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Application of Laplace Transformation to Evaluation of Integrals

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A new procedure for the evaluation of definite and infinite integrals which is connected mainly with the Laplace transformation, but also with the Mellin and Stieltjes integral transforms, is discussed. The advantage of the proposed technique is illustrated by considering a number of different types of integrals. © 1994 Academic Press, Inc.

1. INTRODUCTION

A number of formulas for the evaluation of finite and infinite integrals using the Laplace transform technique are presented in this paper. The application of these formulas is illustrated by the consideration of a variety of integrals. Some of the integrals treated here are related to the polylogarithms [1–6] which are of importance in certain physical applications, in particular in quantum electrodynamics [7–10]. However, the proposed procedure, which is characterized by a simplicity of mathematical operations, is of more general character than the explicit evaluation of integrals involving powers of logarithms and algebraic functions.

It is worthwhile to mention that the Laplace transformation is a standard technique used to evaluate infinite integrals by applying the operation of integration, differentiation, and integration under the integral sign with respect to the transform variable s, frequently in combination with the rules and theorems of the operational calculus [11–14]. Repeated use of different integral transforms and the Laplace transforms expressed in terms of the logarithmic functions which are connected with the evalua-

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tion of several classes of integrals were discussed by us previously [15, 16].

2. Evaluation of Integrals Using the Laplace Transformation

Let us assume that function $f(t)$ possess the Laplace transform

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s). \tag{1}$$

Then by changing the integration variable, $x = e^{-st}$, Eq. (1) becomes

$$\int_0^1 f\left(-\frac{1}{s}\ln x\right) dx = sF(s). \tag{2}$$

This is a starting point for the evaluation of many logarithmic integrals because extensive tables of the function–transform pairs are available in the literature [17–22]. For example, taking into consideration that [22, p. 38, (5.10)]

$$L\left\{\frac{t^n}{1 - e^{-at}}\right\} = (-a)^{-n-1}\psi^{(n)}\left(\frac{s}{a}\right),$$
$$\text{Re } a > 0, \text{ Re } s > 0, n = 1, 2, 3, \ldots \tag{3}$$

and denoting $\alpha = a/s$, we have from (2)

$$\int_0^1 \frac{(\ln x)^n}{1 - x^\alpha} dx = -\frac{1}{\alpha^{n+1}} \psi^{(n)}\left(\frac{1}{\alpha}\right), \qquad \alpha > 0, n = 1, 2, 3, \ldots \tag{4}$$

where $\psi^{(n)}(z)$ are derivatives of the psi (digamma)-function [23].

A well-known result is derived by introducing $n = 1$ and $\alpha = 1$ into (4) [24, p. 577, (4.23.2)]

$$\int_0^1 \frac{\ln x}{1 - x} dx = -\frac{\pi^2}{6} \tag{5}$$

because $\psi^{(1)}(1) = \pi^2/6$.

The second example considered does not include logarithmic integrals. From the function–transform pair [22, p. 81, (7.152)]

$$f(t) = \frac{\cos[a\sqrt{e^t - 1}]}{\sqrt{1 - e^{-t}}} \tag{6a}$$

$$F(s) = \frac{2^{1-s}\sqrt{\pi}\,a^s}{\Gamma(s+1/2)} K_s(a), \qquad \operatorname{Re} s > -\frac{1}{2}, \tag{6b}$$

and using (2) we have

$$\int_0^1 \frac{\cos[a\sqrt{(1/x)^{1/s}-1}]}{\sqrt{1-x^{1/s}}}\,dx = \frac{2^{1-s}\sqrt{\pi}\,sa^s}{\Gamma(s+1/2)} K_s(a). \tag{7}$$

By changing the integration variable, $t = x^{1/\mu}$, $\mu = s$, Eq. (7) takes the form

$$K_\mu(a) = \frac{2^{\mu-1}\Gamma(\mu+1/2)}{\sqrt{\pi}\,a^\mu} \int_0^1 \frac{t^{\mu-1}}{\sqrt{1-t}} \cos\left[a\sqrt{\frac{1}{t}-1}\right] dt, \qquad \operatorname{Re}\mu > -\frac{1}{2}, \tag{8}$$

which can be treated as a new integral representation of the modified Bessel function of the second kind $K_\mu(z)$ [23].

Now, we will apply systematically the rules and theorems of the operational calculus (in the context of Eq. (2)) assuming the existence of the Laplace transform of function $f(t)$ and the permissibility of performed mathematical operations in the cases under consideration. From

$$\int_0^\infty e^{-st} f(t+a)\,dt = e^{as}\left[F(s) - \int_0^a e^{-su} f(u)\,du\right], \qquad a \geq 0 \tag{9}$$

it follows that

$$\int_0^1 f\left(a - \frac{1}{s}\ln x\right) dx = se^{as}\left[F(s) - \int_0^a e^{-su} f(u)\,du\right]. \tag{10}$$

