Noncommutative Fourier Transforms of Bounded Bilinear Forms and Completely Bounded Multilinear Operators

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Let \( G_1, \ldots, G_n \) be locally compact groups and \( H \) a Hilbert space. Any bounded \( n \)-linear operator \( \Phi: \mathbb{C}^*(G_1) \times \cdots \times \mathbb{C}^*(G_n) \to L(H) \) is known to have a unique separately \( \sigma \)-weakly continuous extension \( \tilde{\Phi}: W^*(G_1) \times \cdots \times W^*(G_n) \to L(H) \). The Fourier transform of \( \Phi \) is the function \( \tilde{\Phi}: G_1 \times \cdots \times G_n \to L(H) \) defined by \( \tilde{\Phi}(s_1, \ldots, s_n) = \tilde{\Phi}(\omega_1(s_1), \ldots, \omega_n(s_n)) \), where \( \omega_i: G_i \to W^*(G_i) \subset L(H_{\omega_i}) \) is the universal representation. In the case \( n = 2 \), \( L(H) = \mathbb{C} \), characterizations based on the Grothendieck-Pisier-Haagerup inequality are given for such Fourier transforms in terms of weakly harmonizable and hemihomogeneous random fields and continuous unitary Jordan representations. In the case of arbitrary \( n \), attention is confined to completely bounded \( n \)-linear \( L(H) \)-valued operators. Their Fourier transforms are characterized and a convolution operation for them is defined. In the case of \( L(H) = \mathbb{C} \), the completely bounded \( n \)-linear forms on \( \mathbb{C}^*(G_1) \times \cdots \times \mathbb{C}^*(G_n) \) form a commutative Banach algebra whose maximal ideal space contains the image of \( \Delta(B(G_1)) \times \cdots \times \Delta(B(G_n)) \) under a separately continuous injection with a continuous left inverse, where \( \Delta(B(G_i)) \) is the maximal ideal space of the Fourier-Stieltjes algebra of \( G_i \).

1. INTRODUCTION

Bimeasures and their Fourier transforms (resp. the noncommutative analogues of these notions) play a key role in the study of certain types of stochastic processes as Hilbert space valued functions on the real line; see, e.g., [1, 18, 21] (resp. on an arbitrary locally compact group [26, 30]).

An independent wave of interest in the Fourier analysis of bimeasures—in the setting of abelian and later also arbitrary locally compact groups—began with [10], and continued, e.g., in [9, 11]. It is fair to say that in most of that work the basic ingredient is the famous Grothendieck

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inequality or the "fundamental theorem in the metric theory of tensor products" of [12]. In particular, in [10] it is used in proving the following interesting characterization of the Fourier transforms (to be explained in Section 2) of bounded bilinear forms, i.e., bimeasures, on $C_0(\Gamma_i) \times C_0(\Gamma_2)$, where $C_0(\Gamma_i)$ for $i = 1, 2$ is the space of continuous complex functions vanishing at infinity, defined on the dual group $\Gamma_i$ of a locally compact abelian group $G_i$.

1.1. THEOREM [10, p. 101]. A function $F: G_1 \times G_2 \rightarrow \mathbb{C}$ is the Fourier transform of a bimeasure $B: C_0(\Gamma_1) \times C_0(\Gamma_2) \rightarrow \mathbb{C}$ if and only if there exist a Hilbert space $H$, two continuous unitary representations $\pi_i: G_i \rightarrow L(H)$, $i = 1, 2$, and two vectors $\xi_1, \xi_2 \in H$ such that

$$F(s, t) = (\pi_1(s)\xi_1 | \pi_2(t)\xi_2)$$

for all $s \in G_1$, $t \in G_2$.

The above formulation is by the Pontryagin duality theorem equivalent to that of [10], though we have exchanged the roles of $G_i$ and $\Gamma_i$ to be in harmony with the point of view of the present paper.

It is natural to ask if there is a generalization of Theorem 1.1, when $G_i$ and $G_2$ are not necessarily commutative. Of course, the dual object of $G_i$ is no longer a group $\Gamma_i$, but there is an analogue of $C_0(\Gamma_i)$, namely the group $C^*(G_i)$ of $G_i$. Fourier transforms of bounded bilinear forms on $C^*(G_1) \times C^*(G_2)$ were defined in [26] (and the notion referred to in Theorem 1.1 is a special case). It turns out that (1) again gives the desired characterization provided that now $\pi_i$ for $i = 1, 2$ is replaced by the direct sum of a continuous unitary representation and a continuous unitary antirepresentation (Theorem 4.3). As a by-product, Theorem 4.3 also contains characterizations of the Fourier transforms of bounded bilinear forms on $C^*(G_1) \times C^*(G_2)$ in terms of weakly harmonizable and hemihomogeneous random fields on $G_1$ and $G_2$ (notions to be explained in Section 3), thus providing a link with the work mentioned in the opening paragraph of this introduction.

In [10], Theorem 1.1 was used in defining the convolution of two bimeasures by noting that the representation given by Theorem 1.1 shows that the space of Fourier transforms of bimeasures is closed under pointwise multiplication. (In [9], a more direct approach to convolutions is used.) Unfortunately, the appearance of both representations and antirepresentations in our Theorem 4.3 rules out the direct and complete extension of this method of [10] to the noncommutative case. There is, however, a satisfactory generalization which we study in Section 5. There, Fourier transforms and convolutions are introduced for so-called completely bounded $n$-linear $L(H)$-valued operators for a Hilbert space $H$ (and,
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in particular, C-valued forms), defined on $C^*(G_1) \times \cdots \times C^*(G_n)$, where $G_1, \ldots, G_n$ are arbitrary locally compact groups. Completely bounded multilinear operators were defined and characterized in [2]; for further developments see, e.g., [3, 6, 19]. The so-called completely bounded norm $\|\Phi\|_{cb}$ of such a multilinear operator has a simple characterization which enables us to prove that $\|\Phi_1 \ast \Phi_2\|_{cb} \leq \|\Phi_1\|_{cb} \|\Phi_2\|_{cb}$ for the convolution $\Phi_1 \ast \Phi_2$ of two completely bounded $n$-linear operators $\Phi_1$ and $\Phi_2$. In the case of two commutative $C^*$-algebras, and thus for $C^*(G_1) \times C^*(G_2)$, where $G_1$ and $G_2$ are abelian, the completely bounded linear forms are—by virtue of the Grothendieck inequality, see [2, Corollary 5.6; 17, p. 180]—the bounded ones, and the completely bounded norm is equivalent to the usual supremum norm. Even there it may be argued that the completely bounded norm is the natural one to use, as, e.g., the constant $K_G$ is then not needed in [10, Theorem 2.6]. That the completely bounded $n$-linear forms are the natural generalization of bimeasures is further evidenced by Theorem 6.2, which contains an $n$-variable noncommutative extension of [9, Theorem 6.1] dealing with the maximal ideal space of the commutative Banach algebra of bimeasures under convolution.

In Section 7 we return briefly to random fields. Two new classes of them, the strongly harmonizable and the completely bounded ones, are introduced. Some results of Section 5 are used in characterizing the latter: they are shown to have the same type of dilation relationship to continuous right homogeneous random fields as weakly harmonizable random fields are known [29] to have to the hemihomogeneous ones.

