Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 220 (2008) 175–180
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

A penalty function method for solving inverse optimal value problem ☆

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Received 24 April 2007; received in revised form 31 July 2007

Abstract

In order to consider the inverse optimal value problem under more general conditions, we transform the inverse optimal value problem into a corresponding nonlinear bilevel programming problem equivalently. Using the Kuhn–Tucker optimality condition of the lower level problem, we transform the nonlinear bilevel programming into a normal nonlinear programming. The complementary and slackness condition of the lower level problem is appended to the upper level objective with a penalty. Then we give via an exact penalty method an existence theorem of solutions and propose an algorithm for the inverse optimal value problem, also analysis the convergence of the proposed algorithm. The numerical result shows that the algorithm can solve a wider class of inverse optimal value problem.

MSC: 90C05; 90C26

Keywords: Inverse optimal value problem; Inverse optimal problem; Bilevel programming; Penalty method

1. Introduction

An inverse optimization problem consists of inferring the value of the model parameters, such as objective function and constraint coefficient. A standard inverse optimization problem is as follows: given an optimization problem with a linear objective $P : \min_x \{c^{\mathrm{T}}x | x \in X\}$ and a desired optimal solution $x^* \in X$, find a cost vector c^* such that $x^* \in X$ is an optimal solution of P, and at the same time c^* is required to satisfy some additional conditions. Such that, given a preferred cost c', the deviation $\|c^* - c'\|_p$ is to be minimum under some ℓ_p norm.

Geophysical scientists were the first ones to study inverse problems. The book in [12] gives a comprehensive discussion of the theory of inverse problems in the geophysical sciences. From then on, this challenge topic has attracted more and more attention, which leads to a rapid development in theories, algorithms and applications. For detailed expositions, the reader may consult [2,6,7,15].

☆ Supported by the National Natural Science Foundation of China (50479039 and 70771080).
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0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.08.005

Motivated by an application in telecommunication bandwidth pricing as proposed in [10,1] studied the inverse optimal value problem, that is to say, given the optimization problem P, a desired optimal objective value z^*, and a set of feasible cost vectors C, determine a cost vector $c^* \in C$ such that the corresponding optimal objective value of P is closest to z^*.

In [1], Ahmed and Guan proved that the inverse optimal value problem is NP-hard and based on some assumptions, they got the optimal solutions of the inverse optimal value problem by solving a series of linear and bilinear programming problems. But it deserves pointing out that some assumptions in [1] seem to be rather restrictive, such as the set of cost vectors C must be convex, else most conclusions and the algorithm in [1] would not be valid. Then, it is necessary to study the inverse optimal value problem under more general conditions.

The purpose of this paper is two-fold. One is to transform the inverse optimal value problem into the corresponding nonlinear bilevel programming (BLP) problem, the reason why we do such transformation is that we can solve the inverse optimal value problem under more general conditions; while the other is to give via an exact penalty method an existence theorem of the solutions of the corresponding nonlinear BLP problem and additionally propose a simple algorithm for this problem. Towards these ends, the rest of the paper is organized as follows. In Section 2, we will firstly introduce the inverse optimal value problem, then transform this problem into the corresponding nonlinear BLP problem. Then, in Section 3 for the equivalent nonlinear BLP problem we give via an exact penalty method an existence theorem of solutions. An algorithm is proposed and an example is tested in Section 4. Finally we conclude our paper.

2. Problem statement

In this section, we firstly introduce the inverse optimal value problem, then we describe how to consider this problem using BLP method.

Consider the optimal value function of a linear programming in terms of its cost vector

$$Q(c) = \min_x \left\{ c^{\mathrm{T}} x : Ax \leqslant b, x \geqslant 0 \right\}, \tag{1}$$

where $x \in R^n$, $A \in R^{p \times n}$, $b \in R^p$. Given a set $C \subseteq R^n$ of the objective cost vectors and a real number z^*, then the inverse optimal value problem is to find a cost vector from the set C such that the optimal objective value of the linear programming (1) is "close" to z^*, thus the inverse optimal value problem can be written as [1]

$$\min_c \{ f(c) : c \in C \}, \tag{2}$$

where $f(c) = |Q(c) - z^*|$.

As $Q(c)$ is the optimal value of the linear programming, then problem (2) can be written as

$$\min_c \left[|Q(c) - z^*| : c \in C, Q(c) = \min_x \{ c^{\mathrm{T}} x : Ax \leqslant b, x \geqslant 0 \} \right].$$

Go one step further, it can be written as follows:

$$\begin{aligned} &(\mathrm{UP}) \quad \min_c \quad |Q(c) - z^*|, \\ &\qquad\quad \text{s.t.} \quad c \in C, \\ &(\mathrm{LP}) \qquad\quad Q(c) = \min_x \left\{ c^{\mathrm{T}} x : Ax \leqslant b, x \geqslant 0 \right\}. \end{aligned} \tag{3}$$

Remark 1. In fact, model (3) is called the BLP problem with the optimal value of the lower level problem feeding back to the upper level [14]. (UP) is called the upper level problem and (LP) is called the lower level problem.

