Continuous Selection Theorem, Coincidence Theorem, and Generalized Equilibrium in L-Convex Spaces

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Abstract—In this paper, a new continuous selection theorem is first proved in L-convex spaces without linear structure. By using the continuous selection theorem, some new coincidence theorems, fixed-point theorems, and minimax inequality are proved in L-convex spaces. As applications, some new existence theorems of solutions for generalized equilibrium problems are obtained in L-convex spaces. These theorems improve and generalize some known results in recent literature. © 2002 Elsevier Science Ltd. All rights reserved.

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1. PRELIMINARIES

Let X and Y be two nonempty sets. We denote by \(2^Y\) and \(F(X)\) the family of all subsets of Y and the family of all nonempty finite subsets of X. For any \(A \in F(X)\), we denote by \(|A|\) the cardinality of A. Let \(\Delta_n\) be the standard n-dimensional simplex with vertices \(e_0, e_1, \ldots, e_n\). If \(J\) is a nonempty subset of \(\{0, 1, \ldots, n\}\), we denote by \(\Delta_J\) the convex hull of the vertices \(\{e_j : j \in J\}\). If X and Y are two topological spaces and \(G : X \rightarrow 2^Y\) is a set-valued mapping, \(G\) is said to be transfer open-valued (respectively, transfer closed-valued) on X if for \(x \in X\), \(y \in G(x)\) (respectively, \(y \notin G(x)\)) implies that there exists a point \(x' \in X\) such that \(y \in \text{int} G(x')\) (respectively, \(y \notin \text{cl} G(x')\)) (see Definitions 6 and 7 of [1]). \(G\) is said to have the local intersection property on X if for each \(x \in X\) with \(G(x) \neq \emptyset\), there exists a neighborhood \(N(x)\) of \(x\) in \(X\) such that \(\bigcap_{z \in N(x)} G(z) \neq \emptyset\) (see [2]). The example in [2, p. 63] shows that a set-valued
mapping with the local intersection property may not have the property of open inverse values. Let \( f : X \times Y \to \mathbb{R} \) be a function. \( f(x,y) \) is said to be transfer lower (respectively, upper) semicontinuous in \( y \) if for each \( y \in Y \) and \( r \in \mathbb{R} \), with \( \{x \in X : f(x,y) > r\} \neq \emptyset \) (respectively, \( \{x \in X : f(x,y) < r\} \neq \emptyset \)), there exists \( x' \in X \) such that \( y \in \text{int} \{ \{z \in Y : f(x',z) > r\} \} \) (respectively, \( y \in \text{int} \{ \{z \in Y : f(x',z) < r\} \} \)).

The notion of an \( L \)-convex space was introduced by Ben-Israel et al. [3]. An \( L \)-convexity structure on a topological space \( X \) is given by a set-valued mapping \( \Gamma : \mathcal{F}(X) \to 2^X \) satisfying the following conditions.

1. For each \( A \in \mathcal{F}(X) \) with \( |A| = n + 1 \), there exists a continuous mapping \( \phi_A : \Delta_{n} \to \Gamma(A) \) such that \( B \in \mathcal{F}(A) \) with \( |B| = J + 1 \) implies \( \phi_A(\Delta_J) \subseteq \Gamma(B) \), where \( \Delta_J \) denotes the face of \( \Delta_n \) corresponding to \( B \).

Then the pair \( (X, \Gamma) \) is called an \( L \)-convex space. A set \( D \subseteq X \) is said to be \( L \)-convex if for each \( A \in \mathcal{F}(D) \), \( \Gamma(A) \subseteq D \). For a nonempty subset \( E \) of \( (X, \Gamma) \), we define the \( L \)-convex hull of \( E \), denoted by \( L-co(E) \) as

\[
L-co(E) = \cap \{ D \subseteq X : E \subseteq D \text{ and } D \text{ is } L \text{-convex} \}.
\]

Then \( L-co(E) \) is the smallest \( L \)-convex set containing \( E \).

Let \( (X, \Gamma) \) be an \( L \)-convex space. A function \( f : X \to \mathbb{R} \) is said to be \( L \)-quasiconvex (respectively, \( L \)-quasiconcave) if for each \( r \in \mathbb{R} \), the set \( \{x \in X : f(x) < r\} \) (respectively, \( \{x \in X : f(x) > r\} \)) is \( L \)-convex.

If an \( L \)-convex space \( (X, \Gamma) \) satisfies the following additional condition:

2. for each \( A, B \subseteq \mathcal{F}(X) \), \( A \subseteq B \) implies \( \Gamma(A) \subseteq \Gamma(B) \), then the pair \( (X, \Gamma) \) is called by Park and Kim [4] a generalized convex (or, \( G \)-convex) space. Recently, Park [5] has removed the isotony condition (2) and considered the \( G \)-convex space \( (X, D; \Gamma) \), where \( D \) need not be \( X \). If \( D = X \), then a \( G \)-convex space in [5] is an \( L \)-convex space.