2. PRELIMINARIES

All vector spaces are over the field of complex numbers. For a Banach space $E$, we write $f(x) = \langle x, f \rangle$ when $x \in E$ and $f$ is in the (topological) dual $E^*$ of $E$. The inner product of any Hilbert space $H$ is denoted by $(\cdot, \cdot)$, or sometimes by $(\cdot, \cdot)_H$. If $K$ is a closed subspace of $H$, $P_K$ is the orthogonal projection onto $K$. For two Hilbert spaces $H_1$ and $H_2$, $L(H_1, H_2)$ is the space of bounded linear operators from $H_1$ to $H_2$, and we write $L(H_1, H_2) = L(H)$. The basic theory of $C^*$-algebras, von Neumann algebras, and group representations as expounded, e.g., in [4, 23] will be used freely. If $A$ is a $C^*$-algebra and $H$ is a Hilbert space, a linear map $\pi: A \to L(H)$ is said to be a representation (resp. antirepresentation) if $\pi(x^*) = \pi(x)^*$, and $\pi(xy) = \pi(x) \pi(y)$ (resp. $\pi(xy) = \pi(y) \pi(x)$) for all $x, y \in A$. The words cyclic and nondegenerate will have the same meanings in connection with antirepresentations as with representations (see, e.g., [23, pp. 36, 41]). Let $G$ be an arbitrary locally compact group. We build on the basic
structure of harmonic analysis expounded in [8] and also used, e.g., in [26, 30]. In particular, $C^*(G)$ is the group $C^*$-algebra of $G$, and it contains $L^1(G)$ (integration being with respect to fixed left Haar measure $ds$) as a dense $*$-subalgebra. For a Hilbert space $H$, there is a natural bijective correspondence (by integration) between the continuous unitary representations of $G$ on $H$ and the nondegenerate representations of the involutive Banach algebra $L^1(G)$ on $H$ [4, p. 253], and the latter correspond bijectively (by restriction) to the nondegenerate representations of $C^*(G)$ on $H$ [4, p. 271]. We usually identify a representation with its image under these bijections, and use the same notation for all three corresponding representations. We let $\omega: C^*(G) \to L(H_\omega)$ denote the universal representation, and denote by $W^*(G)$ the von Neumann algebra generated in $L(H_\omega)$ by $\omega(C^*(G))$. Then $\omega(G) \subset W^*(G)$ [8, p. 193]. As usual, $W^*(G)$ is identified with $C^*(G)^{**}$, and $C^*(G)$ is regarded as a subspace of $W^*(G)$, and if $\pi: C^*(G) \to L(H)$ is a representation, $\tilde{\pi}: W^*(G) \to L(H)$ will denote its unique weak*-to-$\sigma$-weak continuous extension [4, Sect. 12; 23, Sect. III.2]. In the case of more than one locally compact group $G_1, \dots, G_n$, $\omega_i$ for $i = 1, \dots, n$ has the obvious meaning.

Now let $G_1$ and $G_2$ be locally compact groups and $B: C^*(G_1) \times C^*(G_2) \to \mathbb{C}$ a bounded bilinear form. Then $B$ has a unique separately weak* continuous extension $\tilde{B}: W^*(G_1) \times W^*(G_2) \to \mathbb{C}$ [16, pp. 75–77; 26, pp. 365–366]. The Fourier transform $\tilde{B}$ of $B$ was defined in [26] to be the function $(s, t) \mapsto \tilde{B}(s, t) = \tilde{B}(\omega_1(s), \omega_2(t))$ on $G_1 \times G_2$. The Fourier transform $\tilde{B}$ is jointly continuous and determines $B$ uniquely [26, p. 366].

Let us assume for a moment that $G_1$ and $G_2$ are abelian, and denote their dual groups by $\Gamma_1$ and $\Gamma_2$, respectively. The Fourier transformation $f \mapsto \hat{f}$,

$$\hat{f}(\gamma) = \int_{G_i} \overline{\gamma(s)} f(s) \, ds,$$

on $L^1(G_i)$ extends uniquely to an isometric $*$-isomorphism $\alpha_i: C^*(G_i) \to C_0(\Gamma_i)$, and so any bounded bilinear form (called here briefly a bimeasure) $B: C_0(\Gamma_1) \times C_0(\Gamma_2) \to \mathbb{C}$ can be assigned a Fourier transform $\hat{B}: G_1 \times G_2 \to \mathbb{C}$ defined as $\hat{B}_{\alpha_1, \alpha_2}$, where

$$B_{\alpha_1, \alpha_2}(x, y) = B(\alpha_1(x), \alpha_2(y))$$

for $x \in C^*(G_1)$, $y \in C^*(G_2)$. In [26, Sect. 5] it was shown that for $s \in G_1$, $t \in G_2$, $\hat{B}(s, t)$ may be obtained by integrating the pair of characters $(s, \cdot)$ (on $\Gamma_1$) and $(t, \cdot)$ (on $\Gamma_2$) with respect to $B$ in a functional sense. (For an equivalent ensemble point of view on integrating with respect to $B$, see [27].) It is easy to show (using, e.g., [9, Lemma 1.3]) that $\hat{B}$ as defined here coincides with the Fourier transform of $B$ defined in [10, p. 97].
3. CONTINUOUS UNITARY JORDAN REPRESENTATIONS AND RANDOM FIELDS

In this section $G$ is a locally compact group and $H$ is a Hilbert space. Recall that for a $C^*$-algebra $A$, a linear map $\pi: A \to L(H)$ is a Jordan morphism if and only if there is a closed subspace $H_1 \subset H$ with orthogonal complement $H_2$ such that, writing $H = H_1 \oplus H_2$, for some representation $\pi_1: A \to L(H_1)$ and some antirepresentation $\pi_2: A \to L(H_2)$ we have $\pi = \pi_1 \oplus \pi_2$ [22]. We could take this decomposition as the definition of a Jordan morphism; the term is only used for euphony, and we always work with such decompositions. By analogy we state:

3.1. DEFINITION. A map $\pi: G \to L(H)$ is called a continuous unitary Jordan representation if there is a closed subspace $H_1 \subset H$ with orthogonal complement $H_2$ such that writing $H = H_1 \oplus H_2$ we have for some continuous unitary representation $\pi_1: G \to L(H_1)$ and some continuous unitary antirepresentation $\pi_2: G \to L(H_2)$ (i.e., $\pi_2(st) = \pi_2(t) \pi_2(s)$),

$$\pi(s) = \pi_1(s) \oplus \pi_2(s) \quad \text{for all } s \in G.$$

The term nondegenerate will have the same meaning for Jordan morphisms as for representations.

3.2. LEMMA. (a) In the situation described before Definition 3.1, $\pi$ is nondegenerate if and only if $\pi_1$ and $\pi_2$ are so.

(b) If $\pi: C^*(G) \to L(H)$ is an antirepresentation (resp. a Jordan morphism), then $\pi$ has a unique weak*-to-$\sigma$-weak continuous extension $\tilde{\pi}: W^*(G) \to L(H)$. This $\tilde{\pi}$ is an antirepresentation (resp. a Jordan morphism), and if $\pi$ is nondegenerate, the mapping $s \mapsto \tilde{\pi}(\omega(s))$ is a continuous unitary antirepresentation (resp. a continuous unitary Jordan representation) of $G$.

Proof. We omit the easy proof of (a). One way of reducing the antirepresentation part of (b) to the corresponding standard statement about representations is the following. Let $J: H \to H$ be a conjugation of $H$, i.e., a conjugate-linear isometry with $J^* = J^{-1} = J$. The map $\alpha: L(H) \to L(H)$ defined by $\alpha(T) = JT^*J$ is a bijective antirepresentation of $L(H)$, $\alpha = \alpha^{-1}$, and $\alpha$ is $\sigma$-weakly bicontinuous. Composing $\alpha$ with the appropriate antirepresentation $\pi$ and again with $(\alpha \circ \pi)^-$ effects the desired reduction. On the other hand, if

$$\pi: C^*(G) \to L(H) = L(H_1 \oplus H_2)$$

is the direct sum of a representation $\pi_1: C^*(G) \to L(H_1)$ and an
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antirepresentation $\pi_2: \mathcal{C}^*(G) \rightarrow L(H_2)$ we can apply the preceding remarks to $\pi_1$ and $\pi_2$ and (using (a)) easily obtain the Jordan morphism part of (b). (Compare also the proof of [22, Lemma 3.1].)

Before turning to random fields and a characterization of a class of them (Proposition 3.4), we prove another lemma. (We give the short proof for completeness, though a closely related result is in [28, p. 133].)

3.3. Lemma. Let $A$ be a $\mathbb{C}^*$-algebra, $K$ a Hilbert space, $f_1, f_2: A \rightarrow \mathbb{C}$ two positive linear forms, and $\Psi: A \rightarrow K$ a linear map such that

$$\langle \Psi x, \Psi y \rangle = f_1(y^*x) + f_2(xy^*)$$

(2)

for all $x, y \in A$. Suppose, moreover, that $\Psi(A)$ is dense in $K$. Then there exist a cyclic representation $\{\pi', K', \xi'\}$ of $A$, a cyclic antirepresentation $\{\pi'', K'', \xi''\}$ of $A$, and an isometric linear map $V: K \rightarrow K' \oplus K''$ such that

$$V\Psi x = (\pi'(x) \xi', \pi''(x) \xi'')$$

for all $x \in A$.

