General Expressions for the 
Moore-Penrose Inverse of a $2 \times 2$ Block Matrix

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ABSTRACT

The Moore-Penrose inverse of a $2 \times 2$ block matrix $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ is discussed. General expressions for the Moore-Penrose inverse for the block matrix $M$ in terms of the individual blocks $A, B, C, D$ are delivered without any restrictions imposed. Under some conditions, the Moore-Penrose inverse can be simplified. The results extend earlier work by various authors.

1. INTRODUCTION

It has been of interest to derive general expressions for generalized inverses for the $2 \times 2$ block matrix

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

in terms of the individual blocks $A, B, C, D$. In [3], [8], [12], general expressions for $1$-inverses or $(1,2)$-inverses for the block matrix $M$ were given without any restrictions. Since the expression for $M^+$ is rather complicated (see [7]), some special restrictions have been imposed in order to get simpler forms of $M^+$. These include the results of [3], [8], and [12]. Most of these results can be found in [5]. For the special case where $C$ and $B^*$ are columns, then $M$ is a bordered matrix. In [6] and [9], five or six basically different cases...
were considered to obtain the MP inverses of a bordered matrix. In this paper, general expressions for the MP inverse for the block matrix $M$ are derived without any restrictions imposed. Since the results of [6] and [9] are rather complicated, general expressions for $M^+$ will be more complicated. However, when some conditions are placed on the blocks of the matrix $M$, the MP inverse can be simplified. The results extend earlier work by various authors.

Let $A \in \mathbb{S}^{m \times n}$, and let $M$ and $N$ be positive definite matrices of order $m$ and $n$ respectively. Then the unique matrix $X$ satisfying

$$AXA = A, \quad XAX = X, \quad (MAX)^* = MAX, \quad (NXA)^* = NXA \quad (1.1)$$

is called the weighted MP inverse of $A$ and is denoted by $X = A_{MN}^+$. When $M = I_m$ and $N = I_n$, $X$ is called the MP inverse of $A$ and denoted by $X = A^+$. In fact, $A_{MN}^+ = Q^{-1}(P*AQ^{-1})^*P*$, where $P = P^* = M^{1/2}$ and $Q = Q^* = N^{1/2}$ (see [2]).

We abbreviate positive definite to p.d. Some basic properties are given in the following lemma.

**Lemma 1.** Let $A \in \mathbb{S}^{m \times n}$, and let $M$ and $N$ be p.d. matrices of order $m$ and $n$ respectively. Then:

(a) $(A_{MN}^+)^* = (A^*)_{N^{-1}, M^{-1}}$.

(b) $A_{MN}^+ = N^{-1}A*(AN^{-1}A^*)^{-1}_M, M^{-1} = (A^*MA)^{-1}_{N^{-1}, N}A^*M$.

(c) Let $E \in \mathbb{S}^{m \times n}$, $F \in \mathbb{S}^{r \times n}$, and let $M$, $N$, and $R$ be p.d. matrices of order $m$, $n$, and $r$ respectively. If $EF_{RN}^+ = 0$, then $FE_{MN}^+ = 0$.

(d) Let $P$ be a idempotent matrix. Then

$$(I - P)A = 0 \iff R(A) \subseteq R(P); \quad A(I - P) = 0 \iff N(P) \subseteq N(A).$$


**Lemma 2** [4]. Let $K = A^+C$, $P = (I - AA^+)C$. Then

$$\left( A, C \right)^+ = \begin{bmatrix} A^+ & KU \\ U \end{bmatrix}, \quad (1.2)$$

where

$$U = P^+ + (I - P^+P)S^{-1}K^*A^+(1 - CP^+), \quad (1.3a)$$

$$S = I(I - P^+P)K^*K(I - P^+P). \quad (1.3b)$$

**Lemma 3** [10]. Let $A_1 = (A, C) \in \mathbb{S}^{m \times n}$, where $A \in \mathbb{S}^{m \times n}$, $M$ and $N$ are
p.d. matrices of order \( m \) and \( n \) respectively, and \( N \) is partitioned as

\[
N = \begin{pmatrix}
N_1 & L \\
L^* & N_2
\end{pmatrix}, \quad \text{where} \quad N_1 \in \mathbb{R}^{n_1 \times n_1}.
\]  