If an \( L \)-convex space \( (X, \Gamma) \) satisfies the following additional condition:

3. for each \( A, B \subseteq \mathcal{F}(X) \), there exists \( A_1 \subseteq A \) such that \( A_1 \subseteq B \) implies \( \Gamma(A_1) \subseteq \Gamma(B) \), then the pair \( (X, \Gamma) \) is called by Verma [6] a generalized \( H \)-space (or, \( G-H \)-space).

It is clear that the notion of \( L \)-convex spaces includes the \( G \)-convex spaces, \( G-H \)-spaces, \( H \)-spaces (see [7,8]), \( B' \)-simplicial convexity spaces (see [3]), and \( B \)-simplicial convexity spaces (see [9]).

The following result is a special case of Lemma 1.1 of [10].

**Lemma 1.1.** Let \( X \) and \( Y \) be topological spaces and \( G : X \to 2^Y \) be a set-valued mapping with nonempty values. Then the following conditions are equivalent:

1. \( G \) has the local intersection property;
2. for each \( y \in Y \), there exists an open subset \( O_y \) of \( X \) (which may be empty) such that \( O_y \subseteq G^{-1}(y) \) and \( X = \bigcup_{y \in Y} O_y \);
3. there exists a set-valued mapping \( F : X \to 2^Y \) such that \( F(x) \subseteq G(x) \) for each \( x \in X \), \( F^{-1}(y) \) is open in \( X \) for each \( y \in Y \), and \( X = \bigcup_{y \in Y} F^{-1}(y) \);
4. \( X = \bigcup_{y \in Y} \text{int} G^{-1}(y) \);
5. \( G^{-1} : Y \to 2^X \) is transfer open-valued on \( X \).

The following Lemma 1.2 can be proved by using (iv) of Lemma 1.1.

**Lemma 1.2.** Let \( X \) be a topological space, \( (Y, \Gamma) \) be an \( L \)-convex space, and \( G : X \to 2^Y \) be a set-valued mapping with nonempty values such that one of Conditions (i)-(v) in Lemma 1.1 is satisfied. Then the mapping \( H : X \to 2^Y \) defined by \( H(x) = L-co G(x) \) also satisfies any one of Conditions (i)-(v) in Lemma 1.1.
2. CONTINUOUS SELECTION AND COINCIDENCE THEOREM

In this section, we first prove a new continuous selection theorem in an L-convex space. As applications, some coincidence theorems, fixed-point theorems, and minimax inequality are obtained.

**Theorem 2.1.** Let $X$ be a normal space, $(Y, \Gamma)$ be an L-convex space, and $G : X \rightarrow 2^Y$ be a set-valued mapping such that

(i) $G$ has nonempty L-convex values and satisfies one of Conditions (i)-(v) in Lemma 1.1, and

(ii) there exists a compact subset $K$ of $X$ and a finite subset $M$ of $Y$ such that $X \setminus K \subset \bigcup_{y \in M} \text{int} G^{-1}(y)$.

Then there exists a continuous selection $f : X \rightarrow Y$ of $G$ such that $f = \phi \circ \psi$ where $\phi : \Delta_n \rightarrow Y$ and $\psi : X \rightarrow \Delta_n$ are both continuous and $n$ is some positive integer.

**Proof.** By (i) and Lemma 1.1, we have $K \subset X = \bigcup_{y \in Y} \text{int} G^{-1}(y)$. Since $K$ is compact, there exists a finite subset $N$ of $Y$ such that $K \subset \bigcup_{y \in N} \text{int} G^{-1}(y)$. By Condition (ii), we have $X = \bigcup_{y \in A} \text{int} G^{-1}(y)$ where $A = M \cup N$ is also a finite subset of $Y$. We can assume $A = \{y_0, y_1, \ldots, y_n\}$. Since $Y$ is an L-convex space, there exists a continuous mapping $\phi : A_n \rightarrow \mathcal{P}(A)$ such that

$$\phi(\Delta_j) \subset \Gamma(B), \quad \forall B \in \mathcal{F}(A), \quad |B| = J + 1.$$  \hspace{1cm} (2.1)

Since $X$ is normal, let $\{\psi_i\}_{i=0}^n$ be the continuous partition of unity subordinated to the open covering $\{\text{int} G^{-1}(y_i)\}_{i=0}^n$; then we have that for each $i \in \{0, 1, \ldots, n\}$ and $x \in X$,

$$\psi_i(x) \neq 0 \iff x \in \text{int} G^{-1}(y_i) \subset G^{-1}(y_i).$$ \hspace{1cm} (2.2)

Define a mapping $\psi : X \rightarrow \Delta_n$ by $\psi(x) = \sum_{i=0}^n \psi_i(x)e_i$; then $\psi$ is continuous, $\psi(x) \in \Delta_{J(x)}$, and $\{y_j : j \in J(x)\} \subset G(x)$ by (2.2), where $J(x) = \{j \in \{0, 1, \ldots, n\} : \psi_j(x) \neq 0\}$. By (2.1) and the L-convexity of $G(x)$, we have

$$f(x) = \phi \circ \psi(x) \in \phi(\Delta_{|J(x)|-1}) \subset \Gamma(\{y_j : j \in J(x)\}) \subset G(x).$$

This shows that $f = \phi \circ \psi$ is a continuous selection of $G$.