Proof. The Gelfand–Naimark–Segal construction yields a cyclic representation $\{\pi', K', \xi'\}$ of $A$ such that $f_1(x) = (\pi'(x) \xi' | \xi')$, $x \in A$. Similarly, applying the GNS construction to the reversed $\mathbb{C}^*$-algebra we obtain a cyclic antirepresentation $\{\pi'', K'', \xi''\}$ of $A$ such that $f_2(x) = (\pi''(x) \xi'' | \xi'')$ for all $x \in A$. For any vector of the form $\Psi x$ in $K$ we make the assignment

$$V(\Psi x) = (\pi'(x) \xi', \pi''(x) \xi'').$$

(3)

From (2) it follows that

$$\langle \Psi x, \Psi y \rangle = ((\pi'(x) \xi', \pi''(x) \xi'') | (\pi'(y) \xi', \pi''(y) \xi''))_{K' \oplus K''},$$

$x, y \in A$, and so

$$\|\Psi x - \Psi y\|^2 = \|\pi'(x) \xi', \pi''(x) \xi''\| - (\pi'(y) \xi', \pi''(y) \xi'')\|$$

for all $x, y \in A$. Thus the definition of $V$ by (3) is independent of the representation of $\Psi x$ in this form, and $V$ is isometric and obviously linear. Finally, extend $V$ by continuity to all of $K$.

Any function $\varphi: G \rightarrow H$ will be called a random field (see, e.g., [30] for motivation). A random field $\varphi: G \rightarrow H$ is said to be weakly harmonizable, if $\varphi$ is the Fourier transform of some bounded linear map $\Phi: \mathcal{C}^*(G) \rightarrow H$ in the sense that $\varphi(s) = \Phi^{**}(\omega(s))$ for all $s \in G$, where $\Phi^{**}: \mathcal{W}^*(G) \rightarrow H^{**} = H$ is the bitranspose of $\Phi$; we then write $\varphi = \hat{\Phi}$. (See [26] for basic
properties and characterizations of such Fourier transforms; in particular, \( \Phi \) is bounded, continuous, and it determines \( \Phi \) uniquely.) A random field \( \varphi: G \to H \) is called hemihomogeneous, if there are two continuous positive-definite functions \( \rho_1, \rho_2: G \to \mathbb{C} \) such that

\[
(\varphi(s) | \varphi(t)) = \rho_1(t^{-1}s) + \rho_2(st^{-1})
\]

(4)

for all \( s, t \in G \). This term was coined and the significance of the notion was discussed in [30]. For example, every weakly harmonizable random field has a hemihomogeneous dilation [29].

Only the implication \( (ii) \Rightarrow (i) \) in the following result will be used in the sequel, but the equivalence \( (i) \Leftrightarrow (ii) \) has some independent interest.

3.4. PROPOSITION. For a random field \( \varphi: G \to H \) the following two conditions are equivalent:

(i) \( \varphi \) is hemihomogeneous;

(ii) there is a Hilbert space \( K \) containing \( H \) as a subspace such that for some continuous unitary Jordan representation \( \pi: G \to L(K) \) and \( \xi \in H \), \( \varphi(s) = \pi(s)\xi \) whenever \( s \in G \).

Proof. \( (i) \Rightarrow (ii) \) Let \( \rho_1, \rho_2: G \to \mathbb{C} \) be continuous positive-definite functions such that (4) holds. From [29, Lemma 2] combined with [26, Remark 6.3 ], it follows that \( \varphi = \Phi \) for some bounded linear map \( \Phi: C^*(G) \to H \). Let \( f_i \in C^*(G) \) be the positive linear form corresponding to \( \rho_i \) (see [4, pp. 257 and 421]); for all \( s \in G \)

\[
\langle f_i, \omega(s) \rangle = \rho_i(s)
\]

(5)

(see [8, p. 194]). Now

\[
(\Phi x | \Phi y^*) = \langle xy, f_1 \rangle + \langle xy, f_2 \rangle
\]

for all \( x, y \in C^*(G) \), as can easily be seen by observing that each side as a function on \( C^*(G) \times C^*(G) \) has the same Fourier transform \( (s, t) \mapsto (\varphi(s) | \varphi(t^{-1})) = \rho_1(ts) + \rho_2(st) \) (see [26, p. 379] and (5)). Write \( H = H_0 \oplus H_1 \), where \( H_0 \) is the closure of \( \Phi(C^*(G)) \) and \( H_1 = H^* \). We apply Lemma 3.3 to \( \Phi \) regarded as a map into \( H_0 \), and so we get a (by Lemma 3.2(a)) nondegenerate Jordan morphism \( \pi: C^*(G) \to L(K_0) \), where \( K_0 \to H_0 \) is a Hilbert space, and for some \( \xi_0 \in K_0 \), \( \Phi(x) = \pi_0(x) \xi_0 \), \( x \in C^*(G) \). Denote also the corresponding continuous unitary Jordan representation \( s \mapsto \tilde{\pi}_0(\omega(s)) \) by \( \pi_0: G \to L(K_0) \) (see Lemma 3.2(b)). To take all of \( H \) into account we define \( K = K_0 \oplus H_1 \), \( \pi_1(s) = I_{H_1} \) for \( s \in G \), and \( \xi_1 = 0 \). Then \( \pi = \pi_0 \oplus \pi_1 \) and \( \xi = (\xi_0, \xi_1) \) satisfy the requirements in (ii). (Note, e.g., that \( \Phi^{**}(u) = \tilde{\pi}_0(u) \xi_0 \) for all \( u \in W^*(G) \), in particular for
u = \omega(s), since both sides are continuous from \sigma(W^*(G), C^*(G)^*) to \sigma(K_0, K_0^*) and agree on C^*(G). Moreover, \xi = \pi(e)\xi = \varphi(e)\in H, where e is the identity of G.)

(ii) \Rightarrow (i) Suppose that \( K = K_1 \oplus K_2, \quad \pi_1: G \to L(K_1) \) is a continuous unitary representation, \( \pi_2: G \to L(K_2) \) is a continuous unitary antirepresentation, and \( \xi = (\xi_1, \xi_2) \in K \) is such that \( \varphi(s) = (\pi_1(s)\xi_1, \pi_2(s)\xi_2) \) for all \( s \in G \). Then for all \( s, t \in G \)

\[
(\varphi(s) | \varphi(t)) = (\pi_1(s)\xi_1 | \pi_1(t)\xi_1) + (\pi_2(s)\xi_2 | \pi_2(t)\xi_2) = \rho_1(t^{-1}s) + \rho_2(st^{-1}),
\]

where \( \rho_i(s) = (\pi_i(s)\xi_i | \xi_i) \) for \( i = 1, 2, \quad s \in G \). Since both \( \rho_1 \) and \( \rho_2 \) are continuous positive-definite functions, \( \varphi \) is hemihomogeneous.

4. Characterizations of the Fourier Transforms of Bounded Bilinear Forms on \( C^*(G_1) \times C^*(G_2) \)

The title of this section refers to Theorem 4.3. We begin, however, with a general inductive dilation argument which will find an application in Section 5 as well.

4.1. Proposition. Let \( H_0, H_1, ..., H_n+1 \) be Hilbert spaces and \( A_1, ..., A_n \) C*-algebras. Suppose that \( V_i \in L(H_{i+1}, H_i) \) for \( i = 0, ..., n \), and \( \| V_i \| \leq 1 \) for \( i = 1, ..., n - 1 \). Let \( \theta_i: A_i \to L(H_i) \) be a Jordan morphism for \( i = 1, ..., n \).

(a) There exist a Hilbert space \( K \), two operators \( R \in L(K, H_0) \) and \( S \in L(H_{n+1}, K) \), and for each \( i = 1, ..., n \) a Jordan morphism \( \pi_i: A_i \to L(K) \) such that \( \| R \| = \| V_0 \|, \quad \| S \| = \| V_n \| \), and

\[
V_0 \theta_1(x_1) V_1 \theta_2(x_2) V_2 \cdots \theta_n(x_n) V_n = R \left( \prod_{i=1}^{n} \pi_i(x_i) \right) S \quad (6)
\]

for all \( x_i \in A_i, \quad i = 1, ..., n \).

(b) If each \( \theta \) is a representation, then each \( \pi_i \) in (a) may be taken to be a representation.

(c) If each \( A_i \) has a nonzero one-dimensional representation, then in both (a) and (b) each \( \pi_i \) may be taken to be nondegenerate, provided that the norm conditions \( \| R \| = \| V_0 \| \) and \( \| S \| = \| V_n \| \) are relaxed to \( \| R \| \leq \| V_0 \| \) and \( \| S \| \leq \| V_n \| \).

