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NOTE

ON A CLIQUE COVERING PROBLEM OF ORLIN

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Let T_{2n} be the complement of a perfect matching in the complete graph on $2n$ vertices, and $\mathrm{cc}(T_{2n})$ be the minimum number of complete subgraphs necessary to cover all the edges of T_{2n}. Orlin posed the problem of determining the asymptotic behaviour of $\mathrm{cc}(T_{2n})$. We show that $\mathrm{cc}(T_{2n}) = \min\{k: n \le \binom{k-1}{\lceil k/2 \rceil}\}$ for all $n > 1$, (which implies that $\lim_{n\to\infty} \mathrm{cc}(T_{2n})/\log_2 n = 1$). This is done by applying a Sperner-type theorem on set families due to Bollobás and Schönheim.

A *Clique covering* of a graph G is a family of complete subgraphs such that every edge of G is in some member of the family. The minimum cardinality of all clique coverings of G is called the *clique covering number* of G and is denoted here by $\mathrm{cc}(G)$. For an outline of the background of this subject see, e.g., [3] or [4].

Let T_{2n} denote the graph obtained by deleting a perfect matching from the complete graph on $2n$ vertices. In [4, p. 411] Orlin presented the following as an unsolved problem: What is the asymptotic behaviour of $\mathrm{cc}(T_{2n})$? He associated the determination of $\mathrm{cc}(T_{2n})$ with the solution of an optimization problem in the theory of Boolean functions [4, Remark 3.7].

Let $\sigma(n) = \min\{k: n \le \binom{k-1}{\lceil k/2 \rceil}\}$. We will show that $\mathrm{cc}(T_{2n}) = \sigma(n)$ for all $n > 1$ (which implies that $\lim_{n\to\infty} \mathrm{cc}(T_{2n})/\log_2 n = 1$).

Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$ is an indexed family of subsets of a set S. Repetitions in $\mathcal{F}$ are allowed. That is, it may happen that $F_i = F_j$ even though $i \ne j$. Let $G(\mathcal{F})$ be the graph on n distinct vertices $v_1, v_2, \ldots, v_n$ with v_i adjacent to v_j whenever $i \ne j$ and $F_i \cap F_j \ne \emptyset$. This is the *intersection graph* of $\mathcal{F}$.

Suppose G is a simple graph (no loops or multiple edges) having n vertices, none of which is isolated, and $\mathcal{C}$ is any clique covering of G. Index the vertices of G: $v_1, v_2, \ldots, v_n$. For $1 \le i \le n$, let F_i consist of those members of $\mathcal{C}$ having v_i as a vertex. Then define $\mathcal{F}(\mathcal{C}) = \{F_1, F_2, \ldots, F_n\}$. It follows that the intersection graph of $\mathcal{F}(\mathcal{C})$ is isomorphic to G. Therefore $\mathrm{cc}(G)$ is the least value of $|S|$ for which a representation of G as an intersection graph of a family of subsets of S exists. This correspondence between set families and graphs is the historical basis for the subject of clique coverings, (see, e.g. [3] or [4]).

A family $\mathcal{F}$ of subsets of a set S is called a *clutter* if none of its members contains another. Call a family $\mathcal{F}=\{F_1, F_2, \ldots, F_n\}$ of subsets of S *complementary*, if for every $1 \leq i \leq n$, there is some $1 \leq j \leq n$ such that $F_j = S \setminus F_i$.

Theorem 1 (Bollobás, Schönheim [1]). *The largest complementary clutters of subsets of a k-element set have* $2\binom{k-1}{\lceil k/2 \rceil}$ *members.*

Call a family $\mathcal{F}$, *balanced* if for every F in $\mathcal{F}$, there is precisely one F' in $\mathcal{F}$ disjoint from F. Notice that every balanced family is a clutter. By replacing each pair $\{F, F'\}$ in a balanced family $\mathcal{F}$ by $\{F, S \setminus F\}$, we obtain a new balanced complementary family of the same size as $\mathcal{F}$. Therefore

Corollary 1. *The largest balanced families of subsets of a k-element set have* $2\binom{k-1}{\lceil k/2 \rceil}$ *members.*

Corollary 2. *For all* $n > 1$, $\mathrm{cc}(T_{2n}) = \sigma(n)$.

Proof. Let $\mathscr{C}$ be a clique covering of T_{2n} of minimum cardinality k and $\mathcal{F} = \mathcal{F}(\mathscr{C})$. Since $G(\mathcal{F})$ is isomorphic to T_{2n} and each vertex of T_{2n} is adjacent to all but one other vertex, the family $\mathcal{F}$ is balanced.

Therefore $|\mathcal{F}| \leq 2\binom{k-1}{\lceil k/2 \rceil}$, and hence $\mathrm{cc}(T_{2n}) \geq \sigma(n)$ by Corollary 1.

Let $k = \sigma(n)$ and choose a balanced family $\mathcal{F}$ in $S = \{1, 2, \ldots, k\}$ with cardinality $2n \leq 2\binom{k-1}{\lceil k/2 \rceil}$, as we may by Corollary 1. Then $G(\mathcal{F})$ is isomorphic to T_{2n} and $\mathrm{cc}(G(\mathcal{F})) \leq k$. Therefore $\mathrm{cc}(T_{2n}) \leq \sigma(n)$. $\square$

Let $k = \sigma(n)$, then the definition of $\sigma(n)$ and some elementary manipulations of binomial coefficients imply that

$$(1-(2/k))\binom{k-1}{\lceil (k-1)/2 \rceil} < n \leq \frac{1}{2}\binom{k}{\lceil k/2 \rceil}.$$

But Wallis' product representation of π (see, e.g., [2, p. 225]) implies that

$$\lim_{k \to \infty} \binom{k}{\lceil k/2 \rceil} 2^{-k} \sqrt{k} = \sqrt{2/\pi}.$$

Therefore

$$0.1995 < n2^{-k}\sqrt{k} < 0.3184$$

and hence

$$1.325 < \sigma(n) - 1/2 \log_2 \sigma(n) - \log_2(n) < 2.326$$

for all sufficiently large n. Dividing by $cr(n)$, we obtain

$$\lim_{n\to\infty} \frac{cc(T_{2n})}{\log_2(n)} = 1,$$

by Corollary 2.

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