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Eternal solutions of the Boltzmann equation near travelling Maxwellians

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Abstract

It is shown in this paper that the Cauchy problem of the Boltzmann equation, with a cut-off soft potential and an initial datum close to a travelling Maxwellian, has a unique positive eternal solution. This eternal solution is exponentially decreasing at infinity for all $t \in (-\infty, \infty)$, consequently the moments of any order are finite. This result gives a negative answer to the conjecture of Villani in the spatially inhomogeneous case.

Keywords: Boltzmann equation; Eternal positive solution; Kaniel–Shinbrot iteration

1. Introduction

In this paper, we discuss the eternal solutions of the Boltzmann equation. A function $f(t, x, \xi)$ defined on $(-\infty, \infty) \times \mathbf{R}^3 \times \mathbf{R}^3$, is called an eternal solution of the Cauchy problem of the Boltzmann equation if it satisfies this equation for all $(t, x, \xi) \in (-\infty, \infty) \times$

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$\mathbf{R}^3 \times \mathbf{R}^3$. It is known that both global Maxwellians and travelling Maxwellians are eternal solutions. A travelling Maxwellian can be expressed by [5]:

$$\mathcal{M}(x - t\xi, \xi) = C \exp\bigl(-\alpha|x - t\xi|^2\bigr) \exp\bigl(-\beta|\xi|^2\bigr),$$

where C, α and β are positive constants. These explicit solutions exhibit several essential properties of a gas, for example: positivity, conservation of (finite) mass, momentum and energy, etc. Except for the above mentioned results, little is known about eternal solutions at present.

On the other hand, the celebrated H-theorem of Boltzmann reveals that the Boltzmann equation has a basic feature of irreversibility: the entropy function $H(f)(t)$ always decreases. So, one cannot hope that the Cauchy problem of the equation with a given initial value must have a positive eternal solution with finite energy. In fact, C. Villani conjectured [10], [11, p. 117–118] that except for Maxwell's distributions, the nonlinear Boltzmann equation has no other type of positive eternal solutions with finite kinetic energy. This problem was first discussed by Bobylev and Cercignani in the spatially homogeneous case [2,3]. They proved that any positive eternal solution with finite moments up to any order for all $t \in (-\infty, \infty)$ must be a global Maxwellian, which means that Villani's conjecture is partially valid in this special case. They also gave some important exact eternal solutions with infinite energy. It should be mentioned here that the validity of Villani's conjecture for spatially homogeneous Landau equation was proved in [10].

As far as we know, this problem is open in the spatially inhomogeneous situation. The aim of the present work is to investigate it. It is proved in this paper that for soft potentials with angular cut-off, there are many positive eternal solutions near a travelling Maxwellian. This result gives a negative answer to Villani's conjecture in the spatially inhomogeneous case.

Strictly speaking, we shall show that there exists a positive constant $\varepsilon_0 \in (0, 1)$ such that for any measurable function $\phi_0(x, \xi)$ satisfying $1 - \varepsilon_0 \leqslant \phi_0(x, \xi) \leqslant 1 + \varepsilon_0$, the Boltzmann equation has a unique positive mild solution $f(t, x, \xi)$ on the time interval $(-\infty, \infty)$ with initial value $f(0, x, \xi) = \phi_0(x, \xi)\mathcal{M}(x, \xi)$. Furthermore, the solution satisfies

$$0 < f(t, x, \xi) < 2\mathcal{M}(x - t\xi, \xi).$$

Consequently, f has finite kinetic energy and moments up to any order at any instant $t \in (-\infty, \infty)$.

The Cauchy problem of the Boltzmann equation reads:

$$\begin{cases} \frac{\partial}{\partial t} f(t, x, \xi) + \xi \cdot \nabla_x f = Q(f, f)(t, x, \xi), \\ f(0, x, \xi) = f_0(x, \xi), \end{cases} \tag{1.1}$$

where $(x, \xi) \in \mathbf{R}^3 \times \mathbf{R}^3$ and $t \in (-\infty, \infty)$. In (1.1), $Q(f, f)$ is the Boltzmann collision operator and can be expressed by

$$Q(f, f)(t, x, \xi) = \int\limits_{\mathbf{R}^3 \times S^2_+} B\bigl(|V|, \theta\bigr)(f' f'_* - f f_*)\, d\xi_*\, d\omega, \tag{1.2}$$

where $V = \xi - \xi_*$, $f' = f(t,x,\xi')$, $f'_* = f(t,x,\xi'_*)$, $f = f(t,x,\xi)$ and $f_* = f(t,x,\xi_*)$. We recall that (ξ,ξ_*) and (ξ',ξ'_*) are the velocities of two colliding molecules before and after a collision, respectively. We have the following relations between (ξ,ξ_*) and (ξ',ξ'_*):

$$\begin{cases} \xi' = \xi + (\xi_* - \xi, \omega)\omega, \\ \xi'_* = \xi_* - (\xi_* - \xi, \omega)\omega, \\ \omega \in S^2_+ = \{\omega \in \mathbf{R}^3\colon\ |\omega| = 1,\ (\xi_* - \xi, \omega) \geqslant 0\}, \\ \xi' + \xi'_* = \xi + \xi_*,\ |\xi'|^2 + |\xi'_*|^2 = |\xi|^2 + |\xi_*|^2. \end{cases} \tag{1.3}$$