**Remark 2.1.** Theorem 2.1 improves and generalizes (i) in Theorem 1 of [5] from compact setting to noncompact setting, and Proposition 1 of [11] from topological vector spaces to L-convex spaces without linear structure. If $X$ is compact, then by letting $X = K$, Condition (ii) is satisfied automatically. Hence, Theorem 2.1 also generalizes Theorem 2.2 of [12] from a compact topological space and $H$-space to a noncompact normal space and L-convex space.

Applying Theorem 2.1 to $H(x) = \text{L-co} F(x)$, we can obtain the following result.

**Corollary 2.1.** Let $X$ and $(Y, \Gamma)$ be as in Theorem 2.1 and $F, G : X \rightarrow 2^Y$ be set-valued mappings such that

(i) $F$ has nonempty values and satisfies one of Conditions (i)-(v) in Lemma 1.1,

(ii) $\text{L-co} F(x) \subset G(x)$ for each $x \in X$, and

(iii) there exists a compact subset $K$ of $X$ and a finite subset $M$ of $Y$ such that $X \setminus K \subset \bigcup_{y \in M} \text{int} F^{-1}(y)$.

Then the conclusion of Theorem 2.1 holds.

**Remark 2.2.** If $X$ is compact, by letting $X = K$, then Condition (iii) is satisfied trivially. Corollary 2.1 improves and generalizes Corollary 2.1 of [12] to L-convex spaces under weaker assumptions.
COROLLARY 2.2. Let \((X, \Gamma)\) be as in Theorem 2.1 and \(G : X \to 2^Y\) be a set-valued mapping satisfying Conditions (i) and (ii) of Theorem 2.1. Then for any continuous mapping \(h : Y \to X\), there exists \(\hat{y} \in Y\) such that \(\hat{y} \in G(h(\hat{y}))\).

**Proof.** By Theorem 2.1, there exists a continuous mapping \(f = \phi \circ \psi\) where \(\phi : \Delta_n \to Y\) and \(\psi : X \to \Delta_n\) are both continuous. Hence, the continuous mapping \(\psi \circ h \circ \phi : \Delta_n \to \Delta_n\) has a fixed point \(z \in \Delta_n\) by the Brouwer fixed-point theorem. Let \(\hat{y} = \phi(z)\). Then \(\phi \circ \psi \circ h(\hat{y}) = \hat{y}\). As \(f = \phi \circ \psi\) is a continuous selection of \(G\), we must have \(\hat{y} \in G(h(\hat{y}))\).

**Remark 2.3.** Corollary 2.2 improves (ii) in Theorem 1 of [5] and generalizes Corollary 2.2 of [12].

**Corollary 2.3.** Let \((X, \Gamma)\) be a normal \(L\)-convex space and \(G : X \to 2^X\) be such that

(i) \(G\) has nonempty values and satisfies one of Conditions (i)-(v) in Lemma 1.1, and

(ii) there exists a compact subset \(K\) of \(X\) and a finite subset \(M\) of \(X\) such that \(X \setminus K \subseteq \bigcup_{y \in M} \text{int} G^{-1}(y)\).

Then there exists \(\hat{x} \in X\) such that \(\hat{x} \in L^\text{co}G(\hat{x})\).

**Proof.** Define a mapping \(H : X \to 2^X\) by \(H(x) = L^\text{co} G(x)\) for each \(x \in X\). Then each \(H(x)\) is nonempty \(L\)-convex. By Condition (i) and Lemma 1.2, \(H\) satisfies any one of the conditions in Lemma 1.1. By letting \(X = Y\), \(h = I\), and the identity mapping on \(X\) and \(Y\) be in place of \(G\) in Corollary 2.2, then by Corollary 2.2, there exists a point \(\hat{x} \in X\) such that \(\hat{x} \in H(\hat{x}) = L^\text{co}G(\hat{x})\).

**Remark 2.4.** If \(G(x)\) is \(L\)-convex for each \(x \in X\), then \(\hat{x}\) is a fixed point of \(G\). If there is a mapping \(H : X \to 2^X\) such that \(L^\text{co} H(x) \subseteq G(x)\) for each \(x \in X\), then \(H\) has a fixed point in \(X\). Therefore, Corollary 2.3 generalizes Corollary 2.3 of [12] and Theorem 2 of [11] from \(H\)-spaces and topological vector spaces to \(L\)-convex spaces under much weaker assumptions.