(1.4)

Let \( K = A_{MN_1}^+C, \quad P = (I - AA_{MN_1}^+)C \). Then

\[
(A_1)^+_{MN} = \left( A_{MN_1}^+ - KV - \left( I - A_{MN_1}^+ A \right) N_1^{-1} LV \right),
\]

(1.5)

where

\[
V = P_{MT}^+ + (I - P_{MT}^+ P)T^{-1}(K^*N_1^{-1} - L^*)A_{MN_1}^+,
\]

(1.6a)

\[
T = N_2 + K^*N_1K - (K^*L + L^*K) - L^*(I - A_{MN_1}^+)N_1^{-1}L.
\]

(1.6b)

Throughout this paper, we need some notation. Let the \( 2 \times 2 \) block matrix \( M \) be

\[
M = \begin{pmatrix}
A & C \\
B & D
\end{pmatrix};
\]

(1.7)

let

\[
K = A^+C, \quad H = BA^+, \quad Z = D - BA^+C,
\]

(1.8a)

\[
P = (I - AA^+)C, \quad Q = B(1 - A^+A).
\]

(1.8b)

In Section 2, we first derive a formula for the case where \( P = 0 \) and \( Q = 0 \). Then in Section 3, we apply it to \((MM^*)(1.3)\) to get general expressions for \( M^+ \) by using \( M^+ = M^*(MM^*)(1.3) \).

2. SPECIAL CASES WHERE \( P = 0 \) AND \( Q = 0 \)

We first consider two special cases according to the Schur complement \( Z \):

Case 1. \( P = 0, Q = 0, \) and \( Z \) is nonsingular. Then we have (see [5])

\[
M^+ = \begin{pmatrix}
A^+ + KZ^{-1}H & -KZ^{-1}H \\
-Z^{-1}H & Z^{-1}
\end{pmatrix}.
\]

(2.1)

Case 2. \( P = 0, Q = 0, \) and \( Z = 0 \). Then we have (see [5])

\[
M^+ = \begin{pmatrix}
(I - K\hat{K}^{-1}K^*)A^+(1 - H^*\hat{H}^{-1}H) & (1 - K\hat{K}^{-1}K^*)A^+H^*\hat{H}^{-1} \\
\hat{K}^{-1}K^*A^+(1 - H^*\hat{H}^{-1}H) & \hat{K}^{-1}K^*A^+A^*\hat{H}^{-1}
\end{pmatrix}.
\]

(2.2)
where

\[ \tilde{K} = I + K^*K, \quad \tilde{H} = I + HH^*. \quad (2.3) \]

If we let

\[ X = \begin{pmatrix} -K \\ I \end{pmatrix}, \quad Y = \begin{pmatrix} -H, I \end{pmatrix}, \quad (2.4) \]

then (2.1) and (2.2) can be rewritten as

\[ M^+ = \begin{pmatrix} A^+ & 0 \\ 0 & 0 \end{pmatrix} + XZ^{-1}Y, \quad (2.5) \]

and

\[ M^+ = (I - XX^+)egin{pmatrix} A^+ & 0 \\ 0 & 0 \end{pmatrix}(I - Y^+Y), \quad (2.6) \]

respectively. We may combine (2.5) and (2.6) as \( M^+ = G \), where

\[ G = \begin{bmatrix} I - X(I - Z^+Z)X^+ \end{bmatrix} \begin{pmatrix} A^+ & 0 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} I - Y^+(I - ZZ^+)Y \end{bmatrix} + XZ^+Y. \quad (2.7) \]

Indeed, we have

**Theorem 1.** If \( P = 0, Q = 0 \), then \( M^+ = G \) if and only if

\[ ZZ^+HH^* = HH^*ZZ^+, \quad Z^+ZK^*K = K^*KZ^+Z. \quad (2.8) \]

**Proof.** Since

\[ MX = \begin{pmatrix} 0 \\ Z \end{pmatrix}, \quad YM = (0, Z), \]

it follows that

\[ MX(I - Z^+Z) = 0, \quad (I - ZZ^+)YM = 0. \quad (2.9) \]