This formula is illustrated by two examples. From [22, p. 16, (2.27)]

$$L\left\{\frac{1}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}} \tag{11}$$

we have

$$\int_0^1 \frac{dx}{\sqrt{a - (1/s)\ln x}} = se^{as}\left[\sqrt{\frac{\pi}{s}} - \int_0^a e^{-su}\frac{du}{\sqrt{u}}\right], \tag{12}$$

but [24, p. 367, (3.361)]

$$\int_0^a e^{-su}\frac{du}{\sqrt{u}} = \sqrt{\frac{\pi}{s}}\operatorname{erf}(\sqrt{as}), \tag{13}$$

where erf z is the error function [23]; denoting $\alpha = a\,s$, the final result is

$$\int_0^1 \frac{dx}{\sqrt{\alpha - \ln x}} = \sqrt{\pi}\, e^{\alpha}\, \mathrm{erfc}(\sqrt{\alpha}), \qquad \alpha > 0, \tag{14}$$

where erfc z is the complementary error function [23]. Two similar integrals

$$\int_0^1 \sqrt{\alpha - \ln x}\, dx = \left[\sqrt{\alpha} + \frac{\sqrt{\pi}}{2} e^{\alpha}\, \mathrm{erfc}(\sqrt{\alpha})\right] \tag{15a}$$

$$\int_0^1 \frac{dx}{(\alpha - \ln x)^{3/2}} = \frac{2}{\sqrt{\alpha}}\, [1 - \sqrt{\pi\alpha}\, e^{\alpha}\, \mathrm{erfc}(\sqrt{\alpha})], \qquad \alpha > 0 \tag{15b}$$

can be evaluated in the same way.

In the second example we will start with [22, p. 48, (6.1)]:

$$L\{\ln t\} = -\frac{1}{s}(\gamma + \ln s), \qquad \mathrm{Re}\, s > 0, \tag{16}$$

where $\gamma = 0.57721\ \ldots$ is the Euler constant; taking into account that

$$\int_0^a e^{-su} \ln u\, du = \frac{1}{s}\, [\mathrm{Ei}(-as) - e^{-as} \ln a - \ln s - \gamma] \tag{17}$$

we have from (10)

$$\int_0^1 \ln\, [a - b \ln x]\, dx = \left[\ln a - e^{a/b}\, \mathrm{Ei}\left(-\frac{a}{b}\right)\right], \tag{18}$$

where $b = 1/s$ and Ei(z) is the exponential integral [23]. Denoting $\alpha = a/b$, Eq. (18) takes the simple form

$$\int_0^1 \ln\, (\alpha - \ln x)\, dx = [\ln \alpha - e^{\alpha}\, \mathrm{Ei}\,(-\alpha)], \qquad \alpha > 0. \tag{19}$$

From the series expansion of the exponential integral [23]

$$\mathrm{Ei}\,(-z) = \gamma + \ln z + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!n} \tag{20}$$

it follows that the integral (19), in the limit, $\alpha \to 0$, becomes

$$\int_0^1 \ln\left[\ln\left(\frac{1}{x}\right)\right] dx = -\gamma, \tag{21}$$

which is a well-known result [24, p. 576, (4.229.1)].

Applying the translation property of the Laplace transformation

$$L\{e^{-at}f(t)\} = F(s + a) \tag{22}$$

we have

$$\int_0^1 x^{a/s} f\left(-\frac{1}{s}\ln x\right) dx = sF(s + a) \tag{23}$$

for $a > 0$ and

$$\int_0^1 x^{-a/s} f\left(-\frac{1}{s}\ln x\right) dx = sF(s - a), \qquad \operatorname{Re}(s - a) > 0 \tag{24}$$

for $a < 0$. For example, applying (23) to (16) we obtain

$$\int_0^1 x^{a/s} \ln\left(-\frac{1}{s}\ln x\right) dx = -\frac{s}{s + a}[\gamma + \ln(s + a)], \tag{25}$$

which, by denoting $\alpha = 1 + a/s$, takes the familiar form [23, p. 620, (4.325.8)]

$$\int_0^1 x^{\alpha-1} \ln\left[\ln\left(\frac{1}{x}\right)\right] dx = -\frac{1}{\alpha}[\gamma + \ln \alpha], \qquad \alpha > 0. \tag{26}$$

The multiplication of function $f(t)$ by powers of t is expressed in the Laplace transformation by

$$L\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}, \qquad n = 1, 2, 3, \ldots \tag{27}$$

and therefore

$$\int_0^1 (\ln x)^n f\left(-\frac{1}{s}\ln x\right) dx = s^{n+1} \frac{d^n F(s)}{ds^n}. \tag{28}$$

In the simplest case $f(t) = 1$ and $F(s) = 1/s$, we have immediately from (28)

$$\int_0^1 (\ln x)^n dx = s^{n+1} \frac{d^n(s^{-1})}{ds^n} = (-1)^n n! \tag{29}$$

A more interesting example is [22, p. 39, (5.16)]

$$L\left\{\frac{1}{1+e^{-t}}\right\}=\frac{1}{2}\left[\psi\left(\frac{s+1}{2}\right)-\psi\left(\frac{s}{2}\right)\right],\qquad \text{Re } s>0, \tag{30}$$

which leads to ($\alpha = 1/s$)

$$\int_0^1 \frac{(\ln x)^n}{1+x^\alpha}\,dx=\frac{1}{(2\alpha)^{n+1}}\left[\psi^{(n)}\left(\frac{\alpha+1}{2\alpha}\right)-\psi^{(n)}\left(\frac{1}{2\alpha}\right)\right],$$
$$\alpha>0,\ n=1,2,3,\ldots \tag{31}$$