From the above description, we can find that the inverse optimal value problem is equivalent to a special class of BLP problem. Then it provides us an alternative approach to consider the inverse optimal value problem. To our knowledge, there are few papers to consider the inverse optimal value problem from the view point of BLP.

3. The exact penalty method

BLP has been widely applied to decentralized planning problems involving a decision progress with a hierarchical structure. It is characterized by the existence of two optimization problems in which the constraint region of the upper level problem is implicitly determined by another optimization problem. It is well known that the BLP problem is NP-hard [3,5], and there are some approaches for solving the nonlinear BLP problem, such as the branch-and-bound approach [4], descent approach [13,11], penalty function approach [8], etc. Even so, there are few approaches with attractive computation perspective. Then proposing efficient algorithms for solving the nonlinear BLP problem is a challenge topic.

In this section, for problem (3) we give via an exact penalty method an existence of theorems of solution, then propose an algorithm with attractive computation perspective, which is different from the algorithms proposed ever before.

Throughout the rest of this paper, we make the following assumptions:

(H_1) The feasible region of the linear programming $\{x : Ax \leqslant b, x \geqslant 0\}$ is nonempty bound.

(H_2) The set of cost vectors C is nonempty and compact.

Remark 2. Compared with the assumptions $A1$–$A4$ in [1], here the cost vectors C need not be convex.

Without losing generality, we replace $f(c) = |Q(c) - z^*|$ with $f'(c) = (Q(c) - z^*)^2$, then problem (3) can be written as

$$\begin{aligned} \min_{c} \quad & (Q(c) - z^*)^2, \\ \text{s.t.} \quad & c \in C, \\ & Q(c) = \min_{x} \left\{ c^{\mathrm{T}} x : Ax \leqslant b, x \geqslant 0 \right\}. \end{aligned} \tag{4}$$

For fixed $c \in C$, the lower level problem is convex programming, we can replace the lower problem with it's Kuhn–Tucker optimality conditions. That is, problem (4) can be written equivalently as

$$\begin{aligned} \min_{(c,u,v,x)} \quad & \left(c^{\mathrm{T}} x - z^*\right)^2, \\ \text{s.t.} \quad & c \in C, \\ & Ax \leqslant b, \\ & uA - v = -c^{\mathrm{T}}, \\ & u(b - Ax) + vx = 0, \\ & u, v, x \geqslant 0, \end{aligned} \tag{5}$$

where $u \in R^p$, $v \in R^n$ are row vectors. Then, we append the complementary and slackness to the upper level objective with a penalty, and we can obtain the following penalized problem:

$$\begin{aligned} \min_{(c,u,v,x)} \quad & \left[(c^{\mathrm{T}} x - z^*)^2 + k(u(b - Ax) + vx)\right], \\ \text{s.t.} \quad & c \in C, \\ & Ax \leqslant b, \\ & uA - v = -c^{\mathrm{T}}, \\ & u, v, x \geqslant 0, \end{aligned} \tag{6}$$

where $k \in R_+$ is a large positive constant. Let $Z = \{(c, u, v) : c \in C, uA - v = -c^{\mathrm{T}}, u, v \geqslant 0\}$, $X = \{x : Ax \leqslant b, x \geqslant 0\}$. For $k \in R_+$, let

$$Q_k(x) = \min_{(c,u,v) \in Z} \left[(c^{\mathrm{T}} x - z^*)^2 + k(u(b - Ax) + vx)\right].$$

Then, we have the following result.

Theorem 1. *Let $k \in R_+$. Suppose that assumptions (H_1) and (H_2) are satisfied. Then the problem*

$$\min_{x\in X} Q_k(x)$$

has optimal solutions.

Proof.

$$\begin{aligned}\min_{x\in X} Q_k(x) &= \min_{x\in X} \min_{(c,u,v)\in Z} \left[(c^{\mathrm{T}}x - z^*)^2 + k(u(b-Ax)+vx)\right]\\ &\geqslant \min_{x\in X}\left[\min_{(c,u,v)\in Z}(c^{\mathrm{T}}x - z^*)^2 + \min_{(c,u,v)\in Z} k(u(b-Ax)+vx)\right]\\ &\geqslant \min_{x\in X,(c,u,v)\in Z}\left(c^{\mathrm{T}}x - z^*\right)^2 + \min_{x\in X,(c,u,v)\in Z}[k(u(b-Ax)+vx)].\end{aligned}$$

Following (H_1) and (H_2), for fixed $c \in C$ the lower level problem (LP) has optimal solutions, then $\min_{x\in X,(c,u,v)\in Z}[k(u(b-Ax)+vx)] = 0$, that is to say,

$$\min_{x\in X} Q_k(x) \geqslant \min_{x\in X,(c,u,v)\in Z}\left(c^{\mathrm{T}}x - z^*\right)^2 = \min_{x\in X,c\in C}\left(c^{\mathrm{T}}x - z^*\right)^2.$$

Then $Q_k(.)$ is bound from below by the finite optimal value of $(c^{\mathrm{T}}x - z^*)^2$. Then the proof is completed. □

Following Theorem 1, we can get the following corollary.