**Theorem 2.2.** Let \((X, \Gamma)\) and \((Y, \Gamma')\) be two \(L\)-convex spaces and \(K\) be a nonempty compact subset of \(X\). Let \(F, G : X \to 2^Y\) and \(S, T : Y \to 2^X\) be set-valued mappings such that

(i) for each \(x \in X\), \(F(x) \neq \emptyset\), and \(L^\text{co} F(x) \subseteq G(x)\);

(ii) \(F\) satisfies one of Conditions (i)-(v) in Lemma 1.1 with \(X = K\);

(iii) \(S\) has nonempty values on each compact subset of \(Y\) and satisfies one of Conditions (i)-(v) in Lemma 1.1 such that for each \(y \in Y\), \(L^\text{co} S(y) \subseteq T(y) \subseteq K\).

Then there exists \((\hat{x}, \hat{y}) \in X \times Y\) such that \(\hat{x} \in T(\hat{y})\) and \(\hat{y} \in G(\hat{x})\).

**Proof.** By Corollary 2.1 with \(X = K\), there exists continuous selection \(f = \phi \circ \psi\) of \(G|_K\), the restriction of \(G\) on \(K\), where \(\phi : \Delta_n \to Y\) and \(\psi : K \to \Delta_n\) are both continuous and \(n\) is some positive integer. Since \(\Delta_n\) is compact and \(\phi\) is continuous, \(\phi(\Delta_n)\) is compact in \(Y\). Again, by using Corollary 2.1 with \(X = \phi(\Delta_n)\) and \(Y = X\), there is a continuous selection \(\beta : \phi(\Delta_n) \to K\) of \(T|_{\phi(\Delta_n)}\). Hence, the mapping \(\psi \circ \beta \circ \phi : \Delta_n \to \Delta_n\) is continuous. By the Brouwer fixed-point theorem, there exists \(z \in \Delta_n\) such that \(z = \psi \circ \beta \circ \phi(z)\). Let \(\hat{y} = \phi(z)\) and \(\hat{x} = \beta(\hat{y})\). Then we have \(\hat{x} \in T(\hat{y})\) and \(\hat{y} = \phi(z) = \phi \circ \psi(\hat{x}) \in G(\hat{x})\). This completes the proof.

**Remark 2.5.** Theorem 2.2 generalizes Lemma 1 of [13] from \(H\)-spaces to \(L\)-convex spaces.

**Theorem 2.3.** Let \((X, \Gamma)\) be a normal \(L\)-convex space, \((Y, \Gamma')\) be an \(L\)-convex space, and \(F, G : X \to 2^Y\) and \(S, T : Y \to 2^X\) be set-valued mappings such that

(i) \(F\) has nonempty values and satisfies one of Conditions (i)-(v) in Lemma 1.1 such that for each \(x \in X\), \(L^\text{co} F(x) \subseteq G(x)\),

(ii) there exist a nonempty compact subset \(K\) of \(X\) and a finite subset \(M\) of \(Y\) such that \(X \setminus K \subseteq \bigcup_{y \in M} \text{int} F^{-1}(y)\), and

(iii) \(S\) has nonempty values and satisfies one of Conditions (i)-(v) in Lemma 1.1 such that for each \(y \in Y\), \(L^\text{co} S(y) \subseteq T(y) \subseteq K\).

Then there exists \((\hat{x}, \hat{y}) \in X \times Y\) such that \(\hat{x} \in T(\hat{y})\) and \(\hat{y} \in G(\hat{x})\).

**Proof.** By Corollary 2.1, there exists a continuous selection \(f : X \to Y\) of \(G\) such that \(f = \phi \circ \psi\) where \(\phi : \Delta_n \to Y\) and \(\psi : X \to \Delta_n\) are both continuous and \(n\) is some positive integer. Since \(\Delta_n\)
is compact and \( \phi \) is continuous, \( \phi(\Delta_n) \) is compact in \( Y \). By Condition (iii), Corollary 2.1, and Remark 2.2, there exists a continuous selection \( \beta : \phi(\Delta_n) \to X \) of \( T|_{\phi(\Delta_n)} \), the restriction of \( T \) on \( \phi(\Delta_n) \). Hence, we have \( \psi \circ \beta \circ \phi : \Delta_n \to \Delta_n \) is continuous. By the Brouwer fixed-point theorem, there exists \( z \in \Delta_n \) such that \( z = \psi \circ \beta \circ \phi(z) \). Let \( \tilde{y} = \phi(z) \) and \( \tilde{x} = \beta(\tilde{y}) \). Then we obtain \( \tilde{x} \in T(\tilde{y}) \) and \( \tilde{y} = \phi(z) = \phi \circ \psi(\tilde{x}) \in G(\tilde{x}) \).

**Remark 2.6.** Theorem 2.3 is a variant form of Theorem 2.2. The assumption "for each \( y \in Y \), \( T(y) \subset K \)" in Theorem 2.2 is dropped, but an additional Condition (ii) is assumed. Theorems 2.2 and 2.3 are different from the coincidence theorems of [10,11,14-21].