Hence,

\[ MG = \begin{pmatrix} AA^+ & 0 \\ H & 0 \end{pmatrix} \begin{bmatrix} I - Y^+(I - ZZ^+)Y \end{bmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} ZZ^+Y, \]
which, on using \( Y^+ = Y^* \bar{H}^{-1} \) and \( AA^+H^* = H^* \), yields

\[
MG = \begin{pmatrix} AA^+ & 0 \\ 0 & I \end{pmatrix} - Y^* \bar{H}^{-1}(I - ZZ^+)Y. \tag{2.10}
\]

Similarly, using (2.9) and noting that \( X^+ = \bar{K}^{-1}X^* \) and \( K^*A^+A = K^* \), we have

\[
GM = \begin{pmatrix} A^+A & 0 \\ 0 & I \end{pmatrix} - X(I - Z^+Z)\bar{K}^{-1}X^*. \tag{2.11}
\]

Finally from (2.9) and (2.10) we have

\[
MGM = (MG)M = M, \quad GMG = G(MG) = G.
\]

Thus \((MG)^* = MC\) iff \(ZZ^+HH^* = HH^*ZZ^+\), and \((CM)^* = CM\) iff \(Z^+ZK^*K = K^*KZ^+Z\).

Clearly (2.1) and (2.2) are two special cases of Theorem 1. We can derive several corollaries from Theorem 1; these include some results of [12], [8], and [5]. The following corollary can be found in [3], [7], and [5].

**Corollary 1.** If \( P = 0 \), \( Q = 0 \), \( (I - ZZ^+)B = 0 \), \( C(I - Z^+Z) = 0 \), then

\[
M^+ = \begin{pmatrix} A^+ & 0 \\ 0 & 0 \end{pmatrix} + XZ^+Y. \tag{2.12}
\]

**Proof.** Since \( B = ZZ^+B \), then \( HH^* = ZZ^+HH^* = (ZZ^+HH^*)^* = HH^*ZZ^+ \). Similarly, from \( C = CZ^+Z \), we have \( K^*K = K^*KZ^+Z = Z^+ZK^*K \). Using Theorem 1 gives (2.12).

Another special case is where \( C \) and \( B^* \) are columns; then \( Z \) is a scalar and (2.8) is satisfied (see [6, 9]).

The next result is the key upon which all later results are based.

**Theorem 2.** If \( P = 0 \), \( Q = 0 \), then

\[
M^+ = \left[I - X(I - Z^eZ)X^+\right]\begin{pmatrix} A^+ & 0 \\ 0 & 0 \end{pmatrix}\left[I - Y^+(I - ZZ^e)Y\right] + XZ^eY, \tag{2.13}
\]
that is,

$$(M^+)_{11} = \left[ I - K (I - Z^gZ) \tilde{K}^{-1} K^* \right] A^+ \left[ I - H^* \tilde{H}^{-1} (I - ZZ^g) H \right] + KZ^gH,$$

$$(M^+)_{12} = \left[ I - K (I - Z^gZ) \tilde{K}^{-1} K^* \right] A^+ H^* \tilde{H}^{-1} (I - ZZ^g) - KZ^g,$$

$$(M^+)_{21} = (I - Z^gZ) \tilde{K}^{-1} K^* A^+ \left[ I - H^* \tilde{H}^{-1} (I - ZZ^g) H \right] - Z^gH,$$

$$(M^+)_{22} = (I - Z^gZ) \tilde{K}^{-1} K^* A^+ H^* \tilde{H}^{-1} (I - ZZ^g) + Z^g, \quad (2.14)$$

where

$$Z^g = Z_{H^{-1}}^+, \quad (2.15)$$

Proof. Let the right hand side of (2.13) equal $G_1$. Since

$$MX(I - Z^gZ) = 0, \quad YY^* = I, \quad \text{and} \quad \begin{pmatrix} AA^* & 0 \\ H & 0 \end{pmatrix} Y^* = Y^* - \begin{pmatrix} 0 \\ I \end{pmatrix},$$

then

$$MG_1 = \begin{pmatrix} AA^* & 0 \\ 0 & I \end{pmatrix} - Y^* \tilde{H}^{-1} (I - ZZ^g) Y. \quad (2.16)$$