For $\alpha = 1$, because the derivatives of psi-function can be expressed in terms of the Riemann zeta function $\zeta(z)$ [23]

$$\psi^{(n)}(1)=(-1)^n n!\zeta(n+1) \tag{32a}$$

$$\psi^{(n)}\left(\frac{1}{2}\right)=(-1)^n(2^{n+1}-1)n!\zeta(n+1), \tag{32b}$$

we have

$$\int_0^1 \frac{(\ln x)^n}{1+x}\,dx=(-1)^{n+1}\left(\frac{1}{2^n}-1\right)n!\zeta(n+1). \tag{33}$$

In particular, for $n = 1$ ($\zeta(2) = \psi^{(1)}(1) = \pi^2/6$, [23]) and for $n = 2$ it follows that

$$\int_0^1 \frac{\ln x}{1+x}\,dx=-\frac{\pi^2}{12} \tag{34a}$$

$$\int_0^1 \frac{(\ln x)^2}{1+x}\,dx=\frac{3}{2}\zeta(3) \tag{34b}$$

The special case of (27) is the multiplication of function $f(t)$ by noninteger powers of t [11, p. 22]

$$L\{t^\mu f(t)\}=-\frac{1}{\Gamma(1-\mu)}\int_0^\infty u^{-\mu}\left\{\frac{dF(s+u)}{d(s+u)}\right\}du,\qquad |\mu|<1, \tag{35}$$

which is equivalent to

$$\int_0^1\left[\ln\left(\frac{1}{x}\right)\right]^\mu f\left(-\frac{1}{s}\ln x\right)dx=-\frac{s^{\mu+1}}{\Gamma(1-\mu)}\int_0^\infty u^{-\mu}\left\{\frac{dF(s+u)}{d(s+u)}\right\}du,$$
$$|\mu|<1. \tag{36}$$

Evidently, if we denote $g(t) = t^{\mu}f(t)$ and

$$L\{g(t)\} = L\{t^{\mu}f(t)\} = G(s) \tag{37}$$

then

$$L\{t^{n+\mu}f(t)\} = (-1)^n \frac{d^nG(s)}{ds^n}, \qquad n = 1, 2, 3, \dots \tag{38}$$

which is the generalization of (27).

If $G(s)$ is known, it is possible to evaluate the right-hand side integrals of (36). For example, from

$$L\{\sin(at)\} = \frac{a}{s^2 + a^2} \tag{39}$$

we have the equality of integrals ($\alpha = a/s$)

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{\mu} \sin(\alpha \ln x)\, dx = -\frac{2\alpha}{\Gamma(1-\mu)} \int_0^{\infty} u^{-\mu} \frac{(u+1)}{[\alpha^2 + (u+1)^2]^2}\, du, \qquad \alpha > 0,\ |\mu| < 1. \tag{40}$$

However, the Laplace transform of $g(t) = t^{\mu} \sin(at)$ is known [21, p. 33, (5.23)]:

$$L\{t^{\mu} \sin(at)\} = \frac{\Gamma(\mu+1)}{(s^2+a^2)^{(\mu+1)/2}} \sin\left[(\mu+1)\tan^{-1}\left(\frac{a}{s}\right)\right]. \tag{41}$$

Therefore

$$\int_0^{\infty} u^{-u} \frac{(u+1)}{[\alpha^2 + (u+1)^2]^2}\, du = \frac{\Gamma(1-\mu)\Gamma(\mu+1)}{2\alpha(\alpha^2+1)^{(\mu+1)/2}} \sin[(\mu+1)\tan^{-1}(\alpha)], \qquad \alpha > 0,\ |\mu| < 1. \tag{42}$$

On the other hand

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{\mu} \sin(\alpha \ln x)\, dx = -\frac{\Gamma(\mu+1)}{(\alpha^2+1)^{(\mu+1)/2}} \sin[(\mu+1)\tan^{-1}(\alpha)], \qquad \alpha > 0,\ |\mu| < 1. \tag{43}$$

In particular, for $\mu = 0$ and $\mu = 1/2$ and $\alpha = 1$ ($\tan^{-1}(1) = \pi/4$), it follows from (43) that

$$\int_0^1 \sin(\ln x)\, dx = -\frac{1}{2} \tag{44a}$$

$$\int_0^1 \sqrt{\ln\left(\frac{1}{x}\right)} \sin(\ln x)\, dx = -\frac{\sqrt{\pi}}{2^{7/4}} \sin\left(\frac{3\pi}{8}\right). \tag{44b}$$