Usually, ω is parameterized as

$$\omega = \{\cos\theta, \sin\theta\cos\epsilon, \sin\theta\sin\epsilon\}, \quad (\theta,\epsilon) \in \left[0, \frac{\pi}{2}\right] \times [0, 2\pi].$$

For inverse power-law potentials, the collision kernel $B(|V|,\theta)$ can be expressed by

$$B\big(|V|,\theta\big) = |V|^\gamma b(\theta) \geqslant 0.$$

The potentials are called "soft" if the exponent $\gamma \in [-1,0)$, and "hard" if $\gamma \in (0,1]$; for Maxwell's molecules we have $\gamma = 0$. For the general presentation of the Boltzmann equation, we refer to [1,4,5,9] and references therein.

In this paper we deal with Maxwell's molecules and soft potentials, so we assume that $-1 \leqslant \gamma \leqslant 0$. Generally speaking, $b(\theta)$ has a strong singularity at $\theta = \frac{\pi}{2}$. Here, we assume that it satisfies the Grad's angular cut-off condition, i.e.:

$$\int_{S^2_+} b(\theta)\, d\omega = b_0 < \infty.$$

Under the above assumptions, the collision operator can be split into the gain and loss terms:

$$Q(f,f) = Q^+(f,f) - Q^-(f,f), \quad Q^-(f,f) = fL(f),$$
$$L(f) = \int_{\mathbf{R}^3\times S^2_+} B\big(|V|,\theta\big) f_*\, d\xi_*\, d\omega = b_0 \int_{\mathbf{R}^3} |V|^\gamma f_*\, d\xi_*.$$

Let $f^\#(t,x,\xi) = f(t, x + t\xi, \xi)$, then we can express Cauchy problem (1.1) in the following mild form:

$$\begin{cases} \frac{\partial}{\partial t} f^\#(t,x,\xi) = Q(f,f)^\#(t,x,\xi), \\ f^\#(0,x,\xi) = f_0(x,\xi). \end{cases} \tag{1.4}$$

With the above notations, we can write down the main result of this paper.

Theorem 1.1. *Let $\mathcal{M}(x,\xi) = C\exp(-\alpha|x|^2)\exp(-\beta|\xi|^2)$ be given. Then there exists a positive number $\varepsilon_0 \in (0,1)$ such that for any initial distribution f_0 satisfying*:

$$(1-\varepsilon_0)\mathcal{M}(x,\xi) \leqslant f_0(x,\xi) \leqslant (1+\varepsilon_0)\mathcal{M}(x,\xi),$$

the Cauchy problem (1.1) *has a unique positive eternal solution $f(t,x,\xi)$ satisfying*

$$\big(1-\varepsilon(t)\big)\mathcal{M}(x-t\xi,\xi) \leqslant f(t,x,\xi) \leqslant \big(1+\varepsilon(t)\big)\mathcal{M}(x-t\xi,\xi),$$

where $\varepsilon(t)$ is a function defined on $(-\infty, \infty)$ which is nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, \infty)$, respectively, and satisfies $\varepsilon(t) \in (0, 1)$ for all $t \in (-\infty, \infty)$.

In order to prove this theorem, it is necessary to recall some early methods and results. Toscani and Palczewski [7,8], using the monotone iteration method of Kaniel and Shinbrot [6], proved part of this theorem, i.e., they proved the existence and uniqueness of a global solution for $t > 0$ (including the case of hard potentials). Hence, to prove Theorem 1.1 it suffices to show the existence and uniqueness of a global solution for negative time. Nevertheless, the original Kaniel–Shinbrot method is not suitable for dealing with the problem for negative time. The main ingredient of our method is a new iterative scheme, which is a variant of the Kaniel–Shinbrot iterative method.

Suppose that $\varepsilon(t)$ is a continuous function defined in $(-\infty, 0]$ such that $0 < \varepsilon(t) < 1$ (the precise definition of $\varepsilon(t)$ will be given in the next section) and let $l_0(t) = (1 - \varepsilon(t)) \times \mathcal{M}(x - t\xi, \xi)$ and $u_0(t) = (1 + \varepsilon(t))\mathcal{M}(x - t\xi, \xi)$, then we define recursively two sequences $\{l_k(t)\}$ and $\{u_k(t)\}$ as the solutions of the equations

$$\begin{cases} \frac{\partial}{\partial t} l^\#_{k+1} - l^\#_{k+1} L(u_k)^\# = Q^+(u_k, u_k)^\# - 2l^\#_k L(l_k)^\#, & t \in (-\infty, 0), \\ \frac{\partial}{\partial t} u^\#_{k+1} - u^\#_{k+1} L(l_k)^\# = Q^+(l_k, l_k)^\# - 2u^\#_k L(u_k)^\#, & t \in (-\infty, 0), \\ l^\#_{k+1}(0) = u^\#_{k+1}(0) = f_0 \end{cases} \tag{1.5}$$

for $k = 0, 1, 2, \ldots$. (1.5) is the new iterative scheme. We will show in the next section that under certain small condition on $\varepsilon_0 = \varepsilon(0)$, the iterative sequences $\{l_k(t)\}$ and $\{u_k(t)\}$ are monotone:

$$l^\#_0(t) \leqslant l^\#_1(t) \leqslant \cdots \leqslant l^\#_{k+1}(t) \leqslant u^\#_{k+1}(t) \leqslant \cdots \leqslant u^\#_1(t) \leqslant u^\#_0(t). \tag{1.6}$$