Similarly, from

$$(I - ZZ^g) YM = 0, \quad X^+ X = I, \quad \text{and} \quad X^+ \begin{pmatrix} A^+A \\ 0 \\ K \end{pmatrix} = X^+ - \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we have

$$G_1 M = \begin{pmatrix} A^+A & 0 \\ 0 & I \end{pmatrix} - X (1 - Z^gZ) \tilde{K}^{-1} X^*. \quad (2.17)$$

By the definition of $Z^g$ in (2.15), $Y^* \tilde{H}^{-1} (I - ZZ^g) - Y$ is Hermitian because $\tilde{H}^{-1} ZZ^g$ is. Similarly, $X (1 - Z^gZ) \tilde{K}^{-1} X^*$ is also Hermitian. Thus

$$MG_1 M = M, \quad G_1 M G_1 = G_1, \quad (MG_1)^* = MG_1, \quad (G_1 M)^* = G_1 M.$$

\[ \blacksquare \]
In particular, if $A$ is invertible, then $P = 0$ and $Q = 0$, and we can apply Theorem 2 to find $M^+$. One of the important applications of Theorem 2 is to find the Moore-Penrose inverse for a p.s.d. matrix without any restrictions. As shown in Albert [1], the conditions $P = 0$ and $Q = 0$ are automatically satisfied.

**Corollary 2.** If

$$M = \begin{pmatrix} A & C \\ C^* & D \end{pmatrix}$$

is a p.s.d. matrix, then

$$M^+ = \left[ I - X \left( I - Z^*Z \right) X^+ \right] \begin{pmatrix} A^+ & 0 \\ 0 & 0 \end{pmatrix} \left[ I - X^{++} \left( I - ZZ^g \right) X^* \right] + XZ^gX^*, \tag{2.18}$$

where

$$Z^g = Z_{K^{-1}, \tilde{K}}^+. \tag{2.19}$$

In addition, a particular $(1,3)$-inverse for $M$ is given by

$$M^{(1,3)} = \begin{pmatrix} A^+ & 0 \\ 0 & 0 \end{pmatrix} \left[ I - X^{++} \left( I - ZZ^g \right) X^* \right] + XZ^gX^*. \tag{2.20}$$

### 3. General Case

For any matrix $M$, the matrix $MM^*$ is p.s.d. and the Moore-Penrose inverse is then calculated from $M^+ = M^* (MM^*)^{(1,3)}$. Let

$$M_1 = MM^* = \begin{pmatrix} A_1 & C_1 \\ C_1^* & D_1 \end{pmatrix}. \tag{3.1}$$

We now shall derive expressions for $Z_1$, $K_1$, and $Z_1Z_1^g$. First we have to calculate the principal Schur complement $Z_1$ for $M_1$. Since

$$\left( A, C \right)^* A_1^+ = \left( A, C \right)^+ = \begin{pmatrix} A^+ & KU \\ U \end{pmatrix}, \tag{3.2}$$
where $U$ is defined by Lemma 2, then

$$Z_1 = D_1 - C_1^*A_1^+C_1 = (B, D)[I - (A, C)^*(A, C)](B, D)^*$$

$$= (B, D)[I - (A, C)^+(A, C)](B, D)^*,$$

which, on using (3.2) and $B(I - A^+A)B^* = B(I - A^+A)(I - A^+A)B^* = QQ^*$, yields

$$Z_1 = QQ^* + Z[(I - UC)D^* - UAB^*]. \quad (3.3)$$

Since $A^+P = 0$, from Lemma 1(c) we have

$$P^+A = 0, \quad P^+P = P^+C. \quad (3.4)$$

By the definition of $S$ in (1.3b), $I - P^+P$ and $S$ commute; then $I - P^+P$ and $S^{-1}$ also commute; thus

$$S^{-1}(I - P^+P)K*K(I - P^+P) = I - S^{-1} \quad (3.5)$$

Using (3.4) and (3.5), we have

$$I - UC = S^{-1}(I - P^+P), \quad UA = S^{-1}(I - P^+P)K^*. \quad (3.6)$$

Then (3.3) becomes

$$Z_1 = QQ^* + ZS^{-1}(I - P^+P)Z^*. \quad (3.7)$$

Since

$$K_1^* = C_1^*A_1^+ = (B, D)(A, C)^*A_1^+ = (B, D)(A, C)^+ = H + ZU, \quad (3.8)$$