The division of function $f(t)$ by powers of t in the Laplace transformation is related by

$$L\{t^{-(n+1)}f(t)\} = \frac{1}{n!}\int_0^\infty u^n F(s+u)\, du, \qquad n = 0, 1, 2, 3, \ldots \tag{45}$$

and therefore

$$\int_0^1 (\ln x)^{-(n+1)} f\left(-\frac{1}{s}\ln x\right) dx = \frac{(-1)^{n+1}}{n!s^n}\int_0^\infty u^n F(s+u)\, du,$$

$$n = 0, 1, 2, 3, \ldots \tag{46}$$

which takes the simplest form for $n = 0$ and $s = 1$

$$\int_0^1 \frac{1}{\ln x} f(-\ln x)\, dx = -\int_1^\infty F(u)\, du. \tag{47}$$

For example using (39) and (46), $\alpha = a/s$, we obtain

$$\int_0^1 \frac{\sin(\alpha \ln x)}{\ln x}\, dx = \alpha \int_0^\infty \frac{du}{[\alpha^2 + (u+1)^2]} \tag{48}$$

but the indefinite integral

$$\int \frac{du}{\alpha^2 + 1 + 2u + u^2} = \frac{1}{\alpha}\tan^{-1}\left(\frac{u+1}{\alpha}\right) + \text{const.} \tag{49}$$

is known, and therefore

$$\int_0^1 \frac{\sin(\alpha \ln x)}{\ln x}\, dx = \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{\alpha}\right)\right] = \tan^{-1}(\alpha), \qquad \alpha > 0. \tag{50}$$

If both integrals in (45) converge for $n = 0$ and $s \to 0$ then

$$\int_0^\infty \left\{\frac{1}{t} f(t)\right\} dt = \int_0^\infty F(u)\, du \tag{51}$$

and

$$\int_0^1 \frac{1}{x \ln x} f(-\ln x)\, dx = -\int_0^\infty F(u)\, du. \tag{52}$$

Combining (47) with (52) we have

$$\int_0^1 \left(\frac{1}{x} - 1\right) \frac{f(-\ln x)}{\ln x}\, dx = -\int_0^1 F(u)\, du. \tag{53}$$

From the operational relation

$$L\{f[a(e^{-t} - 1)]\} = \frac{1}{a\Gamma(s + 1)} \int_0^\infty u^s e^{-au} F\left(\frac{u}{a}\right) du, \qquad a > 0 \tag{54}$$

it follows that

$$\int_0^1 f\left\{a\left[\left(\frac{1}{x}\right)^{1/s} - 1\right]\right\} dx = \frac{a^s s}{\Gamma(s + 1)} \int_0^\infty u^s e^{-as} F(u)\, du. \tag{55}$$

If the parameter s is chosen to be a positive integer, then, applying (27), Eq. (55) can be written in the form

$$\int_0^1 f\left\{a\left[\left(\frac{1}{x}\right)^{1/n} - 1\right]\right\} dx = \frac{(-a)^n}{(n - 1)!} \frac{d^n G(a)}{da^n}, \qquad n = 1, 2, 3, \ldots \tag{56}$$

where $G(a)$ is the Stieltjes transform of function $f(t)$ (i.e., the iterated Laplace transform) [19, Vol. 2, Ch. 14]

$$G(a) = \int_0^\infty e^{-as} F(u)\, du = \int_0^\infty \frac{f(t)}{a + t}\, dt. \tag{57}$$

Let us consider the following example where both the Laplace and Stieltjes transforms are tabulated [22, p. 65, (7.74) and 19, p. 219, (14.2.36)]:

$$f(t) = \sin(\sqrt{t}\,) \tag{58a}$$

$$F(s) = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s} \tag{58b}$$

$$G(a) = \pi e^{-\sqrt{a}}, \qquad a > 0. \tag{58c}$$

From (56) and (58c) we immediately have

$$\int_0^1 \sin\left\{\sqrt{a\left[\left(\frac{1}{x}\right)^{1/n} - 1\right]}\right\} dx = \frac{(-1)^n \pi a^n}{(n-1)!} \frac{d^n(e^{-\sqrt{a}})}{da^n} \tag{59}$$

which is for $n = 1$

$$\int_0^1 \sin\left\{\sqrt{\frac{a(1-x)}{x}}\right\} dx = \frac{\pi\sqrt{a}}{2} e^{-\sqrt{a}}. \tag{60}$$

However, it also follows from (55) and (58b) that

$$\int_0^1 \sin\left\{\sqrt{a\left[\left(\frac{1}{x}\right)^{1/n} - 1\right]}\right\} dx = \frac{\sqrt{\pi}\, a^n}{2(n-1)!} \int_0^\infty u^{n-3/2} e^{-au-1/4u}\, du. \tag{61}$$

The right-hand side integral is known [24, p. 391, (3.471.9)]:

$$\int_0^\infty u^{n-3/2} e^{-au-1/4u}\, du = \frac{1}{2^{n-3/2} a^{n/2-1/4}} K_{n-1/2}(\sqrt{a}). \tag{62}$$

Therefore

$$\int_0^1 \sin\left\{\sqrt{a\left[\left(\frac{1}{x}\right)^{1/n} - 1\right]}\right\} dx = \frac{\sqrt{\pi}\, a^{n/2+1/4}}{2^{n-1/2}(n-1)!} K_{n-1/2}(\sqrt{a}). \tag{63}$$