Proof. The proof proceeds by induction on \( n \). For \( n = 1 \), (a) and (b) are trivial. To prove (c) in the case \( n = 1 \), express \( \theta_1 \) as a direct sum \( \theta_1 = \theta_1' \oplus \theta_1'' \), where \( \theta_1' \) is a representation and \( \theta_1'' \) an antirepresentation. Then \( \theta_1' \) (resp. \( \theta_1'' \)) defines by restriction to its essential space \( K' \) (resp. \( K'' \)) a nondegenerate representation \( \pi_1' \) (resp. antirepresentation \( \pi_1'' \)). Choosing
\( K = K' \oplus K'' \), \( \pi = \pi_1' \oplus \pi_i'' \), \( R = V_0|K \), and \( S = P_K V_1 \) we have \( V_0 \theta_1(x) V_1 = R \pi_1(x) S \), and \( \pi_1 \) is nondegenerate (Lemma 3.2(a)). If \( \theta_1 \) is a representation, then so is \( \pi_1 \).

Suppose now that the Proposition is true for \( n - 1 \). Then the left-hand side of (6) is equal to

\[
V_0 \theta_1(x_1) R' \left( \prod_{i=2}^{n} \pi_i(x_i) \right) S',
\]

where for some Hilbert space \( K' \), \( R' \in L(K', H_1) \), \( \|R'\| \leq 1 \), \( S' \in L(H_{n+1}, K') \), \( \|S'\| = \|V_n\| \), and each \( \pi_i' \colon A_i \to L(K') \) for \( i = 2, \ldots, n \) is a Jordan morphism (a representation in case (b)). Moreover, in (c) each \( \pi_i' \) may be taken to be nondegenerate, though then only \( \|S'\| < \|V_n\| \).

There is a unitary operator \( U \in L(H_1 \oplus K') \) such that \( R' \xi = P_{H_1} U(0, \xi) \) for all \( \xi \in K' \). (The proof of the Halmos dilation theorem in [14, pp. 126–127] or [15, pp. 177–178] also works in the case of a linear contraction from a Hilbert space to a different Hilbert space. That technique was used in [3], and this observation somewhat streamlines my original proof of this part, more akin to an argument in [10, p. 101]. I am indebted to the authors of [3] for a preliminary version of the paper.) We denote \( K = H_1 \oplus K' \). To define \( \pi_i : A_i \to L(K) \) for \( i = 1, \ldots, n \) we choose auxiliary representations \( \alpha_i : L(K') \) and \( \alpha_i : A_i \to L(H_1) \) for \( i = 2, \ldots, n \). In (a) and (b) we may take all \( \alpha_i = 0 \), but in (c) we choose \( \alpha_i(x_i) = f_i(x_i) I_{K'} \) and \( \pi_i(x_i) = \alpha_i(x_i) \oplus \pi_i'(x_i) \) for \( i = 2, \ldots, n \), where \( f_i \) is a nonzero \( \mathbb{C} \)-valued representation of \( A_i \), \( i = 1, \ldots, n \). Denote \( \pi_1(x_1) = U^*(\theta_1(x_1) \oplus \alpha_1(x_1)) U \), and \( \pi_i(x_i) = \alpha_i(x_i) \oplus \pi_i'(x_i) \) for \( i = 2, \ldots, n \). (Note that for (c), \( \theta_1 \) may be assumed to be nondegenerate by applying, if necessary, the technique used at the beginning of this proof. Thus in any case the \( \pi_i \) are the type of morphisms we are looking for.) Finally, define \( R = V_0 P_{H_1} U \in L(K, H_0) \) and \( S\xi = (0, S'\xi) \) for \( \xi \in H_{n+1} \). Then \( \|S\| = \|S'\| \) and \( \|R\| = \|V_0\| \) (except that in (c), \( \|R\| \) may be less than the norm of the original \( V_0 \), see the preceding parenthetical note). Moreover, for all \( x_i \in A_i \), \( i = 1, \ldots, n \), and \( \xi \in H_{n+1} \),

\[
R \pi_1(x_1) \left( \prod_{i=2}^{n} \pi_i(x_i) \right) S\xi
= V_0 P_{H_1} U U^*(\theta_1(x_1) \oplus \alpha_1(x_1)) U \prod_{i=2}^{n} (\alpha_i(x_i) \oplus \pi_i'(x_i))(0, S'\xi)
= V_0 P_{H_1}(\theta_1(x_1) \oplus \alpha_1(x_1)) P_{H_1} U \left( 0 \oplus \prod_{i=2}^{n} \pi_i'(x_i) S'\xi \right)
= V_0 \theta_1(x_1) R' \prod_{i=2}^{n} \pi_i(x_i) S'\xi
= V_0 \theta_1(x_1) V_1 \theta_2(x_2) V_2 \cdots \theta_n(x_n) V_n. \]
Part (a) of the following lemma was already noted in [31, Remark 2.2], but we give a unified proof, which also yields (b).

4.2. **Lemma.** Let $A_1$ and $A_2$ be $C^*$-algebras and $B: A_1 \times A_2 \to \mathbb{C}$ a bounded bilinear form.

(a) There exist a Hilbert space $H$, two Jordan morphisms $\pi_i: A_i \to L(H)$, $i = 1, 2$, and two vectors $\xi, \eta \in H$ such that

$$B(x, y) = (\pi_1(x) \pi_2(y) \xi | \eta)$$

for all $x \in A_1$, $y \in A_2$.

(b) If $A_i = C^*(G_i)$, $i = 1, 2$, where $G_1$ and $G_2$ are locally compact groups, then $\pi_1$ and $\pi_2$ in (a) can be taken to be nondegenerate.

**Proof.** There exist two Hilbert spaces $H_1$ and $H_2$ with vectors $\xi, \eta \in H_i$ and Jordan morphisms $\pi_i': A_i \to L(H_i)$ such that for some linear contraction $V: H_2 \to H_1$

$$B(x, y) = (\pi_1'(x) V \pi_2'(y) \xi | \xi_2), \quad x \in A_1, \quad y \in A_2.$$ 

This is a consequence of the Grothendieck [12]–Pisier [20]–Haagerup [13] inequality [13, Theorem 1.1] as was (for $A_1 = A_2$) observed in [17, Remark 5.3(a)]. A somewhat different derivation (with the roles of $A_1$ and $A_2$ exchanged) is in [31]. Thus (a) follows from Proposition 4.1(a), and (b) from Proposition 4.1(c). Indeed, the condition in Proposition 4.1(c) is satisfied for $A_i = C^*(G_i)$, as the positive linear form on $C^*(G_i)$ corresponding to the constant function 1 on $G_i$ (see [4, pp. 257 and 42]) is a nonzero one-dimensional representation.

4.3. **Theorem.** Let $G_1$ and $G_2$ be locally compact groups. The following eight conditions are equivalent:

(i) (resp. (ii)): $F$ (resp. $\tilde{F}$) is the Fourier transform of some bounded bilinear form on $C^*(G_1) \times C^*(G_2)$.

(iii) (resp. (iv)): There is a Hilbert space $H$ with two continuous unitary Jordan representations $\pi_i: G_i \to L(H)$ and two vectors $\xi, \eta \in H$ such that

$$F(s, t) = (\pi_1(s) \pi_2(t) \xi | \eta)$$

(resp. $F(s, t) = (\pi_2(t) \pi_1(s) \xi | \eta)$) for all $s \in G_1$, $t \in G_2$.

(v) (resp. (vi)): There is a Hilbert space $H$ with two hemi-homogeneous random fields $\varphi_i: G_i \to H$, $i = 1, 2$, such that $F(s, t) = (\varphi_1(s) | \varphi_2(t))$ (resp. $F(s, t) = (\varphi_2(t) | \varphi_1(s))$) for all $s \in G_1$, $t \in G_2$. 
(vii) (resp. (viii)): The same as (v) (resp. (vi)), but with “hemi-
homogeneous” replaced by “weakly harmonizable.”