Applying Theorem 2.3 with \( F = G \), it is easy to obtain the following result.

**Corollary 2.4.** Let \( (X, \Gamma) \) be a normal L-convex space, \( (Y, \Gamma') \) be an L-convex space, and \( G : X \to 2^Y, T : Y \to 2^X \) be set-valued mappings such that

1. \( G \) and \( T \) both have nonempty L-convex values and satisfy one of Conditions (i)--(v) in Lemma 1.1, and
2. there exist a nonempty compact subset \( K \) of \( X \) and a nonempty finite subset \( M \) of \( Y \) such that
   \[ X \setminus K \subset \bigcup_{y \in M} \text{int} G^{-1}(y). \]

Then there exists \( (\tilde{x}, \tilde{y}) \in X \times Y \) such that \( \tilde{x} \in T(\tilde{y}) \) and \( \tilde{y} \in G(\tilde{x}) \).

**Theorem 2.4.** Let \( (X, \Gamma) \) be a normal L-convex space and \( (Y, \Gamma') \) be an L-convex space. Suppose that \( f, g : X \times Y \to \mathbb{R} \) are two functions such that

1. \( f(x, y) \leq g(x, y) \) for all \( (x, y) \in X \times Y \),
2. \( f(x, y) \) is transfer lower semicontinuous in \( x \) and for each \( x \in X \), \( y \mapsto f(x, y) \) is L-quasi-concave,
3. \( g(x, y) \) is transfer upper semicontinuous in \( y \) and for each \( y \in Y, x \mapsto g(x, y) \) is L-quasi-convex, and
4. there exist a nonempty compact subset \( K \) of \( X \) and a finite subset \( M \) of \( Y \) such that for each \( r \in \mathbb{R}, \) if \( \inf_{x \in X} \sup_{y \in Y} f(x, y) > r, \) then \( X \setminus K \subset \bigcup_{y \in M} \text{int} \{ x \in X : f(x, y) > r \} \).

Then
\[ \inf_{x \in X} \sup_{y \in Y} f(x, y) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y). \]

**Proof.** If the conclusion is false, then there exist real numbers \( \alpha \) and \( \beta \) such that
\[ \inf_{x \in X} \sup_{y \in Y} f(x, y) > \alpha > \beta > \sup_{y \in Y} \inf_{x \in X} g(x, y). \] (2.3)

Define mappings \( G : X \to 2^Y \) and \( T : Y \to 2^X \) by
\[ G(x) = \{ y \in Y : f(x, y) > \alpha \}, \quad T(y) = \{ x \in X : g(x, y) < \beta \}. \]

By (2.3), \( G \) and \( T \) both have nonempty values. By (ii) and (iii), \( G(x) \) and \( T(y) \) are both L-convex for each \( x \in X \) and \( y \in Y \), and \( G^{-1} \) and \( T^{-1} \) are both transfer open-valued. By (2.3) and Condition (iv), we have \( X \setminus K \subset \bigcup_{y \in M} \text{int} G^{-1}(y) \). It follows from Corollary 2.4 that there exists \( (\tilde{x}, \tilde{y}) \in X \times Y \) such that \( \tilde{x} \in T(\tilde{y}) \) and \( \tilde{y} \in G(\tilde{x}) \). It follows that \( f(\tilde{x}, \tilde{y}) > \alpha \) and \( g(\tilde{x}, \tilde{y}) < \beta \). By (i), we obtain \( \alpha < \beta \) which contradicts the choices of \( \alpha \) and \( \beta \). Therefore, we must have
\[ \inf_{x \in X} \sup_{y \in Y} f(x, y) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y). \]

**Remark 2.7.** If \( X \) is compact, by letting \( K = X \), then Condition (iv) is satisfied trivially. Theorem 2.4 generalizes Theorem 2 of [13] from compact \( H \)-spaces to noncompact L-convex spaces, and Theorem 3.2 and Corollary 3.5 of [22] in several aspects.
3. GENERALIZED EQUILIBRIUM PROBLEMS

In order to establish some new generalized equilibrium existence theorems in L-convex spaces, we first prove the following nonempty intersection theorem.

**Theorem 3.1.** Let \((X, \Gamma)\) be an L-convex space, \((Y, \Gamma')\) be a compact L-convex space, and \(F, G : X \to 2^Y\) be two set-valued mappings such that

1. \(F(x) \subseteq G(x)\) for all \(x \in X\),
2. for each \(y \in Y\), \(X \setminus G^{-1}(y)\) is L-convex,
3. \(G\) is transfer closed-valued, and
4. \(F\) has nonempty L-convex values and satisfies one of Conditions (i)–(v) in Lemma 1.1.

Then \(\bigcap_{x \in X} G(x) \neq \emptyset\).