then

$$X_1^* = -(H + ZU), I), \quad \tilde{K}_1 = I + (H + ZU)(H + ZU)^*. \quad (3.9)$$
Now from Corollary 2, we have

\[ M^+ = M^* \left( \begin{array}{cc} A_1^+ & 0 \\ 0 & 0 \end{array} \right) \left[ I - X_1^{*+} (I - Z_1Z_1^f) X_1^* \right] + \left( \frac{Q^* - (UA)^*Z^*}{(I - UC)^*Z^*} \right) Z_1^f X_1^*. \]  

(3.10)

Using (3.2) and (3.6) gives

\[ M^+ = \left( \begin{array}{cc} A^+ - KU & 0 \\ U & 0 \end{array} \right) \left[ I - X_1^{*+} (I - Z_1Z_1^f) X_1^* \right] + \left( \frac{Q^* - KS^{-1}(I - P^+P)Z^*}{S^{-1}(I - P^+P)Z^*} \right) Z_1^f X_1^*. \]  

(3.11)

where

\[ Z_1^f = (Z_1)_{\tilde{K}_1; N}. \]  

(3.12)

Let

\[ \tilde{Q} = (Q, Z(I - P^+P)), \quad N = \text{diag}(I, S); \]  

(3.13)

then \( Z_1 \) in (3.7) can be written as

\[ Z_1 = \tilde{Q} N^{-1} \tilde{Q}^*. \]  

(3.14)

Now we shall calculate \( Z_1Z_1^f, S^{-1}(I - P^+P)Z^*Z_1^f, \) and \( Q^*Z_1^f, \) and finally substitute them in (3.11) to obtain \( M^+. \) From Lemma 1(b) we have

\[ N^{-1} \tilde{Q}^*Z_1^f = (\tilde{Q})_{\tilde{K}_1; N}^+ = (Q, Z(I - PP))_{\tilde{K}_1; N}. \]  

(3.15)

Now

\[ N^{-1} \tilde{Q}^*Z_1^f = \left( \begin{array}{c} Q^* \\ S^{-1}(I - P^+P)Z^* \end{array} \right) Z_1^f, \]  

(3.16)
while by Lemma 3 with $M = \tilde{K}_1^{-1}$, $N_1 = I$, $N_2 = S$, $L = 0$, we have

\[
(\tilde{Q})^{+}_{\tilde{K}_1^{-1};N} = \left[ \begin{array}{c} Q^g - Q^g z (I - P^* P) V \\ V \end{array} \right], \tag{3.17}
\]

where

\[
V = W^g + (I - W^g W) T^{-1} (I - P^* P) Z^* Q^g Q^g, \tag{3.18a}
\]

\[
T = S + (I - P^* P) Z^* Q^g Q^g Z (I - P^* P), \tag{3.18b}
\]

\[
W = (I - QQ^g) Z (I - P^* P). \tag{3.18c}
\]

\[
Q^g = Q^g_{\tilde{K}_1^{-1};1}, \quad W^g = W^g_{\tilde{K}_1^{-1};T}. \tag{3.18d}
\]

Hence substituting,

\[
Z_1 Z_1^g = \tilde{Q} (\tilde{Q})^{+}_{\tilde{K}_1^{-1};N} = QQ^g - QQ^g Z (I - P^* P) V + Z (I - P^* P) V
\]

\[
= QQ^g + WV = QQ^g + WW^g. \tag{3.19}
\]

Recalling that $S^{-1}$ and $I - P^* P$ commute, we see from the corresponding submatrices of (3.16) and (3.17)

\[
V = S^{-1} (I - P^* P) Z^* Z_1^g = (I - P^* P) S^{-1} Z^* Z_1^g = (I - P^* P) V,
\]

\[
Q^* Z_1^g = Q^g - Q^g Z^* V. \tag{3.20}
\]