Proof. (i) ⇒ (iii) Assume that $F = \hat{B}$ for some bounded bilinear form $B: C^*(G_1) \times C^*(G_2) \to \mathbb{C}$. Applying Lemma 4.2 we find a Hilbert space $H$, two nondegenerate Jordan morphisms $\pi_i: C^*(G_i) \to L(H)$, and two vectors $\xi, \eta \in H$ such that

$$B(x, y) = (\pi_1(x) \pi_2(y) \xi | \eta)$$

for all $x \in C^*(G_1), y \in C^*(G_2)$. Let $\tilde{\pi}_i: W^*(G_i) \to L(H)$ be the weak*-to-$\sigma$-weak continuous extension of $\pi_i$, so that the mapping $\sigma \mapsto \tilde{\pi}_i(\sigma)$ on $G_i$—we also denote it by $\pi_i$—is a continuous unitary Jordan representation (Lemma 3.2(b)) of $G_i$. By the uniqueness of the separately weak* continuous extension $\tilde{B}$ of $B$ we have $\tilde{B}(u, v) = (\tilde{\pi}_1(u) \tilde{\pi}_2(v) \xi | \eta)$ for all $u \in W^*(G_1), v \in W^*(G_2)$; in particular

$$\tilde{B}(s, t) = (\pi_1(s) \pi_2(t) \xi | \eta), \quad s \in G_1, t \in G_2.$$

(i) ⇒ (iv) Assume that $F = \hat{B}$ as above. Define $B_0: C^*(G_2) \times C^*(G_1) \to \mathbb{C}$ by $B_0(y, x) = B(x, y)$. This time Lemma 4.2 yields a Hilbert space $H$ with vectors $\xi, \eta \in H$ and two nondegenerate Jordan morphisms $\pi'_2: C^*(G_2) \to L(H)$ and $\pi'_1: C^*(G_1) \to L(H)$ such that

$$B(x, y) = B_0(y, x) = (\pi'_2(y) \pi'_1(x) \xi | \eta)$$

for all $x \in C^*(G_1), y \in C^*(G_2)$. Now proceed as above.

(iii) ⇒ (vii), (iv) ⇒ (v) See Proposition 3.4. (Note that if $\pi_i$ is a continuous unitary Jordan representation, then so is $s \mapsto \pi_i(s^{-1})$.)

(v) ⇒ (vii), (vi) ⇒ (viii) See [29, Lemma 2; 26, Remark 6.3].

(vii) ⇒ (i) Suppose $F(s, t) = (\Phi_1(s) | \Phi_2(t))$, where $\Phi_i: C^*(G_i) \to H$ is a bounded linear operator. Then $(s, t) \mapsto F(s, t^{-1})$ is the Fourier transform of the bounded bilinear form $(x, y) \mapsto (\Phi_1(x) | \Phi_2(y^*))$ on $C^*(G_1) \times C^*(G_2)$ (modify the proof of [26, Theorem 6.5]), and so $F$ is also the Fourier transform of some bounded bilinear form by [26, Corollary 4.8].

(viii) ⇒ (i) Suppose $F(s, t) = (\Phi_2(t) | \Phi_1(s))$ for $\Phi_i$ as above. This time $(s, t) \mapsto F(s^{-1}, t)$ is the Fourier transform of the bounded bilinear form $(x, y) \mapsto (\Phi_2(y) | \Phi_1(x^*))$, and [26, Corollary 4.8] again applies.

(i) ⇐ (ii) This equivalence is built into the above chain of implications, since, e.g., the fact that (v) ⇐ (i) ⇐ (vi) shows that the space of Fourier transforms of bounded bilinear forms on $C^*(G_1) \times C^*(G_2)$ is closed under complex conjugation. \[\square\]
4.4. Remark. (a) There is also a direct proof of (i) \(\Longleftrightarrow\) (ii) (which could be used to restructure the above proof) based on [26, Theorem 4.7]. Indeed, in view of that theorem it suffices to show that (in its notation)

\[
\left\| \sum_{i=1}^{n} z_i \delta_{s_i} \right\|' = \left\| \sum_{i=1}^{n} \bar{z}_i \delta_{s_i} \right\|'
\]

for linear combinations of Dirac measures. But (7) follows, e.g., from the proof of [26, Corollary 3.6], since \(\left\| \sum_{i=1}^{n} z_i \delta_{s_i} \right\|' = \left\| \sum_{i=1}^{n} \bar{z}_i \delta_{s_i} \right\|'\).

(b) The equivalence of (i) and (ii) is especially to be noted, as its analogue is conspicuous by its absence in the next section.

5. FOURIER TRANSFORMS AND CONVOLUTIONS OF COMPLETELY BOUNDED MULTILINEAR OPERATORS

One motive for the research reported in this paper was the quest for a convolution of two bounded bilinear forms on \(C^*(G_1) \times C^*(G_2)\) for arbitrary locally compact groups \(G_1\) and \(G_2\). A natural approach would be to declare the convolution of two such bilinear forms \(B_1\) and \(B_2\) to be the bounded bilinear form \(B\) on \(C^*(G_1) \times C^*(G_2)\) having as its Fourier transform the pointwise product of those of \(B_1\) and \(B_2\), if only such a \(B\) could be shown to exist. Contrary to the abelian situation treated in [10], the characterization given in Theorem 4.3 does not, however, seem to be of much help, since both representations and antirepresentations of the groups are involved. In this section we restrict our attention to so-called completely bounded bilinear forms (and their generalizations completely bounded multilinear operators). While falling short of the original goal, we do get a smoothly functioning noncommutative \(n\)-variable extension of the abelian case of two variables.

The notion of a completely bounded multilinear operator and its completely bounded norm were introduced in [2]. We do not repeat those definitions here. Instead, we take an equivalent formulation [2, Theorem 5.2] as the starting point: For \(C^*\)-algebras \(A_1, ..., A_n\) and a Hilbert space \(H\), an \(n\)-linear operator \(\Phi: A_1 \times \cdots \times A_n \to L(H)\) is completely bounded if, and only if, there are Hilbert spaces \(H_0, ..., H_n\), operators \(V_i \in L(H_{i+1}, H_i)\) for \(i = 0, ..., n\), where \(H_0 = H = H_{n+1}\), and representations \(\theta_i: A_i \to L(H_i)\) for \(i = 1, ..., n\), such that

\[
\Phi(x_1, ..., x_n) = V_0 \theta_1(x_1) V_1 \theta_2(x_2) V_2 \cdots \theta_n(x_n) V_n
\]

for \(x_i \in A_i, i = 1, ..., n\). Moreover, the completely bounded norm \(\|\Phi\|_{cb}\) of such a \(\Phi\) equals the infimum of \(\|V_0\| \cdot \|V_1\| \cdots \|V_n\|\) over all representations of \(\Phi\) in the form (8), and in fact this infimum is attained.
As a consequence of (8), by the universal property of the enveloping von Neumann algebras $A_i^{**}$ [4, p. 237], $\Phi$ has a (clearly unique) separately weak*-to-$\sigma$-weak continuous $n$-linear extension $\tilde{\Phi}: A_1^{**} \times \cdots \times A_n^{**} \to L(H)$ (which is also completely bounded [2, Corollary 5.5]).

5.1. Definition. We call the $\tilde{\Phi}$ described above the canonical extension of $\Phi$. If $G_1, \ldots, G_n$ are locally compact groups, and $\Phi: C^*(G_1) \times \cdots \times C^*(G_n) \to L(H)$ for a Hilbert space $H$ is a completely bounded $n$-linear operator, the Fourier transform of $\Phi$ is the function $\hat{\Phi}: G_1 \times \cdots \times G_n \to L(H)$ defined by the formula

$$\hat{\Phi}(s_1, \ldots, s_n) = \tilde{\Phi}(\omega_1(s_1), \ldots, \omega_n(s_n)).$$

5.2. Remark. (a) Since we are only interested in the Fourier transforms of completely bounded multilinear operators in this section, we used the above simple construction of the Fourier transform. It is worth noting, however, that using [16, Theorem 2.3] in the universal representations of $C^*(G_1), \ldots, C^*(G_n)$, we see that an arbitrary bounded $n$-linear operator from $C^*(G_1) \times \cdots \times C^*(G_n)$ to $L(H)$ (or more generally to the dual space of any complex Banach space) has a similar canonical extension and hence the Fourier transform (generalizing the notion studied in Section 4).

(b) Since each $\{\omega_i(s_i) | s_i \in G_i\}$ spans a weak* dense subspace of $W^*(G_i)$, it is clear that $\hat{\Phi}$ determines $\Phi$ uniquely.