For $n = 1$ and taking into account that [23]

$$K_{1/2}(\sqrt{a}) = \sqrt{\frac{\pi}{2}}\, a^{-1/4} e^{-\sqrt{a}} \tag{64}$$

the previous result (60) is derived. By comparing (59) with (63), the modified Bessel function of half odd integer order can be expressed by the relation

$$K_{n-1/2}(\sqrt{a}) = (-1)^n 2^{n-1/2} \sqrt{\pi}\, a^{n/2-1/4} \frac{d^n(e^{-\sqrt{a}})}{da^n}, \qquad n = 1, 2, 3, \ldots \tag{65}$$

By putting $s = 1/n$, $n = 2, 3, 4, \ldots$ in Eq. (55) we have

$$\int_0^1 f\left\{a\left[\left(\frac{1}{x}\right)^n - 1\right]\right\} dx = \frac{a^{1/n}}{\Gamma(1/n)} \int_0^\infty u^{1/n} e^{-au} F(u)\, du, \tag{66}$$

but if $G(a)$, the Stieltjes transform of function $f(t)$ in (57) is available, then it is possible to use (35) with $\mu = 1/n$

$$\int_0^1 f\left\{a\left[\left(\frac{1}{x}\right)^n - 1\right]\right\} dx = \frac{a^{1/n}\sin(\pi/n)}{\pi}\int_0^\infty u^{-1/n}\left\{\frac{dG(a+u)}{d(a+u)}\right\} du, \qquad n = 2, 3, 4, \ldots \quad (67)$$

where $\Gamma(1/n)\Gamma(1 - 1/n) = \pi/\sin(\pi/n)$ [23].

Let us return to the previous example (58); then by applying (67) we have

$$\int_0^1 \sin\left\{\sqrt{a\left[\left(\frac{1}{x}\right)^n - 1\right]}\right\} dx = \frac{a^{1/n}\sin(\pi/n)}{2}\int_0^\infty \frac{u^{-1/n}}{\sqrt{a+u}} e^{-\sqrt{a+u}}\, du, \qquad n = 2, 3, 4, \ldots \quad (68)$$

The infinite integral in (68) can be evaluated because [24, p. 393, (3.479.1)]

$$\int_0^\infty x^{\mu-1}\frac{e^{-\beta\sqrt{1+x}}}{\sqrt{1+x}}\, dx = \frac{2\Gamma(1/\mu)}{\sqrt{\pi}}\left(\frac{\beta}{2}\right)^{1/2-\mu} K_{1/2-\mu}(\beta), \qquad \operatorname{Re}\beta > 0,\ \operatorname{Re}\mu > 0. \quad (69)$$

Denoting $\alpha^2 = a$, and taking into account that $K_{-\mu}(z) = K_\mu(z)$ [23], from (68) and (69) the final result is reached:

$$\int_0^1 \sin\left[\alpha\sqrt{\left(\frac{1}{x}\right)^n - 1}\right] dx = \frac{2^{1/2-1/n}\Gamma(n/n-1)\sin(\pi/n)\,\alpha^{3/n+1/2}}{\sqrt{\pi}} K_{1/2-1/n}(\alpha), \qquad \alpha > 0,\ n = 2, 3, 4, \ldots \quad (70)$$

In particular, for $n = 2$ and $n = 3$

$$\int_0^1 \sin\left[a\sqrt{\left(\frac{1}{x^2} - 1\right)}\right] dx = \frac{\alpha^2}{\sqrt{\pi}} K_0(\alpha) \quad (71a)$$

$$\int_0^1 \sin\left[a\sqrt{\left(\frac{1}{x^3} - 1\right)}\right] dx = \frac{\sqrt{3}\,\alpha^{3/2}}{2^{11/6}} K_{1/6}(\alpha), \qquad \alpha > 0. \quad (71b)$$

The Laplace transform of derivatives of $f(t)$ is given by

$$L\{f^{(n)}(t)\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0), \qquad n = 1, 2, 3, \ldots \quad (72)$$

which is equivalent to

$$\int_0^1 f^{(n)}\left(-\frac{1}{s}\ln x\right) dx = s^{n+1}F(s) - \sum_{k=0}^{n-1} s^{n-k}f^{(k)}(0). \tag{73}$$

For example, taking $f(t)$ and $F(s)$ given in (30), the first two derivatives are

$$f(t) = \frac{1}{1+e^{-t}}, \qquad f(0) = \frac{1}{2} \tag{74a}$$

$$f'(t) = \frac{e^{-t}}{(1+e^{-t})^2}, \qquad f'(0) = \frac{1}{4} \tag{74b}$$

$$f''(t) = -\frac{e^{-t}}{(1+e^{-t})^2} + \frac{2e^{-2t}}{(1+e^{-t})^3}, \tag{74c}$$

and therefore from (2), (30), and (73) we have

$$\int_0^1 \frac{dx}{1+x^\alpha} = \frac{1}{2\alpha}\left[\psi\left(\frac{\alpha+1}{2\alpha}\right) - \psi\left(\frac{1}{2\alpha}\right)\right], \qquad \alpha > 0 \tag{75a}$$

$$\int_0^1 \frac{x^\alpha}{(1+x^\alpha)^2}\, dx = \left\{\frac{1}{2\alpha^2}\left[\psi\left(\frac{\alpha+1}{2\alpha}\right) - \psi\left(\frac{1}{2\alpha}\right)\right] - \frac{1}{2\alpha}\right\} \tag{75b}$$

$$\int_0^1 \frac{x^{2\alpha}}{(1+x^\alpha)^3}\, dx = \left\{\frac{\alpha+1}{4\alpha^3}\left[\psi\left(\frac{\alpha+1}{2\alpha}\right) - \psi\left(\frac{1}{2\alpha}\right)\right] - \frac{2+3\alpha}{8\alpha^2}\right\}. \tag{75c}$$