**Proof.** Define a set-valued mapping \(T : Y \to 2^X\) by

\[ T(y) = X \setminus G^{-1}(y), \quad \forall y \in Y. \]

Then by (ii), \(T(y)\) is L-convex for each \(y \in Y\). From the definition of \(T\) it follows that for each \(x \in X\),

\[ T^{-1}(x) = \{ y \in Y : x \in T(y) \} = \{ y \in Y : x \notin G^{-1}(y) \} = \{ y \in Y : y \notin G(x) \} = Y \setminus G(x). \]

Hence, \(T^{-1}\) is transfer open-valued by (iii). Suppose that for each \(y \in Y\), \(T(y) \neq \emptyset\). Note that \((Y, \Gamma')\) is a compact L-convex space, and by Condition (iv) and Remark 2.7, all conditions of Corollary 2.5 are satisfied. Hence, there exists \((\tilde{x}, \tilde{y}) \in X \times Y\) such that \(\tilde{x} \in T(\tilde{y})\) and \(\tilde{y} \in F(\tilde{x})\). It follows that \(\tilde{y} \notin G(\tilde{x})\) and \(\tilde{y} \in F(\tilde{x})\) which contradicts Condition (i). Therefore, there exists a point \(y \in Y\) such that \(T(y) = \emptyset\); i.e., \(y \notin G(x)\) for all \(x \in X\). Hence, \(\bigcap_{x \in X} G(x) \neq \emptyset\).

**Remark 3.1.** Theorem 3.1 generalizes Lemma 4 of [13] from H-space to L-convex space and our proof is different from those in [13].

**Theorem 3.2.** Let \((X, \Gamma)\) be a normal L-convex space and \((Y, \Gamma')\) be an L-convex space. Let \(T : X \to 2^Y\) be an upper semicontinuous set-valued mapping with nonempty compact L-convex values and \(\phi : X \times Y \times X \to \mathbb{R}\) be a function such that

1. for each \((x, y) \in X \times Y\), \(z \mapsto \phi(x, y, z)\) is lower semicontinuous and L-quasiconvex,
2. for each \((x, z) \in X \times X\), \(y \mapsto \phi(x, y, z)\) is quasiconcave,
3. there exist a nonempty compact subset \(K\) of \(X\) and a finite subset \(M\) of \(X\) such that
   \[ X \setminus K \subseteq \bigcup_{y \in M} \text{int} \{ x \in X : \sup_{y \in T(x)} \phi(x, y, z) < 0 \}, \]
4. for each \(x \in X\), there is a \(y \in T(x)\) such that \(\phi(x, y, x) \geq 0\), and
5. for each \(z \in X\), \((x, y) \mapsto \phi(x, y, z)\) is upper semicontinuous.

Then there exist \(\bar{x} \in X\) and \(\bar{y} \in T(\bar{x})\) such that

\[ \phi(\bar{x}, \bar{y}, z) \geq 0, \quad \forall z \in X. \]

**Proof.** We first show that there exists a point \(\bar{x} \in X\) such that

\[ \sup_{y \in T(\bar{x})} \phi(\bar{x}, y, z) \geq 0, \quad \forall z \in X. \]

If it is false, then for each \(x \in X\), there exists \(z \in X\) such that \(\sup_{y \in T(x)} \phi(x, y, z) < 0\). Define a set-valued mapping \(H : X \to 2^X\) by

\[ H(x) = \left\{ z \in X : \sup_{y \in T(x)} \phi(x, y, z) < 0 \right\}, \quad \forall x \in X. \]
Then for each \( x \in X \), \( H(x) \neq \emptyset \). Now we prove that for each \( x \in X \), \( H(x) \) is L-convex. For each finite set \( A = \{ z_1, z_2, \ldots, z_n \} \subset H(x) \), we have

\[
\sup_{y \in T(x)} \phi(x, y, z_i) < 0, \quad \forall i = 1, 2, \ldots, n.
\]

Hence, there is a real number \( r \) such that

\[
\phi(x, y, z_i) < r < 0, \quad \forall y \in T(x), \quad i = 1, 2, \ldots, n.
\]

By (i), for each \( (x, y) \in X \times Y \), the set \( \{ z \in X : \phi(x, y, z) < r \} \) is L-convex, and hence, we have that for each \( z \in \Gamma(A) \) and \( y \in T(x) \), \( \phi(x, y, z) < r \) and so \( \sup_{y \in T(x)} \phi(x, y, z) < r < 0 \) for all \( z \in \Gamma(A) \). Therefore, \( \Gamma(A) \subset H(x) \); i.e., \( H(x) \) is L-convex for each \( x \in X \). Since \( T \) is upper semicontinuous with nonempty compact values and \( (x, y) \mapsto \phi(x, y, z) \) is upper semicontinuous, by Proposition 3.1.21 of [23], for each \( z \in X \), the function \( x \mapsto \sup_{y \in T(x)} \phi(x, y, z) \) is upper semicontinuous. Hence, for each \( z \in X \),