Actually every completely bounded $n$-linear operator has even a representation of the form (8) where $H_1 = \cdots = H_n$, and each of the bridging maps $V_1, \ldots, V_{n-1}$ is the identity operator ([3, 5, 32] and Proposition 4.1(b)). The proof in [32] uses an inductive argument based on the Halmos dilation theorem for a contractive endomorphism, generalizing an approach in [10, p. 101; 31]. The present version of the proof of Proposition 4.1 was influenced by the somewhat more economical inductive dilation proof in [3], the original form having been an elaboration of [32]. Paulsen and Smith [19] have proved a representation theorem for completely contractive multilinear operators on the product of subspaces of $C^*$-algebras; cf. [19, Theorem 3.2]. It is clear from the proofs referred to above that there, too, as well as in [19, Theorem 2.9 and Corollary 2.10], one can get rid of the appropriate bridging maps. We now apply the nondegeneracy part of Proposition 4.1 to represent the Fourier transforms of completely bounded multilinear operators.

5.3. Theorem. Let $G_1, \ldots, G_n$ be locally compact groups and $H$ a Hilbert space. A function $F: G_1 \times \cdots \times G_n \to L(H)$ is the Fourier transform of some completely bounded $n$-linear operator $\Phi: C^*(G_1) \times \cdots \times C^*(G_n) \to L(H)$ if,
and only if, there exist a Hilbert space $K$, two operators $R \in L(K, H)$, $S \in L(H, K)$, and for each $i = 1, \ldots, n$ a continuous unitary representation $\pi_i : G_i \to L(K)$ such that

$$F(s_1, \ldots, s_n) = R \left( \prod_{i=1}^{n} \pi_i(s_i) \right) S$$

(9)

for all $s_i \in G_i$, $i = 1, \ldots, n$. If this is the case, then $\Phi$ satisfies

$$\Phi(x_1, \ldots, x_n) = R \left( \prod_{i=1}^{n} \pi_i(x_i) \right) S$$

(10)

for all $x_i \in C^*(G_i)$, $i = 1, \ldots, n$. Moreover, $\|\Phi\|_{cb}$ equals the infimum of $\|R\|\|S\|$ over all representations in the form (9), and the infimum is attained.

Proof. (1°) Let first $\Phi : C^*(G_1) \times \cdots \times C^*(G_n) \to L(H)$ be a completely bounded $n$-linear operator represented in the form (8) (with $A_i = C^*(G_i)$), where $\|\Phi\|_{cb} = \prod_{i=0}^{n} \|V_i\|$. We may exclude the trivial case $\Phi = 0$ and assume that

$$\|V_1\| = \cdots = \|V_{k-1}\| = 1, \quad \|V_0\| \|V_n\| = \|\Phi\|_{cb}.$$  

Since each $C^*(G_i)$ satisfies the condition in Proposition 4.1(c) (see the proof of Lemma 4.2), we can find a Hilbert space $K$, nondegenerate representations $\pi_i : C^*(G_i) \to L(K)$, $i = 1, \ldots, n$, and operators $R \in L(K, H)$, $S \in L(H, K)$ such that $\|R\| \leq \|V_0\|$, $\|S\| \leq \|V_n\|$, and (10) holds. By the infimum characterization of $\|\Phi\|_{cb}$ we must have $\|R\| = \|V_0\|$ and $\|S\| = \|V_n\|$. Clearly, $\Phi(s_1, \ldots, s_n) = R(\prod_{i=1}^{n} \pi_i(s_i))S$ for all $s_i \in G_i$, $i = 1, \ldots, n$.

(2°) Suppose, conversely, that (9) holds for $F$. Then (10) defines a completely bounded $n$-linear operator $\Phi$ whose Fourier transform is $F$. Thus also $\|\Phi\|_{cb} \leq \|R\|\|S\|$.

Combining (1°) and (2°), we see that the theorem, including the last sentence about $\|\Phi\|_{cb}$, has been proved.$\blacksquare$

5.4. COROLLARY. Let $G_1, \ldots, G_n$ be locally compact groups.

(a) If $F_1, F_2 : G_1 \times \cdots \times G_n \to \mathbb{C}$ are Fourier transforms of completely bounded $n$-linear forms on $C^*(G_1) \times \cdots \times C^*(G_n)$, then so is their pointwise product $F_1F_2$.

(b) If $H_1$ and $H_2$ are Hilbert spaces, and $F_i : G_1 \times \cdots \times G_n \to L(H_i)$ for $i = 1, 2$ are Fourier transforms of completely bounded $n$-linear operators, then so is the function $F : G_1 \times \cdots \times G_n \to L(H_1 \otimes H_2)$ defined by $F(s_1, \ldots, s_n) = F_1(s_1, \ldots, s_n) \otimes F_2(s_1, \ldots, s_n)$, where $H_1 \otimes H_2$ is the Hilbert space tensor product of $H_1$ and $H_2$. 
Proof. Since (a) is a special case of (b), we prove (b). Using for $F_j$, $j = 1, 2$, the type of representation afforded by Theorem 5.3 we get

$$F_1(s_1, ..., s_n) \otimes F_2(s_1, ..., s_n) = (R_1 \otimes R_2) \left( \prod_{i=1}^{n} \pi_{1}^{(1)}(s_i) \otimes \pi_{1}^{(2)}(s_i) \right) (S_1 \otimes S_2).$$

The above corollary is the basis of the following definition.

5.5. Definition. In the situation of Corollary 5.4(a) (resp. (b)), the completely bounded $n$-linear form (resp. $L(H_1 \otimes H_2)$-valued $n$-linear operator) $\Phi$ having $F_1 F_2$ (resp. $F$) as its Fourier transform is called the convolution of those $\Phi_1$ and $\Phi_2$, having $F_1$ and $F_2$ as their Fourier transforms. We denote $\Phi = \Phi_1 * \Phi_2$.

5.6. Corollary. In the situation of the above definition,

$$\|\Phi_1 * \Phi_2\|_c b \leq \|\Phi_1\|_c b \|\Phi_2\|_c b.$$

Proof. In the proof of Corollary 5.4 we may take $\|R_j\| \|S_j\| = \|\Phi_j\|_c b$, and so

$$\|\Phi_1 * \Phi_2\|_c b \leq \|R_1 \otimes R_2\| \|S_1 \otimes S_2\| \leq \|\Phi_1\|_c b \|\Phi_2\|_c b.$$

5.7. Corollary. Let $G_1, ..., G_n$ be locally compact groups.

(a) If $F: G_1 \times \cdots \times G_n \to \mathbb{C}$ is a linear combination of continuous positive-definite functions, then $F$ is the Fourier transform of some completely bounded $n$-linear form on $C^*(G_1) \times \cdots \times C^*(G_n)$.

(b) If $H$ is a Hilbert space, and $F: G_1 \times \cdots \times G_n \to L(H)$ is the Fourier transform of some completely bounded linear (i.e., 1-linear) operator $\Phi: C^*(G_1) \times \cdots \times C^*(G_n) \to I(H)$, then $F$ is the Fourier transform of some completely bounded $n$-linear operator $\Psi: C^*(G_1) \times \cdots \times C^*(G_n) \to L(H)$.

Proof. Again (a) is a special case of (b) (see [8, p. 191]), so let us prove (b). By Theorem 5.3 there is a Hilbert space $K$ with a continuous unitary representation $\pi: G_1 \times \cdots \times G_n \to L(K)$ and operators $R \in L(K, H)$, $S \in L(H, K)$ such that

$$F(s) = R \pi(s) S \quad \text{for all} \quad s \in G_1 \times \cdots \times G_n.$$

Defining $\pi_i(s_i) = \pi(e_1, ..., e_{i-1}, s_i, e_{i+1}, ..., e_n)$ for $s_i \in G_i$, where $e_j$ is the
neutral element of $G_j$, we get continuous unitary representations
\[ \pi_i: G_i \to L(K) \]
such that for all $s_i \in G_i$, $i = 1, \ldots, n$,
\[
F(s_1, \ldots, s_n) = R \left( \prod_{i=1}^{n} \pi_i(s_i) \right) S,
\]
and so the assertion follows from Theorem 5.3.

6. THE BANACH ALGEBRA OF COMPLETELY BOUNDED MULTILINEAR FORMS

In this section $G_1, \ldots, G_n$ are locally compact groups, and we denote by
\[ CB(C^*(G_1) \times \cdots \times C^*(G_n)) \]
or by $CB$ for short—the space of all completely bounded $n$-linear forms $\Phi: C^*(G_1) \times \cdots \times C^*(G_n) \to \mathbb{C}$ equipped
with the norm $\|\cdot\|_{cb}$. The normed space $CB$ is known to be complete. In
fact, it is the dual of a normed space (see [6, 7, 19] for arguments leading
to this and more general results); the completeness of $CB$ also follows in a
standard fashion from the original definition of $[2]$. We shall regard $CB$ as
a commutative Banach algebra with respect to convolution (see
Corollary 5.6). Note that $CB$ has the identity $(x_1, \ldots, x_n) \mapsto \prod_{i=1}^{n} e_i(x_i)$ of
norm 1, where $e_i: C^*(G_i) \to \mathbb{C}$ corresponds to the constant one-dimensional
continuous unitary representation of $G_i$.