From integrals in (75), the first integral is known [24, p. 345, (3.24.1)], the second integral can be evaluated only for $\alpha = 2$ from [24, p. 348, 3.251.7)], and the last integral is new.

There are a number of substitution formulas of the Laplace transformation which can serve in evaluation of rather complex infinite integrals, this time if logarithmic integrals are tabulated.

From the operational relation

$$L\{f(t^2)\} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{u}}\, e^{-s^2/4u} F(u)\, du \tag{76}$$

$$\int_0^1 f\left[\left(\frac{\ln x}{s}\right)^2\right] dx = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{u}}\, e^{-s^2/4u} F(u)\, du. \tag{77}$$

If the left-hand side integral is known, then it is easy to recognize function $f(t)$, and having $F(s)$, it is possible to evaluate the infinite integral in (77).

For example, let us start with the logarithmic integral [24, p. 569, (4.213.1)]

$$\int_0^1 \frac{dx}{[a^2 + \ln x)^2]} = \frac{1}{a}\,[\sin(a)\,\mathrm{ci}(a) - \cos(a)\,\mathrm{si}(a)], \qquad a > 0 \tag{78}$$

or, in a more familiar form ($a = \alpha s$),

$$\int_0^1 \frac{dx}{\left[\alpha^2 + \left(\frac{\ln x}{s}\right)^2\right]} = \frac{s}{\alpha}\,[\sin(\alpha s)\,\mathrm{ci}(\alpha s) - \cos(\alpha s)\,\mathrm{si}(\alpha s)] \tag{79}$$

it is clear that the desired function is $f(t) = 1/(\alpha^2 + t)$; its Laplace transform is [22, p. 13, (2.9)]

$$F(s) = -\,e^{\alpha^2 s}\mathrm{Ei}(-\alpha^2 s). \tag{80}$$

From (77), (79), and (80) with $\beta = s$, we have

$$\int_0^\infty \frac{1}{\sqrt{u}}\,e^{\alpha^2 u - \beta^2/4u}\,\mathrm{Ei}(-\alpha^2 u)\,du = \frac{2\sqrt{\pi}}{\alpha}\,[\cos(\alpha\beta)\,\mathrm{si}(\alpha\beta) - \sin(\alpha\beta)\,\mathrm{ci}(\alpha\beta)],$$
$$\alpha > 0,\ \beta > 0, \tag{81}$$

where $\mathrm{si}(z)$ and $\mathrm{ci}(z)$ are the sine and cosine integrals [23]. The evaluated integral (81) is identical with that given in [24, Vol. 2, p. 29, (6.229)] if $\alpha = 1/2$, $\beta^2 = 4\mu$, and the integration variable is changed to $u = 1/x^2$.

In the same manner it is possible to evaluate infinite integrals of the parabolic cylinder $D_\mu(z)$ [23] and Bessel functions

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^\mu f\left[\left(\frac{\ln x}{s}\right)^2\right] dx = \frac{s^{\mu+1}}{\sqrt{2\pi}} \int_0^\infty u^{\mu-2} e^{-s^2u^2/4} D_\mu(su)\,F\left(\frac{1}{2u^2}\right) du \tag{82a}$$

$$\int_0^1 f\left[\left(-\frac{\ln x}{s}\right)^3\right] dx = \frac{s^{3/2}}{3\pi} \int_0^\infty \frac{1}{\sqrt{u}}\,K_{1/3}\left[2\left(\frac{s}{3u}\right)^{3/2}\right] F(u)\,du \tag{82b}$$

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^\mu f\left[-\frac{s}{\ln x}\right] dx = s^{\mu/2} \int_0^\infty u^{\mu/2} J_\mu(2\sqrt{su})\,F(u)\,du,$$
$$\mathrm{Re}\,\mu > -1 \tag{82c}$$

$$\int_0^1 f\left\{\frac{a}{2}\left[\left(\frac{1}{x}\right)^{1/s} - x^{1/s}\right]\right\} dx = s \int_0^\infty J_s(au)\,F(u)\,du, \qquad a > 0. \tag{82d}$$

If μ is a positive integer, the parabolic cylinder functions can be replaced in (82a) by the Hermite polynomials [21]. However, the value of (82) is rather limited because the known left-hand side integrals of this type are rare.

