k-1})$. The right side has two nonvanishing terms indexed by $\alpha = \mu = (a^k)$ and $\alpha = \mu + (1^k) = ((a+1)^k)$. The first of these terms is $H^q_{(a^k)} H^q_{(a^k)}$. The second is $(-q)^k\, H^q_{((a+1)^k)} H^q_{(a-k,\, a^{k-1})}$. But $H^q_{(a-k,\, a^{k-1})} = (-1)^{k-1}\, H^q_{((a-1)^k)}$ by Corollary 4. ∎

This identity can be viewed as the trace of a short exact sequence that resolves the ideal of a nilpotent conjugacy class closure over the coordinate ring of a minimally larger one [7]. The version of this identity for the fermionic form of generalized Kostka polynomials appears in [10]. This q-character identity is put in a more general context in [5].

9. GENERALIZATION OF GARSIA–PROCESI DEFINING RECURRENCE FOR KOSTKA–FOULKES POLYNOMIALS

Manipulations of the definition allow us to derive commutation relations between the H_ν^q and the operators $e_k^\perp$. As a consequence we obtain a generalization of a defining recurrence for the Kostka–Foulkes polynomials given in [4].

Let $E(u)$ be the generating function of operators on Λ defined by

$$E(u)\, P[X] = P[X-u]. \tag{29}$$

By (4) we have

$$P[X-u] = \sum_{\lambda \in \mathscr{P}} s_\lambda^\perp(P)[X]\, s_\lambda[-u] = \sum_{k \geqslant 0} e_k^\perp(P)[X](-u)^k.$$

In other words

$$e_k^\perp P[X] = (-1)^k\, P[X-u]|_{u^k}. \tag{30}$$

The commutation relation of $H(Z^n)$ and $E(u)$ is

$$\begin{aligned} E(u)\, H(Z^n)\, P[X] &= E(u)\, P[X-(1-q)\, Z^*]\, \Omega[XZ]\, R(Z^n) \\ &= P[X-u-(1-q)\, Z^*]\, \Omega[(X-u)\, Z]\, R(Z^n) \\ &= \Omega[-uZ]\, H(Z^n)\, E(u)\, P[X]. \end{aligned}$$

Taking the coefficient of $(-u)^k z^\lambda$ on both sides of this equation we obtain the following relation.

Proposition 18. *Let $\lambda \in \mathbb{Z}^n_\geqslant$. Then*

$$e_k^\perp H_\lambda^q = \sum_{\substack{\beta \in \mathbb{Z}^n_\geqslant \\ \lambda/\beta \in \mathscr{V}}} H_\beta^q e^\perp_{k-|\lambda|+|\beta|}.$$

Let $\eta = (\eta_1, \eta_2, ..., \eta_t)$ be a fixed sequence of positive integers summing to n. For any weight $\gamma \in \mathbb{Z}^n$, write $\gamma^{(1)} \in \mathbb{Z}^{\eta_1}$ for the first η_1 parts of γ, $\gamma^{(2)} \in \mathbb{Z}^{\eta_2}$ for the next η_2 parts of γ, etc.

PROPOSITION 19. *Let k be a fixed positive integer, $\eta=(\eta_1, \eta_2, ..., \eta_t)$ a sequence of positive integers summing to n, $\alpha \in \mathbb{Z}^n_{\geqslant}$, and $\gamma \in \mathbb{Z}^n$ such that $\gamma^{(i)} \in \mathbb{Z}^{\eta_i}_{\geqslant}$ for all i and $|\gamma|-|\alpha|=k$. Then*

$$\sum_{\substack{\nu \in \mathbb{Z}^n \\ |\gamma|-|\nu|=k \\ \nu^{(i)} \in \mathbb{Z}^{\eta_i}_{\geqslant} \\ \gamma^{(i)}/\nu^{(i)} \in \mathscr{V}}} K_{\alpha, \nu, \eta}(q) = \sum_{\substack{\lambda \in \mathbb{Z}^{\eta}_{\geqslant} \\ |\lambda|-|\alpha|=k \\ \lambda/\alpha \in \mathscr{V}}} K_{\lambda, \gamma, \eta}(q).$$