Using the boundedness and monotone property, we define two positive functions $l(t)$ and $u(t)$ by

$$l(t, x, \xi) = \lim_{k\to\infty} l_k(t, x, \xi), \qquad u(t, x, \xi) = \lim_{k\to\infty} u_k(t, x, \xi).$$

Then, by means of Gronwall's lemma we show that $l \equiv u$, so $f(t, x, \xi) = l(t, x, \xi)$ is the unique solution of (1.1) for $t \in (-\infty, 0)$.

2. Proof of Theorem 1.1

The first auxiliary tool, which will be used in the proof of Theorem 1.1, was shown by Toscani [1,8] for $t \geqslant 0$. For $-\infty < t < \infty$, the proof is similar to that of $t \geqslant 0$. For the sake of completeness, we will give a short proof below.

Lemma 2.1. *Let $-1 \leqslant \gamma \leqslant 0$ and let $\mathcal{M}(x, \xi)$ be the travelling Maxwellian given above. Then for any $t \in (-\infty, \infty)$,*

$$\int_{\mathbf{R}^3 \times S^2_+} B(|V|, \theta)\mathcal{M}(x + t\xi', \xi')\mathcal{M}(x + t\xi'_*, \xi'_*)\, d\xi_*\, d\omega$$
$$= \int_{\mathbf{R}^3 \times S^2_+} B(|V|, \theta)\mathcal{M}(x + t\xi, \xi)\mathcal{M}(x + t\xi_*, \xi_*)\, d\xi_*\, d\omega.$$

Furthermore, let

$$F(t,x,\xi)=\int\limits_{\mathbf{R}^3\times S_+^2} B\big(|V|,\theta\big)\mathcal{M}(x-tV,\xi_*)\,d\xi_*\,d\omega$$
$$=b_0C\int\limits_{\mathbf{R}^3} B\big(|V|,\theta\big)\mathcal{M}(x-tV,V+\xi)\,dV$$

and

$$M(t)=\sup\big\{F(t,x,\xi)\colon\ (x,\xi)\in\mathbf{R}^3\times\mathbf{R}^3\big\}.$$

Then $M(t)\in L^1(-\infty,\infty)$.

Proof. The first part of the theorem follows directly from the conservation laws: $\xi+\xi_*=\xi'+\xi_*'$ and $|\xi|^2+|\xi_*|^2=|\xi'|^2+|\xi_*'|^2$. For the second part of this theorem, we first notice that $F(t,x,\xi)=F(-t,-x,\xi)$ for $t\in(-\infty,\infty)$, which implies that $M(t)=M(-t)$ (i.e., $M(t)$ is an even function). So, it is enough to show $M(t)\in L^1(0,\infty)$. Let $t\geqslant 0$, then

$$\alpha(x-tV)^2+\beta(V+\xi)^2=(-\sqrt{\alpha}\,x+\sqrt{\alpha}\,tV)^2+(\sqrt{\beta}\,V+\sqrt{\beta}\,\xi)^2$$
$$\geqslant\frac{1}{2}\big|(-\sqrt{\alpha}\,x+\sqrt{\beta}\,\xi)+(\sqrt{\alpha}\,t+\sqrt{\beta})V\big|^2=\frac{1}{2}\big|z+\eta(t)V\big|^2,$$

where $z=-\sqrt{\alpha}\,x+\sqrt{\beta}\,\xi$, $\eta(t)=\sqrt{\alpha}\,t+\sqrt{\beta}$. Let us compute the integral

$$\int\limits_{\mathbf{R}^3} B\big(|V|,\theta\big)\mathcal{M}(x-tV,V+\xi)\,dV$$

in spherical coordinates:

$$F(t,x,\xi)=b_0C\int\limits_{\mathbf{R}^3}|V|^\gamma\exp\big[-\alpha(x-tV)^2-\beta(V+\xi)^2\big]\,dV$$
$$\leqslant b_0C\int_0^{2\pi}d\phi\int_0^{\pi}\sin\psi\,d\psi$$
$$\times\int_0^\infty|V|^{\gamma+2}\exp\bigg(-\frac{1}{2}\big[z^2+2\eta(t)|z||V|\cos\psi+\eta(t)^2V^2\big]\bigg)\,dV$$
$$=4\pi b_0C\int_0^\infty|V|^{\gamma+2}\exp\bigg(-\frac{1}{2}\big[z^2+\eta(t)^2V^2\big]\bigg)\frac{\sinh[\eta(t)|z||V|]}{\eta(t)|z||V|}\,dV$$
$$=4\pi b_0C\exp\bigg(-\frac{1}{2}z^2\bigg)\int_0^\infty|V|^{\gamma+2}\exp\bigg(-\frac{1}{2}\eta(t)^2V^2\bigg)$$
$$\times\sum_{k=0}^\infty\frac{[\eta(t)|z||V|]^{2k}}{(2k+1)!}\,dV\quad\text{(using Levi's lemma)}$$