Since in many cases both Laplace transforms $L\{f(t)\} = F(s)$ and $L\{f(t)/t\} = H(s)$ are tabulated, it is possible to evaluate some complex integrals, because for $n = 0$, Eq. (45) becomes

$$\int_0^\infty e^{-st} \left\{\frac{f(t)}{t}\right\} dt = \int_s^\infty F(u)\, du, \tag{83}$$

which is of special interest for $s = 1$. For example from [22, p. 55, (7.12) and (7.14)]

$$F(s) = L\left\{\frac{\sin(at)}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{2}}\sqrt{\sqrt{a^2+s^2} - s/a^2 + s^2} \tag{84a}$$

$$H(s) = L\left\{\frac{\sin(at)}{t^{3/2}}\right\} = \sqrt{2\pi}\sqrt{\sqrt{a^2+s^2} - s} \tag{84b}$$

we have immediately from (83)

$$\int_1^\infty \sqrt{\sqrt{a^2+u^2} - u/a^2 + u^2}\, du = 2\sqrt{\sqrt{a^2+1} - 1}\,. \tag{85}$$

Evaluation of integrals via the Laplace transformation is demonstrated here by numerous examples, but other integral transforms can also be applied for the same purposes. Closely related to the Laplace transformations are the Mellin, Stieltjes, and generalized Stieltjes transforms [19]

$$M\{f(t)\} = \int_0^\infty t^{s-1} f(t)\, dt = M(s) \tag{86a}$$

$$S\{f(t)\} = \int_0^\infty \frac{f(t)}{a+t}\, dt = G(a) \tag{86b}$$

$$S_\rho\{f(t)\} = \int_0^\infty \frac{f(t)}{(a+t)^\rho}\, dt = G(a;\rho), \tag{86c}$$

which after a change of integration variable are

$$\int_0^1 \frac{1}{x}\left[\ln\left(\frac{1}{x}\right)\right]^{s-1} f(-\ln x)\, dx = M(s) \tag{87a}$$

$$\int_0^1 \frac{1}{x(1-\ln x)} f(-a\ln x)\, dx = G(a), \qquad a > 0 \tag{87b}$$

$$\int_0^1 \frac{1}{x(1-\ln x)^\rho} f(-a \ln x)\, dx = a^{\rho-1} G(a; \rho), \qquad a > 0 \tag{87c}$$

or in the form

$$\int_1^\infty \frac{1}{x} (\ln x)^{s-1} f(\ln x)\, dx = M(s) \tag{88a}$$

$$\int_1^\infty \frac{1}{x(1+\ln x)} f(a \ln x)\, dx = G(a), \qquad a > 0 \tag{88b}$$

$$\int_1^\infty \frac{1}{x(1+\ln x)^\rho} f(a \ln x)\, dx = a^{\rho-1} G(a; \rho), \qquad a > 0. \tag{88c}$$

The properties of these integral transforms as related to the evaluation of integrals will be discussed elsewhere, but three simple examples are given here.

From [19, pp. 312–313 (6.3.6) and (6.3.10)]

$$M\left\{\frac{1}{1+e^t}\right\} = \Gamma(s)(1 - 2^{1-s})\zeta(s), \qquad \text{Re } s > 0 \tag{89a}$$

$$M\left\{\frac{1}{(e^t - 1)^2}\right\} = \Gamma(s)[\zeta(s-1) - \zeta(s)], \qquad \text{Re } s > 2. \tag{89b}$$

Using (87a) with $\alpha = s - 1$, we have

$$\int_0^1 \frac{1}{1+x}\left[\ln\left(\frac{1}{x}\right)\right]^\alpha dx = \Gamma(\alpha + 1)(1 - 2^{-\alpha})\zeta(\alpha + 1), \qquad \alpha > -1 \tag{90a}$$

$$\int_0^1 \frac{1}{(1-x)^2}\left[\ln\left(\frac{1}{x}\right)\right]^\alpha dx = \Gamma(\alpha + 1)[\zeta(\alpha) - \zeta(\alpha + 1), \qquad \alpha > 1. \tag{90b}$$

For $\alpha = n$; $n = 1, 2, 3, \ldots$ (90a) is identical with (33).

Using the Stieltjes transform, from [19, Vol. 2, p. 216, (14.2.2)]

$$S\left\{\frac{1}{1+t}\right\} = \frac{1}{(a-1)} \ln a, \qquad a > 0 \tag{91}$$

it follows that

$$\int_0^1 \frac{dx}{x(1-\ln x)(1 - a \ln x)} = \frac{1}{(a-1)} \ln a, \qquad a > 0, \tag{92}$$

which becomes for $a = 1$

$$\int_0^1 \frac{dx}{x(1 - \ln x)^2} = 1. \tag{93}$$

The generalized Stieltjes transform of the same function is [19, Vol. 2, p. 233, (14.4.9)]

$$S_\rho\left\{\frac{1}{1+t}\right\} = \frac{a^{1-\rho}}{\rho}\,{}_2F_1(1, 1; \rho + 1; 1 - a), \qquad \operatorname{Re}\rho > 0, \tag{94}$$

and therefore

$$\int_0^1 \frac{dx}{x(1 - a\ln x)(1 - \ln x)^\rho} = \frac{1}{\rho}\,{}_2F_1(1, 1; \rho + 1; 1 - a), \qquad a > 0,\ \rho > 0, \tag{95}$$

which is identical with (92) for $\rho = 1$, because the hypergeometric function in this case is given by ${}_2F_1\,(1, 1; 2; z) = (1/z)\ln(1 - z)$.