For any locally compact group $G$, $B(G)$ will denote the Fourier–Stieltjes
algebra described in [5]: $B(G)$ is the commutative Banach algebra of linear
combinations of continuous positive-definite functions on $G$ equipped
with the norm derived from the identification of $B(G)$ with $C^*(G)^*$. We let
\[ \Delta(B(G)) (\subset W^*(G)) \]
denote its spectrum (i.e., the set of nonzero multiplicative linear functionals with the weak* topology). Similarly, $\Delta(CB)$ is
the spectrum of $CB(C^*(G_1) \times \cdots \times C^*(G_n))$.

We let \[ B(G_1) \otimes \cdots \otimes B(G_n) \]
denote the (completed) projective tensor
product of $B(G_1), \ldots, B(G_n)$. In a standard way (see [24]) the product
space \[ \Delta(B(G_1)) \times \cdots \times \Delta(B(G_n)) \]
will be identified with the spectrum of the commutative Banach algebra $B(G_1) \otimes \cdots \otimes B(G_n)$ by making $(\psi_1, \ldots, \psi_n)$
for $\psi_i \in \Delta(B(G_i))$ correspond to the element of $\Delta(B(G_1) \otimes \cdots \otimes B(G_n))$ which takes the value $\prod_{i=1}^{n} \langle f_i, \psi_i \rangle$
for any $f_1 \otimes \cdots \otimes f_n$, where $f_i \in B(G_i)$, $i = 1, \ldots, n$.

6.1. PROPOSITION. There are unique contractive, unital algebra
morphisms
\[ \alpha: B(G_1) \otimes \cdots \otimes B(G_n) \to B(G_1 \times \cdots \times G_n) \]
\[ \beta : B(G_1 \times \cdots \times G_n) \to CB(C^*(G_1) \times \cdots \times C^*(G_n)) \]

such that

\[ \alpha(f_1 \otimes \cdots \otimes f_n)(s_1, \ldots, s_n) = \prod_{i=1}^{n} f_i(s_i) \quad (11) \]

for all \( f_i \in B(G_i), \ s_i \in G_i, \ i = 1, \ldots, n \), and the Fourier transform of \( \beta(f) \) is \( f \) for all \( f \in B(G_1 \times \cdots \times G_n) \).

**Proof.** For each \( i = 1, \ldots, n \), if \( f_i \in B(G_i) \), there is a continuous unitary representation \( \pi_i : G_i \to L(H_i) \) such that for some vectors \( \xi_i, \eta_i \in H_i \),

\[ \|f_i\| = \|\xi_i\| \|\eta_i\| \quad \text{and} \quad f_i(s_i) = (\pi_i(s_i) \xi_i, \eta_i), \quad s_i \in G_i \quad [8, \ p. 195]. \]

Denote \( \pi_i(s_1, \ldots, s_n) = \pi_i(s_i) \), so that \( \pi_i \) is a continuous unitary representation of \( G_1 \times \cdots \times G_n \), and for \( s = (s_1, \ldots, s_n) \)

\[ \prod_{i=1}^{n} f_i(s_i) = ((\pi_1'(s_1) \otimes \cdots \otimes \pi_n'(s_n))(\xi_1 \otimes \cdots \otimes \xi_n) \eta_1 \otimes \cdots \otimes \eta_n). \]

Thus the function \( (s_1, \ldots, s_n) \mapsto \prod_{i=1}^{n} f_i(s_i) \) is in \( B(G_1 \times \cdots \times G_n) \), and its norm is at most \( \|\xi_1 \otimes \cdots \otimes \xi_n\| \|\eta_1 \otimes \cdots \otimes \eta_n\| = \prod_{i=1}^{n} \|f_i\| \quad [8, \ p. 195]. \)

It follows that there is a bounded \( n \)-linear operator from \( B(G_1) \times \cdots \times B(G_n) \) to \( B(G_1 \times \cdots \times G_n) \), of norm at most one, such that (11) holds for the corresponding linear contraction \( \alpha : B(G_1) \otimes \cdots \otimes B(G_n) \to B(G_1 \times \cdots \times G_n) \). Obviously \( \alpha(1) = 1 \). To prove that \( \alpha \) is multiplicative, it suffices to observe that

\[ \alpha((f_1 \otimes \cdots \otimes f_n)(g_1 \otimes \cdots \otimes g_n)) = \alpha(f_1 g_1 \otimes \cdots \otimes f_n g_n) \]

takes the value

\[ \prod_{i=1}^{n} f_i(s_i) g_i(s_i) = \prod_{i=1}^{n} f_i(s_i) \prod_{i=1}^{n} g_i(s_i) \]

for \( (s_1, \ldots, s_n) \in G_1 \times \cdots \times G_n \), and so does \( \alpha(f_1 \otimes \cdots \otimes f_n) \alpha(g_1 \otimes \cdots \otimes g_n) \), \( f_i, g_i \in B(G_i) \), \( i = 1, \ldots, n \). Clearly, \( \alpha \) is uniquely determined by (11). The existence and uniqueness of a unital algebra morphism \( \beta : B(G_1 \times \cdots \times G_n) \to CB(C^*(G_1) \times \cdots \times C^*(G_n)) \) with \( \beta(f) = f \) follows from Corollary 5.7(a) and Remark 5.2(b). That \( \beta \) is contractive is an easy consequence of [8, Lemma 2.14], the proof of Corollary 5.7, and Theorem 5.3. \( \Box \)

The following theorem generalizes and augments [9, Theorem 6.1] and an observation made in [9, p. 158] in the case of two abelian locally compact groups. (Recall from the Introduction that in the abelian case every bounded bilinear form on \( C^*(G_1) \times C^*(G_2) \) is completely bounded, and the completely bounded norm is equivalent to the usual supremum norm.) We retain the notation of Proposition 6.1.
6.2. Theorem. (a) If $\psi_i \in \Delta(B(G_i))$ for $i = 1, \ldots, n$, denote for all $\Phi \in CB$

$$[[\Phi(\psi_1, \ldots, \psi_n)](\Phi) = \mathcal{F}(\psi_1, \ldots, \psi_n). \quad (12)$$

Then (12) defines a separately continuous map $1: \Delta(B(G_1) \times \cdots \times \Delta(B(G_n)) \rightarrow \Delta(CB)$.

(b) If $\beta^*: \Delta(CB) \rightarrow \Delta(B(G_1) \times \cdots \times G_n))$ (resp. $\alpha^*: \Delta(B(G_1) \times \cdots \times G_n)) \rightarrow \Delta(B(G_1)) \times \cdots \times \Delta(B(G_n))$ is (defined by) the transpose of $\beta$ (resp. $\alpha$), then $\alpha^* \circ \beta^* \circ 1$ is the identity map of $\Delta(B(G_1)) \times \cdots \times \Delta(B(G_n))$. In particular, $1$ is injective.

(c) The map $\beta^* \circ 1: \Delta(B(G_1)) \times \cdots \times \Delta(B(G_n)) \rightarrow \Delta(B(G_1) \times \cdots \times G_n))$ is a separately continuous injection having $\alpha^*$ as a continuous left inverse.