$$
\begin{aligned}
&= 4\pi b_0 C \exp\left(-\frac{1}{2}z^2\right) \sum_{k=0}^{\infty} \int_0^{\infty} |V|^{\gamma+2} \exp\left(-\frac{1}{2}\eta(t)^2 V^2\right) \\
&\quad \times \frac{[\eta(t)|z||V|]^{2k}}{(2k+1)!}\, dV \quad \left(\text{setting } s = \frac{1}{2}\eta(t)^2 V^2\right) \\
&= 4\pi b_0 C \exp\left(-\frac{1}{2}z^2\right) \frac{1}{\eta(t)^{\gamma+3}} \sum_{k=0}^{\infty} \frac{|z|^{2k} 2^{k+\frac{\gamma+1}{2}}}{(2k+1)!} \\
&\quad \times \int_0^{\infty} s^{k+\frac{\gamma+1}{2}} \exp(-s)\, ds \\
&= 4\pi b_0 C \exp\left(-\frac{1}{2}z^2\right) \frac{1}{\eta(t)^{\gamma+3}} \sum_{k=0}^{\infty} \frac{|z|^{2k} 2^{k+\frac{\gamma+1}{2}}}{(2k+1)!} \Gamma\left(k + \frac{\gamma+3}{2}\right) \\
&\leqslant 4\pi b_0 C \exp\left(-\frac{1}{2}z^2\right) \frac{2^{\frac{\gamma+1}{2}}}{\eta(t)^{\gamma+3}} \sum_{k=0}^{\infty} \frac{2^{2k} \Gamma(k+1+\frac{1}{2})}{(2k+1)!} \left(\frac{|z|^2}{2}\right)^k \\
&\leqslant C(\gamma) \exp\left(-\frac{1}{2}z^2\right) \frac{1}{\eta(t)^{\gamma+3}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{|z|^2}{2}\right)^k = \frac{C(\gamma)}{\eta(t)^{\gamma+3}}.
\end{aligned}
$$

In the above calculations, we have used the following facts:

$$
\Gamma\left(k + \frac{\gamma+3}{2}\right) \leqslant \Gamma\left(k + 1 + \frac{1}{2}\right), \quad \gamma \geqslant -1,\ k \geqslant 1,
$$

$$
\Gamma\left(k + 1 + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot \dots \cdot (2k+1)}{2^{k+1}} \sqrt{\pi}, \quad k \geqslant 0,
$$

and

$$
\frac{2^{k-1} \cdot 1 \cdot 3 \cdot \dots \cdot (2k+1)}{(2k+1)!} = \frac{1}{2k!}, \quad k \geqslant 0.
$$

So,

$$
M(t) = \sup\left\{F(t,x,\xi)\colon (x,\xi) \in \mathbf{R}^3 \times \mathbf{R}^3\right\} \leqslant \frac{C(\gamma)}{\eta(t)^{\gamma+3}}.
$$

Since $\gamma \geqslant -1$, $M(t) \in L^1(0,\infty)$. $\square$

Thanks to Lemma 2.1 and the choice of u_0, we have for $t < 0$,

$$
\begin{aligned}
&\int_t^0 Q^+(u_0,u_0)^\#(s,x,\xi)\, ds \\
&\quad = \int_t^0 ds \int_{\mathbf{R}^3 \times S_+^2} B\big(|V|,\theta\big) u_0(s, x+s\xi, \xi') u_0(s, x+s\xi, \xi'_*)\, d\xi_*\, d\omega
\end{aligned}
$$

$$
\begin{aligned}
&= \int_t^0 ds \int_{\mathbf{R}^3\times S^2_+} B\big(|V|,\theta\big) u_0^{\#}\big(s, x+s(\xi-\xi'),\xi'\big) u_0^{\#}\big(s, x+s(\xi-\xi'_*),\xi'_*\big)\, d\xi_*\, d\omega \\
&\leqslant 4\int_t^0 ds \int_{\mathbf{R}^3\times S^2_+} B\big(|V|,\theta\big)\mathcal{M}\big(x+s(\xi-\xi'),\xi'\big)\mathcal{M}\big(x+s(\xi-\xi'_*),\xi'_*\big)\, d\xi_*\, d\omega \\
&= 4\int_t^0 ds \int_{\mathbf{R}^3\times S^2_+} B\big(|V|,\theta\big)\mathcal{M}(x,\xi)\mathcal{M}(x-sV,\xi_*)\, d\xi_*\, d\omega \\
&\leqslant 4\mathcal{M}(x,\xi)\int_t^0 M(s)\, ds \leqslant 4\mathcal{M}(x,\xi)\int_{-\infty}^0 M(s)\, ds = 4K\mathcal{M}(x,\xi),
\end{aligned}
$$

where $K = \int_{-\infty}^0 M(s)\, ds$. Hence, for any $(x,\xi)\in \mathbf{R}^3\times\mathbf{R}^3$, we have

$$
Q^+(u_0,u_0)^{\#}(s,x,\xi), \qquad Q^+(l_0,l_0)^{\#}(s,x,\xi) \in L^1_{\mathrm{loc}}(-\infty,0).
$$

This shows that iterative equation (1.5) has a unique solution (l_1,u_1) for $k=0$. The next step is crucial for the proof of Theorem 1.1.