The last example is [19, Vol. 2, pp. 217 and 233, (14.4.11) and (14.4.10)]

$$S\,\{e^{-t}\} = -e^a\,\mathrm{Ei}(-a) \tag{96a}$$

$$S_\rho\{e^{-t}\} = e^a\Gamma(1 - \rho, a), \qquad \operatorname{Re}\rho > 0, \tag{96b}$$

which, from (87b) and (87c) becomes

$$\int_0^1 \frac{x^{a-1}}{(1 - \ln x)}\,dx = -e^a\,\mathrm{Ei}(-a), \qquad a > 0 \tag{97a}$$

$$\int_0^1 \frac{x^{a-1}}{(1 - \ln x)^\rho}\,dx = e^a\Gamma(1 - \rho, a), \qquad a > 0,\ \rho > 0. \tag{97b}$$

Both results are identical for $\rho = 1$, because the incomplete gamma function then becomes $\Gamma(0, a) = -\mathrm{Ei}(-a)$.

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References

1. M. J. Levine, "Dilogarithms and Associated Functions," Elsevier/North Holland, New York, 1981.
2. K. S. Kölbig, J. A. Mignaco, and E. Remiddi, On Nielsen's generalized polylogarithms and their numerical calculations, *Nordisk. Tidskr. Inform. (BIT)* **10** (1970), 38–74.
3. R. Gastmans and W. Troost, On the evaluation of polylogarithmic integrals, *Simon Stevin* **55** (1981), 205–219.
4. K. S. Kölbig, Closed expressions for the integral $\int_0^1 t^{-1} \log^{n-1} t \log^p(1-t)\, dt$, *Math. Comp.* **39** (1982), 647–654.
5. K. S. Kölbig, On the integral $\int_0^\infty e^{-\mu t} t^{\nu-1} \log^m t\, dt$, *Math. Comp.* **41** (1983), 171–182.
6. K. S. Kölbig, On the integral $\int_0^\infty x^{\nu-1} (1+\beta x)^{-\lambda} \ln^m x\, dx$, *J. Comput. Appl. Math.* **14** (1983), 319–344.
7. J. A. Mignaco and E. Remiddi, Fourth-order vacuum polarization contribution in the 6th order electron magnetic moment, *Nuovo Cimento A* **60** (1969), 519–529.
8. R. Barbieri, J. A. Mignaco, and E. Remiddi, Electron form factors up to the fourth order I, *Nuovo Cimento A* **11** (1972), 824–864.
9. M. J. Levine, E. Remiddi, and R. Roskies, Analytical contributions to the *g* factor of the electron in sixth order. *Phys. Rev. D* **20** (1979), 2068–2077.
10. A. Devoto and D. W. Duke, Table of integrals and formulae for Feynman diagram calculations, *Riv. Nuovo Cimento* **7** (1984), 1–39.
11. M. Parodi, "Introduction a l'Étude de l'Analyse Symbolique," pp. 20–21, Gauthier-Villars, Paris, 1957.
12. N. W. McLachlan, "Laplace Transforms and Their Applications to Differential Equations," pp. 14–38, 100–111, 155–176, Dover, New York, 1962.
13. B. van der Pol and H. Bremmer, "Operational Calculus Based on the Two-Sided Laplace Integral," Cambridge Univ. Press, London/New York, 1964.
14. J. Hladik, "La Transformation de Laplace á Plusieurs Variables," pp. 69–73, Masson et Cie Editeurs, Paris, 1969.
15. A. Apelblat, Repeating use of integral transforms—A new method for evaluation of some infinite integrals, *IMA J. Appl. Math.* **27** (1981), 481–496.
16. A. Apelblat, Some integrals of gamma, polygamma and Volterra functions, *IMA J. Appl. Math.* **34** (1985), 173–186.
17. N. W. McLachlan, and P. Humbert, "Formulaire pour le Calcul Symbolique," Mémorial des Sciences Mathématiques, Fascicule 100, Gauthier–Villars, Paris, 1950.
18. N. W. McLachlan, P. Humbert, and L. Poli, "Supplement au Formulaire pour le Calcul Symbolique," Mémorial des Sciences Mathématiques, Fascicule 103, Gauthier-Villars, Paris, 1950.
19. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, "Tables of Integral Transforms," Vols, 1 and 2, McGraw–Hill, New York, 1954.
20. V. A. Ditkin and A. P. Prudnikov, "Integral Transforms and Operational Calculus," Gosfizmatizdat, Moscow, 1961. [in Russian]
21. G. E. Roberts and H. Kaufman, "Table of Laplace Transforms," Saunders, Philadelphia, 1968.
22. F. Oberhettinger and L. Badii, "Tables of Laplace Transforms," Springer-Verlag, Berlin, 1973.
23. A. Abramowitz and I. E. Stegun, "Handbook of Mathematical Functions," U.S. National Bureau of Standards, Washington, DC, 1965.
24. I. S. Gradshtein and I. Ryzhik, "Tables of Series, Products and Integrals," Vols. 1 and 2, Verlag Harri Deutsch, Frankfurt/M, 1964.