
Spines and Homology of Thin Riemannian Manifolds with Boundary

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I. INTRODUCTION

1.1. Cut Loci, Spines, and Simple Polyhedra

This paper establishes a connection between Riemannian geometry and the notion in PL-topology of collapse to a simple polyhedral spine. In particular, a class of spines that has previously received little attention is shown to be geometrically natural.

One of the aims of modern differential geometry is to extract topological consequences from geometric bounds. Our program for doing this builds on our previous work, according to which if a Riemannian manifold $M$ with boundary $B$ has sufficiently small inradius relative to its curvature, then the cut locus of $B$ exhibits canonical branching behavior of arbitrarily low branching number. Thus $M$ is forced by geometric bounds to collapse to a polyhedron with certain canonical singularities. In this paper, we examine topological consequences of such a collapse. Thus, although our original motivation was geometric, many of our arguments are in the setting of PL topology.

The following geometric theorem is an application of our work. It states that a simply connected Riemannian manifold with connected and simply connected boundary must be large relative to curvature. While our inradius bound may not be sharp, it misses by at most a factor of 2 (see below).

THEOREM 1.1. Suppose a Riemannian manifold $M$ with connected boundary $B$ satisfies $|K_M| \leq 1$ and $|\kappa_B| \leq 1$, where $K_M$ is sectional curvature of the interior and $\kappa_B$ is normal curvature of the boundary. If $M$ and $B$ are simply connected boundary.

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connected, then \( M \) has inradius at least 0.108. More generally, if \( \pi_1(B) \) and \( \pi_1(M) \) are isomorphic under the inclusion map, then \( M \) has inradius at least 0.108.

The work in this paper and [AB1] may be viewed as a response to the following challenge of Berger: “One can ask why study only two objects: euclidean domains with boundary and Riemannian manifolds without boundary. There exist of course the notion of Riemannian manifolds with boundary. But the idea is this: in the Euclidean domain the inside geometry is given, say flat and trivial and the interesting phenomena are coming from the shape of the boundary. On the other hand in Riemannian manifolds there is no boundary and the geometry phenomena are those of the inside. If you ask both at the same time you risk having too much to handle.” [B, p. 2]

The definitions of \( k \)-branched simple polyhedron and collapse are given in Section 2.1. In the 2-dimensional case, these polyhedra have the singularities of a soap film. They have also been called fake surfaces and standard complexes, and collapsing to them has been extensively studied; see [BP2] for a bibliography. For higher dimensions, see [M, GMR, LF1]. For instance, Matveev has proved that every compact PL \( n \)-manifold \( M \) with nonvoid boundary collapses to a simple \((n-1)\)-dimensional polyhedron whose \((n-1)\)-dimensional strata are open cells [M]. The result of a collapse is referred to as a spine. Our geometric bounds yield spines with low branching numbers. For example, a 3-branched spine is locally either an \((n-1)\)-manifold or the product of an \((n-2)\)-manifold with a cone over three points. The topological consequences of having such a spine are far from clear and are a main topic of this paper.

Now we state the geometric theorem that initiated this study. The inradius, \( \mathcal{I}nr \), of a Riemannian manifold \( M \) with boundary \( B \) is the supremum on \( M \) of the distance to \( B \); or equivalently, the supremum of radii of metric balls in \( M \) that do not intersect \( B \). The curvature-normalized inradius is the scale-free invariant

\[
\mathcal{I}nr \max \{ \sup \sqrt{|K_M|}, \sup |\kappa_B| \}.
\]

As will be shown in Section 2, the following theorem is a corollary of our work in [AB1]:

**Theorem 1.2.** There exists a sequence of universal constants \( a_2 < a_3 < \cdots \) (independent of dimension \( n \)), such that if a Riemannian manifold \( M \) with boundary \( B \) has curvature-normalized inradius less than \( a_k \), then the cut locus of \( B \) is a \( k \)-branched simple polyhedron of dimension \( n-1 \), and is a spine of \( M \). Here \( a_2 \approx 0.075 \) and \( a_3 \approx 0.108 \).
This theorem extends a theorem stated by Gromov in his 1978 address to the International Congress [G]. That theorem concerns the case $k = 2$, and establishes a constant $a_2$ that depends on the dimension $n$ and goes to 0 on order $n^{-\omega}$. Note that a 2-branched simple polyhedron is a manifold without boundary; if $M$ has such a spine, then $M$ is the product of a manifold without boundary and a closed interval or else is doubly covered by such.