\[
H^{-1}(z) = \{ x \in X : z \in H(x) \} = \left\{ x \in X : \sup_{y \in T(x)} \phi(x, y, z) < 0 \right\}
\]

is open in \( X \). By (iii), we have \( X \setminus K \subset \operatorname{int} H^{-1}(z) \). By Corollary 2.3 and Remark 2.4, there exists a point \( \bar{x} \in X \) such that \( \bar{x} \in H(\bar{x}) \), i.e., \( \sup_{y \in T(\bar{x})} \phi(\bar{x}, y, \bar{x}) < 0 \), which contradicts Condition (iv). Therefore, there exists \( \bar{x} \in X \) such that

\[
\sup_{y \in T(\bar{x})} \phi(\bar{x}, y, z) \geq 0, \quad \forall z \in X.
\]

By (v) and the compactness of \( T(\bar{x}) \), for each \( z \in X \), there exists \( y(z) \in T(\bar{x}) \) such that \( \phi(\bar{x}, y(z), z) > 0 \). For any given \( \epsilon > 0 \), define set-valued mappings \( F, G : X \to 2^{T(x)} \) by

\[
F(z) = \{ y \in T(\bar{x}) : \phi(\bar{x}, y, z) > \epsilon \}, \quad G(z) = \{ y \in T(\bar{x}) : \phi(\bar{x}, y, z) \geq \epsilon \}.
\]

Then \( F(z) \neq \emptyset \) and \( F(z) \subset G(z) \) for each \( z \in X \). Note that \( T(\bar{x}) \) is a compact L-convex space; it follows from Condition (ii) that \( F(z) \) is L-convex for each \( z \in X \). By (i), for each \( y \in T(\bar{x}) \),

\[
F^{-1}(y) = \{ z \in X : y \in F(z) \} = \{ z \in X : \phi(\bar{x}, y, z) > \epsilon \}
\]

is open in \( X \) and so \( F^{-1} \) is transfer open-valued, and \( X \setminus G^{-1}(y) = \{ z \in X : \phi(\bar{x}, y, z) < 0 \} \) is L-convex. By (v), each \( G(z) \) is closed and so \( G \) is transfer closed-valued. By Theorem 3.1, for each \( \epsilon > 0 \), there exists \( y_\epsilon \in \bigcap_{z \in X} G(z) \); i.e., \( y_\epsilon \in T(\bar{x}) \) and \( \phi(\bar{x}, y_\epsilon, z) \geq \epsilon \) for each \( z \in X \). Since \( T(\bar{x}) \) is compact, we may assume \( y_\epsilon \to \bar{y} \in T(\bar{x}) \) as \( \epsilon \to 0 \). By (v), we obtain

\[
\phi(\bar{x}, \bar{y}, z) \geq \lim_{\epsilon \to 0} \phi(\bar{x}, y_\epsilon, z) \geq 0, \quad \forall z \in X.
\]

**Remark 3.2.** If \( X \) is compact, by letting \( K = X \), Condition (iii) is satisfied trivially. Hence, Theorem 3.2 generalizes Theorem 5 of [13] from compact \( H \)-spaces to noncompact L-convex spaces, and in turn generalizes Theorem 6 of [24] in many aspects.

**Theorem 3.3.** Let \( (X, \Gamma) \) be a normal L-convex space, \( (Y, \Gamma') \) be an L-convex space, and \( T : X \to 2^Y \) be a set-valued mapping with nonempty compact L-convex values. Let \( \phi : X \times Y \to \mathbb{R} \) be a function such that

(i) for each \( x \in X \), \( y \mapsto \phi(x, y) \) is upper semicontinuous and L-quasiconcave,

(ii) for each \( y \in Y \), \( x \mapsto \phi(x, y) \) is lower semicontinuous and L-quasiconvex,

(iii) there exist a nonempty compact subset \( K \) of \( X \) and a finite subset \( M \) of \( X \) such that \( X \setminus K \subset \bigcup_{x \in M} \text{int} \{ z \in X : \max_{y \in T(z)} \phi(x, y) < c \} \) where \( c \) is a constant, and

(iv) for each \( x \in X \), there exists \( y \in T(x) \) such that \( \phi(x, y) \geq c \).