Corollary 1. *Let the assumptions (H_1) and (H_2) be satisfied, then the problem* (6) *has optimal solutions.*

Furthermore, we will show that a finite value of k would yield an exact solution to the problem (6), where the $k(u(b-Ax)+vx)$ becomes zero.

Theorem 2. *Let assumptions (H_1) and (H_2) be satisfied. Let $\{(c_k, u_k, v_k, x_k)\}$ be a sequence of solutions of* (6), *then there exists $k_1 \in R_+$ such that for all $k \geqslant k_1$, $u_k(b - Ax_k) + v_k x_k = 0$.*

Proof. Suppose (c^*, u^*, v^*, x^*) is the optimal solution of the problem (3), from the Kuhn–Tucker optimality condition of the low level problem it is obvious that the complementary and slackness condition is satisfied. That is, $u^*(b - Ax^*) + v^*x^* = 0$.

For $(c_k, u_k, v_k, x_k) \in arg\,min[(c^{\mathrm{T}}x - z^*)^2 + k(u(b-Ax)+vx) : x \in X, (c,u,v) \in Z]$, we have

$$\left(c_k^{\mathrm{T}}x - z^*\right)^2 + k(u_k(b-Ax_k)+v_k x_k) \leqslant \left((c^*)^{\mathrm{T}}x^* - z^*\right)^2.$$

Thus,

$$\begin{aligned}u_k(b-Ax_k)+v_kx_k &\leqslant \frac{\left((c^*)^{\mathrm{T}}x^* - z^*\right)^2 - \left(c_k^{\mathrm{T}}x_k - z^*\right)^2}{k}\\ &\leqslant \frac{\max\left[((c^*)^{\mathrm{T}}x^* - z^*)^2 - (c_k^{\mathrm{T}}x_k - z^*)^2 : x_k \in X, c_k \in C\right]}{k}\\ &\leqslant \frac{m}{k}.\end{aligned}$$

Where m is some constant. Note that $u_k(b-Ax_k)+v_kx_k \geqslant 0$ for all (c_k, u_k, v_k, x_k). Thus as $k \to \infty$, $u_k(b-Ax_k) + v_kx_k = 0$. That is, there exists $k_1 \in R_+$, such that $u_k(b-Ax_k)+v_kx_k = 0$ for all $k \geqslant k_1$. □

Remark 3. From the property of the penalized function method, we can find that the upper level objective $(c^{\mathrm{T}}x - z^*)^2$ and $u(b-Ax)+vx$ are both monotonically nondecreasing in the value of the penalty parameter k. Then we can get the optimal solutions of problem (3) by solving problem (6). In fact, we can get the following theorem, which shows that the penalty method is exact.

Theorem 3. *Suppose that assumptions* (H_1) *and* (H_2) *are satisfied. Let* $\{(c_k, u_k, v_k, x_k)\}$ *be a sequence of solutions of* (6), $k \in R_+$. *Then there exists* $k^* \in R_+$, *such that for all* $k \geqslant k^*$, (c_k, x_k) *solves problem* (3).

Proof. Let $k^* = k_1$, following Theorem 2 and Remark 3, Theorem 3 is obvious. □

The above theorems explore the relationship between the inverse optimal value problem (2) and the problem (6). In the following section, we will propose an algorithm for the inverse optimal value problem and also give the convergence of the algorithm proposed.

4. The algorithm

From Theorem 3, and inspired from the algorithm given in [9], we can propose an algorithm to solve problem (3), the algorithm proposed by us needs only to solve a series of differential nonlinear programming problems.

Step 0: Set $i = 0$, choose $k > 0$ (k large) and $\lambda > 0$.

Step 1: Solve the problem (6), get the optimal solution $(c_k^i, x_k^i, u_k^i, v_k^i)$.

Step 2: If $u_k^i(b - Ax_k^i) + v_k^i x_k^i = 0$, then the optimal solution of the inverse optimal value problem is $(c^*, x^*) = (c_k^i, x_k^i)$. If $u_k^i(b - Ax_k^i) + v_k^i x_k^i > 0$. Set $k = k + \lambda$, $i = i + 1$, go to step 1.