A final substitution in (3.11) yields

\[
M^+ = \begin{pmatrix} A^+ - KU & 0 \\ U & 0 \end{pmatrix} [I - X_1^* (I - QQ^g - WW^g) X_1^*]
\]

\[
+ \begin{pmatrix} Q^g - Q^g Z^* V - KV \\ V \end{pmatrix} X_1^*. \tag{3.21}
\]
that is,

\[
\begin{align*}
(M^+)_{11} &= (A^+ - KU)[I - (H + ZU)^*\tilde{K}_1^{-1}(I - QQ^g - WW^g)(H + ZU)] \\
&\quad - (Q^g - Q^gZV - KV)(H + ZU), \\
(M^+)_{12} &= (A^+ - KU)(H + ZU)^*\tilde{K}_1^{-1}(I - QQ^g - WW^g) \\
&\quad + Q^g - Q^gZV - KV, \\
(M^+)_{21} &= U[I - (H + ZU)^*\tilde{K}_1^{-1}(I - QQ^g - WW^g)(H + ZU)] \\
&\quad - V(H + ZU), \\
(M^+)_{22} &= U(H + ZU)^*\tilde{K}_1^{-1}(I - QQ^g - WW^g) + V.
\end{align*}
\]

We shall discuss several special cases when the Moore-Penrose inverse can be reduced to simpler forms. It is not difficult to see that 

\[P = 0 \iff R(C) \subseteq R(A), \quad \text{and} \quad Q = 0 \iff R(B^*) \subseteq R(A^*); \quad \text{so,} \quad P \text{ is of full column rank iff} \ C \text{ is of full column rank and} \ R(C) \cap R(A) = \{0\}, \quad \text{and} \quad Q \text{ is of full row rank iff} \ B \text{ is of full row rank and} \ R(B^*) \cap R(A^*) = \{0\}.

Case 1. \ P \text{ is of full column rank.} \ Since \ W = 0, \ V = 0, \ U = P^+, \ then \ \tilde{K}_1 = I + HH^* + Z(P^*P)^{-1}Z^*, \ and

\[
\begin{align*}
(M^+)_{11} &= A^+ - KP^+ - Q^gH - Q^gZP^+ \\
&\quad - \left[A^+H^* - K(P^*P)^{-1}Z^*\right]\tilde{K}_1^{-1}(I - QQ^g)(H + ZP^+), \\
(M^+)_{12} &= Q^g + \left[A^+H^* - K(P^*P)^{-1}Z^*\right]\tilde{K}_1^{-1}(I - QQ^g), \\
(M^+)_{21} &= P^+ - (P^*P)^{-1}Z^*\tilde{K}_1^{-1}(I - QQ^g)(H + ZP^+), \\
(M^+)_{22} &= (P^*P)^{-1}Z^*\tilde{K}_1^{-1}(I - QQ^g).
\end{align*}
\]

Case 2. \ P \text{ is of full column rank and} \ Q \text{ is of full row rank.} \ Then

\[M^+ = \begin{pmatrix} A^+ - KP^+ - Q^+H - Q^+ZP^+ & Q^+ \\ P^+ & 0 \end{pmatrix}. \tag{3.24}\]

Case 3. \ P \text{ is of full column rank and} \ Q = 0. \ In this case \ \tilde{K}_1 = I + HH^*
\[ M^+ = \begin{pmatrix} \left( A^+ - KP^+ - \left[ A^+H^* - K(P^+P)^{-1}Z^* \right] \tilde{K}_1^{-1}(H + ZP^+) \right) & \left[ A^+H^* - K(P^+P)^{-1}Z^* \right] \tilde{K}_1^{-1} \\ P^+ - (P^+P)^{-1}Z^* \tilde{K}_1^{-1}(H + ZP^+) & (P^+P)^{-1}Z^* \tilde{K}_1^{-1} \end{pmatrix}. \] (3.25)

Case 4. \( Q \) is of full row rank. In this case \( Q^* = Q^+ \), \( W = 0 \), \( V = T^{-1}(I - P^+P)Z^*(QQ^*)^{-1} \), and

\[ M^+ = \begin{pmatrix} A^+ - KU - (Q^- - Q^+Z(V - KV))(H + ZU) & Q^+ - Q^+Z(V - KV) \\ U - V(H + ZU) & V \end{pmatrix}. \] (3.26a)