Proof. Let η and γ be as above. For the moment assume that the entries of γ are positive. Apply $e_k^{\perp}$ to (16) and apply Proposition 18 to both sides to commute $e_k^{\perp}$ to the right of the H operators,

$$\sum_{\substack{\nu \in \mathbb{Z}^n \\ \nu^{(i)} \in \mathbb{Z}^{\eta_i}_{\geqslant} \\ \gamma^{(i)}/\nu^{(i)} \in \mathscr{V}}} H^q_{\nu^{(1)}} H^q_{\nu^{(2)}} \cdots H^q_{\nu^{(t)}} e^{\perp}_{k-|\gamma|+|\nu|}$$

$$= \sum_{\lambda \in \mathbb{Z}^n_{\geqslant}} \sum_{\substack{\beta \in \mathbb{Z}^n_{\geqslant} \\ \lambda/\beta \in \mathscr{V}}} K_{\lambda, \nu, \eta}(q)\, H^q_{\beta} e^{\perp}_{k-|\lambda|+|\beta|}.$$

Since γ has positive parts, all the ν have nonnegative parts. Expanding the left hand side using (16) and applying the resulting operators to $1 \in \Lambda$, we obtain

$$\sum_{\substack{\nu \in \mathbb{Z}^n \\ |\gamma|-|\nu|=k \\ \nu^{(i)} \in \mathbb{Z}^{\eta_i}_{\geqslant} \\ \gamma^{(i)}/\nu^{(i)} \in \mathscr{V}}} \sum_{\mu \in \mathbb{Z}^n_{\geqslant}} K_{\mu, \nu, \eta}(q)\, H^q_{\mu} 1 = \sum_{\lambda \in \mathbb{Z}^n_{\geqslant}} \sum_{\substack{\beta \in \mathbb{Z}^n_{\geqslant} \\ |\lambda|-|\beta|=k \\ \lambda/\beta \in \mathscr{V}}} K_{\lambda, \nu, \eta}(q)\, H^q_{\beta} 1.$$

Assume for the moment that α has nonnegative parts. For such α, $H^q_{\alpha} 1 = s_{\alpha}[X]$. Taking the coefficient of $s_{\alpha}[X]$ on both sides, we obtain the desired relation.

The statement is true in general since $K_{\lambda-(a^n), \gamma-(a^n), \eta}(q) = K_{\lambda, \gamma, \eta}(q)$ for all integers a. ∎

The recurrence for the Kostka–Foulkes polynomials given in [4] is recovered by setting $\eta=(1^n)$ in Proposition 19, as $H^q_{(m)}$ in our notation is H_m in theirs. In the case $\eta=(1^n)$ the situation is particularly nice; the nondominant operators can be made dominant using the relation $H^q_{(m)} H^q_{(m+1)} = q H^q_{(m+1)} H^q_{(m)}$. For general η the identities needed to rewrite the nondominant terms as dominant ones, can get complicated and produce negative signs.

10. JING'S OPERATORS

We give corresponding constructions for Jing's Hall–Littlewood vertex operators [6, 9, Ex. III.5.8]. Define

$$\bar{B}(Z^k)\, P[X] = P[X - Z^*]\, \Omega[XZ(1-q)]$$
$$B(Z^k)\, P[X] = P[X - Z^*]\, \Omega[XZ(1-q)]\, R(Z^k)$$
$$B^q_\nu = B(Z^k)|_{z^\nu}$$

for $\nu \in \mathbb{Z}^k$. The generating series of operators $B(z)$ (for a single variable z) and $\bar{B}(Z^k)$ defined here, coincide with the operators $B(z)$ and $B(z_1, z_2, ..., z_k)$ in the notation of [9, Ex. III.5.8]. $B(z)$ in our notation is the operator $H(z)$ in Jing [6]. However, the operators B_ν themselves are not studied by Jing or Macdonald.

Let F be the plethystic operator $FP[X] = P[X(1-q)]$, with inverse operator $F^{-1}P[X] = P[X/(1-q)]$. Then it is not hard to show that

$$B(Z^k) = F \circ H(Z^k) \circ F^{-1}. \tag{31}$$

This gives the following analogue of (16) for $B(Z^k)$.

$$B^q_{\gamma(1)} B^q_{\gamma(2)} \cdots B^q_{\gamma(t)} = \sum_{\lambda \in \mathbb{Z}^n_{\geqslant}} K_{\lambda,\gamma,\eta}(q)\, B^q_\lambda. \tag{32}$$

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