Proof. (a) To show that $i(\psi_1, \ldots, \psi_n)$ is multiplicative, take $\Phi_1, \Phi_2 \in CB$, and represent their Fourier transforms in accordance with Theorem 5.3 as

$$\hat{\Phi}_j(s_1, \ldots, s_n) = \left(\prod_{j=1}^n \pi_i^{(j)}(s_i) \right) \xi_j | \eta_j,$$

where for $j = 1, 2$, $i = 1, \ldots, n$, $\pi_i^{(j)}: G_i \rightarrow L(K_j)$ are continuous unitary representations and $\xi_j, \eta_j \in K_j$. By definition,

$$(\Phi_1 \ast \Phi_2)(s_1, \ldots, s_n)$$

$$= \prod_{j=1}^2 \left(\prod_{i=1}^n \pi_i^{(j)}(s_i) \right) \xi_j | \eta_j$$

$$= \left(\prod_{i=1}^n (\pi_i^{(1)}(s_i) \otimes \pi_i^{(2)}(s_i)) \right) (\xi_1 \otimes \xi_2) | \eta_1 \otimes \eta_2,$$

so that when we denote $\pi_i = \pi_i^{(1)} \otimes \pi_i^{(2)}: G_i \rightarrow L(K_1 \otimes K_2)$ we get, by Theorem 5.3,

$$(\Phi_1 \ast \Phi_2)(u_1, \ldots, u_n) = \left(\prod_{i=1}^n \tilde{\pi}_i(u_i) \right) (\xi_1 \otimes \xi_2) | \eta_1 \otimes \eta_2$$

for all $u_i \in W^*(G_i), i = 1, \ldots, n$. But if $\psi_i \in \Delta(B(G_i))$, then $\tilde{\pi}_i(\psi_i) = \tilde{\pi}_i^{(1)}(\psi_i) \otimes \tilde{\pi}_i^{(2)}(\psi_i)$ (see [25, p. 25]), and so

$$(\Phi_1 \ast \Phi_2)(\psi_1, \ldots, \psi_n)$$

$$= \left(\prod_{i=1}^n (\tilde{\pi}_i^{(1)}(\psi_i) \otimes \tilde{\pi}_i^{(2)}(\psi_i)) \right) (\xi_1 \otimes \xi_2) | \eta_1 \otimes \eta_2$$

$$= \prod_{j=1}^2 \left(\prod_{i=1}^n (\tilde{\pi}_i^{(j)}(\psi_i)) \xi_j | \eta_j \right)$$

$$= \mathcal{F}_1(\psi_1, \ldots, \psi_n) \mathcal{F}_2(\psi_1, \ldots, \psi_n).$$
Thus $\ell(\psi_1, \ldots, \psi_n)$ is a multiplicative (clearly linear) functional. It is nonzero since for the identity of $CB$ it takes the value $\prod_{i=1}^n \ell_i(\psi_i) = 1$. The separate continuity is obvious from (11), since $\Phi$ is separately weak* continuous.

(b) Choose $\psi_i \in \mathcal{A}(B(G_i))$ and $f_i \in B(G_i)$ for $i = 1, \ldots, n$. There are Hilbert spaces $H_i$ with vectors $\xi_i, \eta_i \in H_i$ and continuous unitary representations $\pi_i: G_i \to L(H_i)$ such that $f_i(s) = (\pi_i(s) \xi_i | \eta_i)$, $s \in G_i$, $i = 1, \ldots, n$ \cite[p. 195]{8}. Denote $\Phi = \beta \alpha(f_1 \otimes \cdots \otimes f_n)$. Now $\Phi$ is the map $(u_1, \ldots, u_n) \mapsto \prod_{i=1}^n (\pi_i(u_i) \xi_i | \eta_i)$ on $W^*(G_1) \times \cdots \times W^*(G_n)$, since both are separately weak* continuous and agree on the Cartesian product of the linear spans of $\omega_i(G_i)$ which are weak* dense in $W^*(G_i)$. In particular,

$$[\alpha^* \circ \beta^* \circ \iota(\psi_1, \ldots, \psi_n)](f_1 \otimes \cdots \otimes f_n) = [\iota(\psi_1, \ldots, \psi_n)](\Phi) = \Phi(\psi_1, \ldots, \psi_n)$$

$$= \prod_{i=1}^n (\pi_i(\psi_i) \xi_i | \eta_i) = \prod_{i=1}^n \langle f_i, \psi_i \rangle$$

$$= (\psi_1, \ldots, \psi_n)(f_1 \otimes \cdots \otimes f_n).$$

From this the assertion follows.

Part (c) is an immediate consequence of (a) and (b).

7. Completely Bounded and Related Random Fields

In this section $G$ is a locally compact group and $H$ is a Hilbert space.

7.1. DEFINITION. Let $\varphi: G \to H$ be a random field and $R: G \times G \to \mathbb{C}$ its covariance function, i.e., $R(s, t) = (\varphi(s) | \varphi(t))$, $s, t \in G$. If the function $(s, t) \mapsto R(s, t^{-1})$ is the Fourier transform of some completely bounded bilinear form on $C^*_r(G) \times C^*_r(G)$, we say that the random field $\varphi$ is completely bounded. If $(s, t) \mapsto R(s, t^{-1})$ belongs to the Fourier–Stieltjes algebra $B(G \times G)$, we say that $\varphi$ is strongly harmonizable.

7.2. THEOREM. Let $\varphi: G \to H$ be a random field.

(a) If $\varphi$ is strongly harmonizable, then $\varphi$ is completely bounded.

(b) If $\varphi$ is completely bounded, then $\varphi$ is weakly harmonizable.

Proof. Part (a) follows from Corollary 5.7(a). Now assume that $\varphi$ is completely bounded. Since $R$ is continuous and bounded, $\varphi$ is easily seen to be bounded and weakly continuous. Applying \cite[Theorem 6.5 and Remark 6.3]{26} we see that $\varphi$ is weakly harmonizable.

7.3. Remark. The definition of strong harmonizability above is readily seen to generalize the corresponding notion in the abelian case (discussed,
e.g., in [21]). If $G$ is abelian, the complete boundedness of $\varphi$ is equivalent to its weak harmonizability (see [26, p. 379]), since for commutative $C^*$-algebras every bounded bilinear form is completely bounded.

We conclude with a theorem which characterizes completely bounded random fields in the spirit of [29]. A random field $\varphi: G \to H$ is right homogeneous, if $(\varphi(st) | \varphi(tu)) = (\varphi(s) | \varphi(t))$ for all $s, t, u \in G$.

7.4. Theorem. (a) Let $\Phi: C^*(G) \to H$ be a bounded linear operator. Its Fourier transform is a completely bounded random field if, and only if, there is a positive linear form $f: C^*(G) \to \mathbb{C}$ such that

$$\|\Phi x\|^2 \leq f(x^*x) \quad (13)$$

for all $x \in C^*(G)$.

(b) A random field $\varphi: G \to H$ is completely bounded if, and only if, there exist a Hilbert space $K$ containing $H$ as a subspace and a continuous right homogeneous random field $\psi: G \to K$ such that $\varphi = P_K \circ \psi$.

Proof. (a) Denote $B(x, y) = (\Phi x | \Phi y^*)$ for $x, y \in C^*(G)$. The Fourier transform of $B$ is the function $(s, t) \mapsto (\hat{\Phi}(s) | \hat{\Phi}(t^{-1}))$, and so $\hat{\Phi}$ is completely bounded if, and only if, $B$ is a completely bounded bilinear form. If the latter condition holds, there are positive linear forms $f_1, f_2: C^*(G) \to \mathbb{C}$ such that $\|B(x, y)\|^2 \leq f_1(x^*x)f_2(y^*y)$ for all $x, y \in C^*(G)$. (This follows at once from the representation $B(x, y) = (\pi_2(y) \xi | \pi_1(x^*) \eta)$ or already from $B(x, y) = (\theta_1(x) V_1 \theta_2(y) \xi | \eta)$; cf. [6, Theorem 2.1].) Taking $f = f_1 + f_2$ we get (13). Conversely, if (13) holds, then $\|B(x, y)\| \leq \|\Phi x\| \|\Phi y^*\| \leq f(x^*x)^{1/2}f(y^*y)^{1/2}$, and so $B$ is completely bounded (see [17, p. 185] or [6, Theorem 2.1]).

(b) If $\psi: G \to K$ is continuous and right homogeneous, there is a continuous positive-definite function $\rho: G \to \mathbb{C}$ with $(\psi(s) | \psi(t)) = \rho(st^{-1})$, and so there is a Hilbert space $H_\rho$ with a continuous unitary representation $\pi_\rho: G \to L(H_\rho)$ and a vector $\xi_\rho \in H_\rho$ such that $(\psi(s) | \psi(t^{-1})) = (\pi_\rho(s) \pi_\rho(t) \xi_\rho | \xi_\rho)$ for all $s, t \in G$. Thus $\psi$ is completely bounded by Theorem 5.3. In view of Theorem 7.4(a) it is clear that $T \circ \psi$ is completely bounded if $T$ is any bounded linear map from $K$ to a Hilbert space. Conversely, if $\varphi: G \to H$ is completely bounded, then $\varphi = \hat{\Phi}$, where $\Phi: C^*(G) \to H$ is a bounded linear map (Theorem 7.2) satisfying (13). A continuous right homogeneous dilation $\psi$ can now be constructed by modifying arguments in [29].
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