Lemma 2.2. *If we have*

$$
0 < l_{k-1}^{\#}(t) \leqslant l_k^{\#}(t) \leqslant u_k^{\#}(t) \leqslant u_{k-1}^{\#}(t), \quad t<0,
$$

for a fixed k, then

$$
0 < l_k^{\#}(t) \leqslant l_{k+1}^{\#}(t) \leqslant u_{k+1}^{\#}(t) \leqslant u_k^{\#}(t), \quad t<0.
$$

Proof. Integrating (1.5) from $t<0$ to 0, we obtain

$$
\begin{aligned}
&l_{k+1}^{\#}(t) + \int_t^0 l_{k+1}^{\#}(s) L(u_k)^{\#}(s)\, ds \\
&\quad = 2\int_t^0 l_k^{\#}(s) L(l_k)^{\#}(s)\, ds - \int_t^0 Q^+(u_k,u_k)^{\#}(s)\, ds + f_0,
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
&u_{k+1}^{\#}(t) + \int_t^0 u_{k+1}^{\#}(s) L(l_k)^{\#}(s)\, ds \\
&\quad = 2\int_t^0 u_k^{\#}(s) L(u_k)^{\#}(s)\, ds - \int_t^0 Q^+(l_k,l_k)^{\#}(s)\, ds + f_0.
\end{aligned}
\tag{2.2}
$$

From (2.1), we obtain

$$(l_{k+1}-l_k)^{\#}(t)+\int_t^0 l_{k+1}^{\#}(s)L(u_k)^{\#}(s)\,ds-\int_t^0 l_k^{\#}(s)L(u_{k-1})^{\#}(s)\,ds$$
$$=2\int_t^0\big[l_k^{\#}(s)L(l_k)^{\#}(s)-l_{k-1}^{\#}(s)L(l_{k-1})^{\#}(s)\big]\,ds$$
$$+\int_t^0\big[Q^+(u_{k-1},u_{k-1})^{\#}(s)-Q^+(u_k,u_k)^{\#}(s)\big]\,ds.$$

Then we get

$$(l_{k+1}-l_k)^{\#}(t)+\int_t^0[l_{k+1}-l_k]^{\#}(s)L(u_k)^{\#}(s)\,ds=g_1(t), \tag{2.3}$$

where

$$g_1(t)=2\int_t^0\big[l_k^{\#}(s)L(l_k)^{\#}(s)-l_{k-1}^{\#}(s)L(l_{k-1})^{\#}(s)\big]\,ds$$
$$+\int_t^0 l_k^{\#}(s)L(u_{k-1}-u_k)^{\#}(s)\,ds$$
$$+\int_t^0\big[Q^+(u_{k-1},u_{k-1})^{\#}(s)-Q^+(u_k,u_k)^{\#}(s)\big]\,ds.$$

Obviously, $g_1'(t)\leqslant 0$, hence (2.3) implies that $0<l_k^{\#}(t)\leqslant l_{k+1}^{\#}(t)$.

Similar to the above procedure, we get from (2.1) and (2.2):

$$(u_{k+1}-l_{k+1})^{\#}(t)+\int_t^0[u_{k+1}-l_{k+1}]^{\#}(s)L(l_k)^{\#}(s)\,ds=g_2(t), \tag{2.4}$$
$$(u_k-u_{k+1})^{\#}(t)+\int_t^0[u_k-u_{k+1}]^{\#}(s)L(l_{k-1})^{\#}(s)\,ds=g_3(t), \tag{2.5}$$

where

$$g_2(t)=2\int_t^0\big[u_k^{\#}(s)L(u_k)^{\#}(s)-l_k^{\#}(s)L(l_k)^{\#}(s)\big]\,ds$$
$$+\int_t^0 l_{k+1}^{\#}(s)L(u_k-l_k)^{\#}(s)\,ds$$
$$+\int_t^0\big[Q^+(u_k,u_k)^{\#}(s)-Q^+(l_k,l_k)^{\#}(s)\big]\,ds,$$

$$g_3(t) = 2\int_t^0 \bigl[u_{k-1}^{\#}(s)L(u_{k-1})^{\#}(s) - u_k^{\#}(s)L(u_k)^{\#}(s)\bigr]\,ds$$
$$+ \int_t^0 u_{k+1}^{\#}(s)L(l_k - l_{k-1})^{\#}(s)\,ds$$
$$+ \int_t^0 \bigl[Q^+(l_k, l_k)^{\#}(s) - Q^+(l_{k-1}, l_{k-1})^{\#}(s)\bigr]\,ds.$$

Thanks to $0 < l_k^{\#}(t) \leqslant l_{k+1}^{\#}(t)$, we have $g_2'(t) \leqslant 0$. Consequently, we get by (2.4) that $0 < l_{k+1}^{\#}(t) \leqslant u_{k+1}^{\#}(t)$. This result also implies that $g_3'(t) \leqslant 0$. Combining $g_3'(t) \leqslant 0$ with (2.5), we obtain that $u_k^{\#}(t) \geqslant u_{k+1}^{\#}(t)$. $\square$