We can also find \( M^+ \) by using left-right symmetry and case 1. Having found \((M^+)^*\), we obtain \( M^+ \) from \( M^+ = (M^+)^* \). In fact, if \( Q \) is of full row rank, let \( \tilde{H}_1 = I + K*K + Z^*(QQ^*)^{-1} \), \( P^g = P^g_1 \tilde{H}_1 \); then

\[ (M^+)_{11} = A^+ - KP^g - Q^+H - Q^+ZP^g \]
\[ - (K + Q^+Z)(1 - P^gP)\tilde{H}_1^{-1}\left[ K^*A^+ - Z^*QQ^*^{-1}H \right], \]

\[ (M^+)_{12} = Q^+ - (K + Q^+Z)(1 - P^gP)\tilde{H}_1^{-1}Z^*(QQ^*)^{-1}, \] (3.26b)

\[ (M^+)_{21} = P^g + (I - P^gP)\tilde{H}_1^{-1}\left[ K^*A^+ - Z^*(QQ^*)^{-1}H \right], \]

\[ (M^+) = (I - P^gP)\tilde{H}_1^{-1}Z^*(QQ^*)^{-1}. \]

Case 5. \( Q \) is of full row rank, and \( P = 0 \). In this case, let \( \tilde{H}_1 = I + K*K + Z^*(QQ^*)^{-1} \tilde{H}_1 \); then

\[ M^+ = \begin{pmatrix} A^+ - Q^+H - (K + Q^+Z)\tilde{H}_1^{-1}\left[ K^*A^+ - Z^*(QQ^*)^{-1}H \right] & Q^+ - (K + Q^+Z)\tilde{H}_1^{-1}Z^*(QQ^*)^{-1} \\ \tilde{H}_1^{-1}\left[ K^*A^+ - Z^*(QQ^*)^{-1}H \right] & \tilde{H}_1^{-1}Z^*(QQ^*)^{-1} \end{pmatrix}. \] (3.27)
If $C$ and $B$ are vectors, then $P$ and $Q$ are also vectors; thus $P$ (or $Q$) is either zero or of full rank. Then we can find $M^+$ according as $P$ and $Q$ are either zeros or of full rank. These cases were discussed by Hartwig [6].

It must be noted that using the weighted MP inverses does nothing more than using square roots of positive definite matrices, which can't be expressed in terms of the original matrices. Thus it does not allow generalization to more general settings.

4. AN EXAMPLE

In [7], Hung and Markham present a method of computing the MP inverse of

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}.$$ 

I shall use my results to indicate how the MP inverse is obtained.

In my method, $M^+$ can be greatly simplified when $P$ (or $Q$) = 0 or $P$ (or $Q$) is of full rank. By the discussion in Section 3, $P$ (or $Q$) has clear geometrical meaning, and it is easy to determine whether $P$ (or $Q$) = 0 or $P$ (or $Q$) is of full rank. In particular, if $A$ is invertible, then $P = (I - AA^+)C = 0$ and $Q = B(I - A^+A) = 0$; thus Theorem 2 is applicable.

Let's consider the same example as that in [7].

**EXAMPLE.**

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$ 

First we calculate $P$ and $Q$. Since $A$ is invertible, when $P = 0$ and $Q = 0$; thus we can apply Theorem 2. Now since $Z = D - BA^+C = 0_{1 \times 2}$, (2.7) can
be reduced to (2.6). Then using the notation of (1.8) and (2.4), we get

\[ K = A^+ C = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad H = BA^+ \begin{pmatrix} 1 & 1 \end{pmatrix}, \]

\[ x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} -H & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix}. \]

Finally, after some simple computations, we obtain

\[ M^+ = \begin{pmatrix} \frac{1}{5} & 0 & \frac{1}{5} \\ -\frac{1}{15} & \frac{1}{3} & \frac{4}{15} \\ \frac{4}{15} & -\frac{1}{3} & -\frac{1}{15} \\ \frac{1}{5} & 0 & \frac{1}{5} \end{pmatrix}. \]

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