Then there exist \( \bar{x} \in X \) and \( \bar{y} \in T(\bar{x}) \) such that \( \phi(x, \bar{y}) \geq c \) for all \( x \in X \).
PROOF. We first prove that there exists \( \bar{z} \in X \) such that
\[
\max_{y \in T(\bar{z})} \phi(x, y) \geq c, \quad \forall x \in X.
\]
If it is false, then for each \( z \in X \), there exists \( x \in X \) such that
\[
\max_{y \in T(z)} \phi(x, y) < c.
\]
Define a set-valued mapping \( H : X \to 2^X \) by
\[
H(z) = \left\{ x \in X : \max_{y \in T(z)} \phi(x, y) < c \right\}, \quad \forall z \in X.
\]
Then each \( H(z) \) is nonempty. By (ii),
\[
H(z) = \left\{ x \in X : \max_{y \in T(z)} \phi(x, y) < c \right\} = \bigcap_{y \in T(z)} \left\{ x \in X : \phi(x, y) < c \right\}
\]
is \( L \)-convex. Let \( f(x, z) = \max_{y \in T(z)} \phi(x, y) \). For each fixed \( x \in X \), and \( r \in \mathbb{R} \), let
\[
D = \{ z \in X : f(x, z) \geq r \}.
\]
If \( \{ z_\alpha : \alpha \in I \} \) is a net in \( D \) such that \( z_\alpha \to z^* \), then we have
\[
\max_{y \in T(z_\alpha)} \phi(x, y) \geq r, \quad \forall \alpha \in I.
\]
Since \( T(z_\alpha) \) is compact and \( y \mapsto \phi(x, y) \) is upper semicontinuous, therefore, for each \( \alpha \in I \), there exists \( y_\alpha \in T(z_\alpha) \) such that \( \phi(x, y_\alpha) \geq r \). Since \( T \) is upper semicontinuous with compact values, it follows from Proposition 3.1.11 of [23] that the set \( T(\{ z_\alpha \}_{\alpha \in I} \cup \{ z^* \}) \) is compact. Noting \( \{ y_\alpha \}_{\alpha \in I} \subset T(\{ z_\alpha \}_{\alpha \in I} \cup \{ z^* \}) \), without loss of generality, we may assume that \( y_\alpha \to y^* \). By the upper semicontinuity of \( T \), we obtain \( y^* \in T(z^*) \). By (i) we have \( \phi(x, y^*) \geq r \), and hence, we have \( \max_{y \in T(z^*)} \phi(x, y) \geq r \); i.e., \( f(x, z^*) \geq r \). Hence, \( z^* \in D \) and \( D \) is closed. It follows that \( z \mapsto f(x, z) = \max_{y \in T(z)} \phi(x, y) \) is upper semicontinuous. Hence, for each \( x \in X \),
\[
H^{-1}(x) = \left\{ z \in X : \max_{y \in T(z)} \phi(x, y) < c \right\}
\]
is open in \( X \). By Condition (iii), we have \( X \setminus K \subset \bigcup_{x \in M} \text{int} \ H^{-1}(x) \). By using a special case of Corollary 2.3, there exists \( \hat{z} \in X \) such that \( \hat{z} \in H(\hat{z}) \), i.e., \( \max_{y \in T(\hat{z})} \phi(\hat{z}, y) < c \), and so \( \phi(\hat{z}, y) < c \) for all \( y \in T(\hat{z}) \) which contradicts Condition (iv). Hence, there exists \( \bar{z} \in X \) such that
\[
\max_{y \in T(\bar{z})} \phi(x, y) \geq c, \quad \forall x \in X.
\]
For any given \( \epsilon < c \), define set-valued mappings \( F, G : X \to 2^{T(\bar{z})} \) by
\[
F(x) = \{ y \in T(\bar{z}) : \phi(x, y) > \epsilon \}, \quad G(x) = \{ y \in T(\bar{z}) : \phi(x, y) \geq \epsilon \}.
\]
Then \( F(x) \neq \emptyset \) and \( F(x) \subset G(x) \) for each \( x \in X \). Since \( T(\bar{z}) \) is a compact \( L \)-convex space, by (i), \( F(x) \) is \( L \)-convex for each \( x \in X \). For each \( y \in T(\bar{z}) \),
\[
F^{-1}(y) = \{ x \in X : y \in F(x) \} = \{ x \in X : \phi(x, y) > \epsilon \}
\]
is open in \( X \) by (ii) and so \( F^{-1} \) is transfer open-valued. By (ii), \( X \setminus G^{-1}(y) = \{ x \in X : \phi(x, y) < \epsilon \} \) is \( L \)-convex for each \( y \in T(\bar{z}) \). By (i), each \( G(x) \) is closed in \( T(\bar{z}) \). It follows from Theorem 3.1 that for each \( \epsilon < c \), there exists \( y_\epsilon \in \bigcap_{x \in X} G(x) \); i.e., \( y_\epsilon \in T(\bar{z}) \) and \( \phi(x, y_\epsilon) \geq \epsilon \) for all \( x \in X \). Since \( T(\bar{z}) \) is compact, we can assume \( y_\epsilon \to \bar{y} \in T(\bar{z}) \) as \( \epsilon \to c \). By (i), we obtain
\[
\phi(x, \bar{y}) \geq c, \quad \forall x \in X.
\]
REMARK 3.3. If \( X \) is compact, then by letting \( K = X \), Condition (iii) is satisfied trivially. Theorem 3.3 generalizes Theorem 6 of [13] from compact \( H \)-spaces to noncompact \( L \)-convex spaces, and in turn generalizes Theorem 5.7.1 of [25] in several aspects.
REFERENCES