Lemma 2.3. *Let* $0 < \varepsilon_0 < \frac{3\exp(-6K)}{4-\exp(-6K)}$ *and let*

$$\varepsilon(t) = \frac{3\varepsilon_0}{(3+\varepsilon_0)\exp(6\int_0^t M(s)\,ds) - \varepsilon_0} \in C(-\infty, 0].$$

Furthermore, assume that $l_0(t) = (1-\varepsilon(t))\mathcal{M}(x-t\xi, \xi)$ *and* $u_0(t) = (1+\varepsilon(t))\mathcal{M}(x-t\xi, \xi)$. *Then the beginning condition is satisfied*:

$$0 < l_0^{\#}(t) \leqslant l_1^{\#}(t) \leqslant u_1^{\#}(t) \leqslant u_0^{\#}(t).$$

Proof. From (2.1) and (2.2) we obtain, by setting $k = 0$,

$$l_1^{\#}(t) - l_0^{\#}(t) = G_1(t, x, \xi) - \int_t^0 \bigl(l_1^{\#}(s) - l_0^{\#}(s)\bigr)L(u_0)^{\#}(s)\,ds, \tag{2.6}$$

$$u_0^{\#}(t) - u_1^{\#}(t) = G_2(t, x, \xi) - \int_t^0 \bigl(u_0^{\#}(s) - u_1^{\#}(s)\bigr)L(l_0)^{\#}(s)\,ds, \tag{2.7}$$

where

$$G_1(t, x, \xi) = f_0 - l_0^{\#}(t) + 2\int_t^0 l_0^{\#}(s)L(l_0)^{\#}(s)\,ds$$
$$- \int_t^0 l_0^{\#}(s)L(u_0)^{\#}(s)\,ds - \int_t^0 Q^+(u_0, u_0)^{\#}(s)\,ds,$$

$$G_2(t, x, \xi) = u_0^{\#}(t) - f_0 - 2\int_t^0 u_0^{\#}(s)L(u_0)^{\#}(s)\,ds$$

$$+\int_t^0 u_0^{\#}(s)L(l_0)^{\#}(s)\,ds+\int_t^0 Q^+(l_0,l_0)^{\#}(s)\,ds.$$

Since $f_0(x,\xi)\geqslant(1-\varepsilon_0)\mathcal{M}(x,\xi)=l_0^{\#}(0,x,\xi)$ and $f_0(x,\xi)\leqslant(1+\varepsilon_0)\mathcal{M}(x,\xi)=u_0^{\#}(0,x,\xi)$, in order to show that $l_1^{\#}(t)\geqslant l_0^{\#}(t)$ and $u_0^{\#}(t)\geqslant u_1^{\#}(t)$, it suffices to prove the following inequalities:

$$\frac{\partial}{\partial t}G_j(t,x,\xi)\leqslant 0,\quad (x,\xi)\in\mathbf{R}^3\times\mathbf{R}^3,\ t<0,\ j=1,2. \tag{2.8}$$

Since $l_0(t)=(1-\varepsilon(t))\mathcal{M}(x-t\xi,\xi)$ and $u_0(t)=(1+\varepsilon(t))\mathcal{M}(x-t\xi,\xi)$, it follows from Lemma 2.1 that

$$\frac{\partial}{\partial t}G_1(t,x,\xi)=\varepsilon'(t)+\big(6\varepsilon(t)-2\varepsilon^2(t)\big)F(t,x,\xi),$$
$$\frac{\partial}{\partial t}G_2(t,x,\xi)=\varepsilon'(t)+\big(6\varepsilon(t)+2\varepsilon^2(t)\big)F(t,x,\xi).$$

Consequently, (2.8) will be valid if $\varepsilon(t)$ satisfies

$$\begin{cases}\varepsilon'(t)+(6\varepsilon(t)+2\varepsilon^2(t))M(t)=0,\\ \varepsilon(0)=\varepsilon_0.\end{cases} \tag{2.9}$$

The unique solution of (2.9) for $t\in(-\infty,0)$ is

$$\varepsilon(t)=\frac{3\varepsilon_0}{(3+\varepsilon_0)\exp(6\int_0^t M(s)\,ds)-\varepsilon_0}.$$

Obviously, $\varepsilon(t)$ is nonincreasing on $(-\infty,0)$ and

$$\varepsilon(-\infty)=\lim_{t\to-\infty}\varepsilon(t)=\frac{3\varepsilon_0}{(3+\varepsilon_0)\exp(-6K)-\varepsilon_0}.$$

Since $0<\varepsilon_0<\frac{3\exp(-6K)}{4-\exp(-6K)}<1$, we get

$$0<\varepsilon(t)\leqslant\varepsilon(-\infty)<1,\quad t\leqslant 0.$$

Then we have proved that $l_1^{\#}(t)\geqslant l_0^{\#}(t)$ and $u_0^{\#}(t)\geqslant u_1^{\#}(t)$. The proof of $l_1^{\#}(t)\leqslant u_1^{\#}(t)$ is contained in the proof of Lemma 2.2. This completes the proof. □