Theorem 4. *Let* (H_1) *and* (H_2) *be satisfied, then the sequence* $\{(c^k, x^k)\}$, *which comes from the above algorithm, converges to the optimal solution of problem* (3).

Proof. Firstly, if the sequence $\{(c^k, x^k)\}$ is finite. From the termination of the algorithm, it is obvious that the last point in the sequence $\{(c^k, x^k)\}$ solves problem (3).

Secondly, if the sequence $\{(c^k, x^k)\}$ is infinite. We have that $\{(c^k, x^k)\} \subset \{(c, x) | c \in C, Ax \leqslant b, x \geqslant 0\}$. Following assumptions (H_1) and (H_2), we have that the sequence $\{(c^k, x^k)\}$ exists an accumulation point $(c^*, x^*) \in \{(c, x) | c \in C, Ax \leqslant b, x \geqslant 0\}$. Corresponding to the point (c^*, x^*), there exists (u^*, v^*), such that (c^*, x^*, u^*, v^*) solves problem (6). Following Theorem 2, $u^*(b - Ax^*) + v^* x^* = 0$, then (c^*, x^*, u^*, v^*) is the optimal solution of problem (5), also solves problem (3). □

To illustrate the algorithm, we solve the following inverse optimal value problem.

Consider the following inverse optimal value problem, where $c \in R^2$, $x \in R^2$.

$$\min_c \left\{ |Q(c) - z^*| : 10 \leqslant c_1^2 + c_2^2 \leqslant 13, c_1 \geqslant 0, c_2 \geqslant 0 \right\}, \tag{7}$$

where $z^* = 14$ and $Q(c) = \max_x \{c_1 x_1 + c_2 x_2 : x_1 + 2x_2 \leqslant 8, x_1 \leqslant 4, x_2 \leqslant 3, x \geqslant 0\}$.

It is obvious that the set of cost vector C is not convex, then the above inverse optimal value problem could not be solved using the algorithm proposed in [1].

Now we solve problem (9) using the algorithm proposed in this paper.

Following problem (4), we can write the above inverse optimal value problem as the following BLP.

$$\begin{aligned} \min_{c \geqslant 0} \quad & (c_1 x_1 + c_2 x_2 - 14)^2, \\ \text{s.t.} \quad & 10 \leqslant c_1^2 + c_2^2 \leqslant 13, \\ & Q(c) = \max_{x \geqslant 0} \{c_1 x_1 + c_2 x_2 : x_1 + 2x_2 \leqslant 8, x_1 \leqslant 4, x_2 \leqslant 3\}. \end{aligned} \tag{8}$$

Step 0: Set $i = 0$, $k = 100$, $\lambda = 10$.

Step 1: Solve the following nonlinear programming problem:

$$\begin{aligned}
\min \quad & \left[(c_1x_1+c_2x_2-14)^2+100((8-x_1-2x_2)u_1+(4-x_1)u_2+(3-x_2)u_3+v_1x_1+v_2x_2)\right],\\
\text{s.t.} \quad & c_1^2+c_2^2\geqslant 10,\\
& c_1^2+c_2^2\leqslant 13,\\
& x_1+2x_2\leqslant 8,\\
& x_1\leqslant 4,\\
& x_2\leqslant 3,\\
& u_1+u_2-v_1-c_1=0,\\
& 2u_1+u_3-v_2-c_2=0,\\
& c,x,u,v\geqslant 0. \qquad (9)
\end{aligned}$$

The optimal solution of problem (11) is $c_{100}^0=(2,3)$, $x_{100}^0=(4,2)^{\mathrm{T}}$, $u_{100}^0=(1.5,0.5,0)$, $v_{100}^0=(0,0)$.

Step 2: As $u_{100}^0(b-Ax_{100}^0)+v_{100}^0x_{100}^0=0$, then the optimal solution of the problem (8) is $(c^*,x^*)=(2,3,4,2)$.

For this example, we get the optimal solution after one iteration. The main reason why such thing happens is that we choose an appropriate penalty k. If the set of cost vector C is convex, which is the assumption in [1], then we can get the optimal solution through solving a series of convex programming problems.

5. Conclusion

In this paper, we consider the inverse optimal value problem under more general conditions using BLP approach. For the corresponding nonlinear BLP problem, we replace the lower level problem with its Kuhn–Tucker optimality condition and append the complementary and slackness condition to the upper level objective with a penalty, then we give via an exact penalty method an existence theorem of solutions of the inverse optimal value problem. The numerical result shows that the algorithm proposed in this paper can solve a wilder class of inverse optimal value problem. It deserves pointing out that the algorithm needs only to solve a series of differential nonlinear programming problems, then it has attractive computation perspective. Moreover the algorithm in this paper can also be applied to a special class of nonlinear BLP problem.

Acknowledgements

The authors thank anonymous reviewers for valuable comments to improve this paper.

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