Proof of Theorem 1.1. For measurable function $f(t,x,\xi)$ satisfying $|f^{\#}(t,x,\xi)|\leqslant 2\mathcal{M}(x,\xi)$, we denote

$$\|f\|(t)=\sup\big\{\big|f^{\#}(t,x,\xi)\big|\mathcal{M}^{-1}(x,\xi)\colon (x,\xi)\in\mathbf{R}^3\times\mathbf{R}^3\big\}.$$

Suppose that the initial value f_0 satisfies

$$(1-\varepsilon_0)\mathcal{M}(x,\xi)\leqslant f_0(x,\xi)\leqslant(1+\varepsilon_0)\mathcal{M}(x,\xi)$$

and let

$$l_0(t)=\big(1-\varepsilon(t)\big)\mathcal{M}(x-t\xi,\xi),\qquad u_0(t)=\big(1+\varepsilon(t)\big)\mathcal{M}(x-t\xi,\xi),$$

where

$$0<\varepsilon_0<\frac{3\exp(-6K)}{4-\exp(-6K)}<1, \qquad \varepsilon(t)=\frac{3\varepsilon_0}{(3+\varepsilon_0)\exp(6\int_0^t M(s)\,ds)-\varepsilon_0},$$

then it follows from Lemmas 2.3 and 2.2 that the iterative sequences $\{l_k(t)\}$ and $\{u_k(t)\}$ defined by (1.5) satisfy (1.6). Hence, we have

$$l_k^\#(t)\uparrow l^\#(t), \quad u_k^\#(t)\downarrow u^\#(t), \quad k\to\infty,$$

and

$$\big(1-\varepsilon(t)\big)\mathcal{M}(x,\xi)=l_0^\#(t)\leqslant l^\#(t)\leqslant u^\#(t)\leqslant u_0^\#(t)=\big(1+\varepsilon(t)\big)\mathcal{M}(x,\xi).$$

Consequently, we get $\|l\|(t)\leqslant\|u\|(t)\leqslant 2$. Letting $k\to\infty$ in (2.1) and (2.2), we obtain by Lebesgue's convergence theorem

$$\begin{aligned} &l^\#(t)+\int_t^0 l^\#(s)L(u)^\#(s)\,ds\\ &\quad=2\int_t^0 l^\#(s)L(l)^\#(s)\,ds-\int_t^0 Q^+(u,u)^\#(s)\,ds+f_0, \end{aligned}\tag{2.10}$$

$$\begin{aligned} &u^\#(t)+\int_t^0 u^\#(s)L(l)^\#(s)\,ds\\ &\quad=2\int_t^0 u^\#(s)L(u)^\#(s)\,ds-\int_t^0 Q^+(l,l)^\#(s)\,ds+f_0. \end{aligned}\tag{2.11}$$

Equations (2.10) and (2.11) lead to

$$\begin{aligned} (u-l)^\#(t)&=2\int_t^0 u^\#(s)L(u)^\#(s)\,ds-2\int_t^0 l^\#(s)L(l)^\#(s)\,ds\\ &\quad+\int_t^0 Q^+(u,u)^\#(s)\,ds-\int_t^0 Q^+(l,l)^\#(s)\,ds\\ &\quad+\int_t^0 l^\#(s)L(u)^\#(s)\,ds-\int_t^0 u^\#(s)L(l)^\#(s)\,ds\\ &=2\int_t^0 u^\#(s)L(u-l)^\#(s)\,ds+2\int_t^0 l^\#(s)L(u-l)^\#(s)\,ds\\ &\quad+\int_t^0 (u-l)^\#(s)L(l)^\#(s)\,ds+\int_t^0 l^\#(s)L(l-u)^\#(s)\,ds \end{aligned}$$

$$+\int_t^0 Q^+(u,u-l)^\#(s)\,ds+\int_t^0 Q^+(u-l,l)^\#(s)\,ds.$$

On the other hand, we get from Lemma 2.1 that

$$\begin{aligned}
&Q^+(u,u-l)^\#(s,x,\xi)\\
&\quad=\int_{\mathbf{R}^3\times S^2_+} B\big(|V|,\theta\big)u^\#\big(s,x+s(\xi-\xi'),\xi'\big)(u-l)^\#\big(s,x+s(\xi-\xi'_*),\xi'_*\big)\,d\xi_*\,d\omega\\
&\quad\leqslant\int_{\mathbf{R}^3\times S^2_+} B\big(|V|,\theta\big)\|u\|(s)\big\|(u-l)\big\|(s)\mathcal{M}(x,\xi)\mathcal{M}(x-sV,\xi_*)\,d\xi_*\,d\omega\\
&\quad\leqslant 2\mathcal{M}(x,\xi)F(s,x,\xi)\big\|(u-l)\big\|(s).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&Q^+(u-l,l)^\#(s)\leqslant 2\mathcal{M}(x,\xi)F(s,x,\xi)\big\|(u-l)\big\|(s),\\
&u^\#(s)L(u-l)^\#(s)\leqslant 2\mathcal{M}(x,\xi)F(s,x,\xi)\big\|(u-l)\big\|(s),\\
&l^\#(s)L(u-l)^\#(s)\leqslant 2\mathcal{M}(x,\xi)F(s,x,\xi)\big\|(u-l)\big\|(s),\\
&(u-l)^\#(s)L(u)^\#(s)\leqslant 2\mathcal{M}(x,\xi)F(s,x,\xi)\big\|(u-l)\big\|(s).
\end{aligned}$$

Hence,

$$(u-l)^\#(t)\leqslant 14\mathcal{M}(x,\xi)\int_t^0 F(s,x,\xi)\|u-l\|(s)\,ds.$$

This inequality implies that

$$\|u-l\|(t)\leqslant 14\int_t^0 M(s)\|u-l\|(s)\,ds,\qquad t\leqslant 0.$$

Then Gronwall's inequality implies that $\|u-l\|(t)=0$ ($\forall t\leqslant 0$). So, we have $l\equiv u$ and $f(t,x,\xi)=l(t,x,\xi)$ is the desired solution for $t\in(-\infty,0)$.

In order to prove uniqueness, we assume that $g(t,x,\xi)$ is another solution of (1.4) and satisfies

$$\big(1-\varepsilon(t)\big)\mathcal{M}(x-t\xi,\xi)\leqslant g(t,x,\xi)\leqslant\big(1+\varepsilon(t)\big)\mathcal{M}(x-t\xi,\xi).$$

So, we have

$$\begin{cases}\frac{\partial}{\partial t}g^\#(t,x,\xi)=Q(g,g)^\#(t,x,\xi),\\ g^\#(0,x,\xi)=f_0(x,\xi).\end{cases}\tag{2.12}$$

Subtracting (2.12) from (1.4) and then integrating from t to 0, we obtain

$$
\begin{aligned}
&f^{\#}(t,x,\xi)-g^{\#}(t,x,\xi)\\
&\quad=\int_t^0 Q(f-g,f+g)^{\#}(s,x,\xi)\,ds+\int_t^0 Q(f+g,f-g)^{\#}(s,x,\xi)\,ds.
\end{aligned}
$$

Using Lemma 2.1 and the definition of the norm, we have

$$
\begin{aligned}
&\bigl|f^{\#}(t,x,\xi)-g^{\#}(t,x,\xi)\bigr|\\
&\quad\leqslant\int_t^0 Q^{+}\bigl(|f-g|,f+g\bigr)^{\#}(s,x,\xi)\,ds+\int_t^0 Q^{+}\bigl(f+g,|f-g|\bigr)^{\#}(s,x,\xi)\,ds\\
&\qquad+\int_t^0 Q^{-}\bigl(|f-g|,f+g\bigr)^{\#}(s,x,\xi)\,ds+\int_t^0 Q^{-}\bigl(f+g,|f-g|\bigr)^{\#}(s,x,\xi)\,ds\\
&\quad\leqslant 4\int_t^0 ds\int_{\mathbf{R}^3\times S_+^2} B\bigl(|V|,\theta\bigr)|f-g|^{\#}\bigl(s,x+s(\xi-\xi'),\xi'\bigr)\\
&\qquad\times\mathcal{M}\bigl(x+s(\xi-\xi_*'),\xi_*'\bigr)\,d\xi_*\,d\omega\\
&\qquad+4\int_t^0 ds\int_{\mathbf{R}^3\times S_+^2} B\bigl(|V|,\theta\bigr)\\
&\qquad\times\mathcal{M}\bigl(x+s(\xi-\xi'),\xi'\bigr)|f-g|^{\#}\bigl(s,x+s(\xi-\xi_*'),\xi_*'\bigr)\,d\xi_*\,d\omega\\
&\qquad+4\int_t^0 ds\int_{\mathbf{R}^3\times S_+^2} B\bigl(|V|,\theta\bigr)|f-g|^{\#}(s,x,\xi)\mathcal{M}\bigl(x+s(\xi-\xi_*),\xi_*\bigr)\,d\xi_*\,d\omega\\
&\qquad+4\int_t^0 ds\int_{\mathbf{R}^3\times S_+^2} B\bigl(|V|,\theta\bigr)\mathcal{M}(x,\xi)|f-g|^{\#}\bigl(s,x+s(\xi-\xi_*),\xi_*\bigr)\,d\xi_*\,d\omega\\
&\quad\leqslant 16\int_t^0 \|f-g\|(s)\,ds\int_{\mathbf{R}^3\times S_+^2} B\bigl(|V|,\theta\bigr)\mathcal{M}(x,\xi)\mathcal{M}\bigl(x+s(\xi-\xi_*),\xi_*\bigr)\,d\xi_*\,d\omega\\
&\quad=16\mathcal{M}(x,\xi)\int_t^0 \|f-g\|(s)F(s,x,\xi)\,ds\leqslant 16\mathcal{M}(x,\xi)\int_t^0 \|f-g\|(s)M(s)\,ds.
\end{aligned}
$$

Consequently,

$$
\|f-g\|(t)\leqslant 16\int_t^0 M(s)\|f-g\|(s)\,ds,\qquad t\leqslant 0.
$$

Then Gronwall's lemma implies that $\|f - g\|(t) = 0$ for all $t \leqslant 0$, hence $f(t,x,\xi) = g(t,x,\xi)$ for all $(t,x,\xi) \in (-\infty, 0) \times \mathbf{R}^3 \times \mathbf{R}^3$. This completes the proof of Theorem 1.1. □

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