We have an algorithm for generating the constants $a_k$. They are probably not sharp, but examples based on hyperbolic geometry show that the sharp constants $a_k$ satisfy

$$a_k \leq \frac{1}{2} \log \frac{2k}{k+1}$$

(so $a_2 \leq 0.144$ and $a_3 \leq 0.203$). An upper bound for all $a_k$ is $(\log 2)/2 \approx 0.347$; our constants $a_k$ increase to 0.195. While we believe that the $a_k$ are strictly increasing, we have not proved this; however, our estimates show, for example, that $a_2 < a_6$.

By Theorem 1.2, any Riemannian manifold $M$ having $|K_M|$ and $|\kappa_B|$ at most 1 and inradius less than $1/10$ has a 3-branched simple polyhedron as spine. Since simple bounds on curvature and size force the existence of these spines, it becomes important to investigate their topological implications. In the following theorem, we consider manifolds with simply connected boundary; this case includes manifolds without boundary, by removal of a ball. We calculate the $H_1$ obstruction for these manifolds to have a given 3-branched spine (see Section 5 for a more detailed statement) and show that if the boundary is connected, the vanishing of $H_1$ is an obstruction to having any 3-branched spine:

**Theorem 1.3.** Suppose a manifold $M$ with boundary $B$ has a 3-branched simple polyhedron as spine, and $H_1(B, \mathbb{Z}) = 0$.

(a) If $B$ is connected, then $H_1(M, \mathbb{Z}) \neq 0$.

(b) If $M$ is compact, then $H_1(M, \mathbb{Z})$ is a direct sum of copies of $\mathbb{Z}, \mathbb{Z}_2$, and $\mathbb{Z}_3$, and depends only on the number of components of the boundary and a bipartite graph representing the combinatorial structure of the spine.

The first conclusion of Theorem 1.1 follows immediately from Theorems 1.2 and 1.3(a). The Euclidean case of Theorem 1.1 was treated by Fet and Lagunov. They showed that when $M$ is a simply connected domain in $E^n, n \geq 3$, bounded by a connected, simply connected hypersurface $B$ with $|\kappa_B| \leq 1$, then the inradius is at least $\sqrt{3/2} - 1 \approx 0.225$ [LF1, Theorem 1]. Moreover, this bound is sharp [LF2]. (For $n = 2$, the sharp bound is 1 [PI].) Fet and Lagunov proved the Euclidean version of the second
conclusion of Theorem 1.1 under the added assumption that $\pi_2(M) = 0$ [LF1, Theorem 2].

It can be shown that in the example constructed in [LF2], the spine is Bing's "house with two rooms" (illustrated in [RS, p. 2]). In a less restrictive setting, Itoh constructs a Riemannian metric on the 3-ball for which the spine is a "dunce hat" [It].

It is a pleasure to acknowledge our debt to the paper [LF1] by Fet and Lagunov. While their main interest is in Euclidean domains, many of their topological arguments are presented in the more general setting of 3-branched simple polyhedral spines. We build on this strong foundation. Modern readers may find [LF1] challenging to read because the language and techniques of PL topology and bundles were not available when the paper was written. Specific cross-references to [LF1] are provided below whenever our work overlaps theirs.

Concerning other related work, we remark that by theorems of Weinstein [W] and Buchner [Bu], any compact $n$-manifold without boundary carries a metric for which the cut locus of some point is a simple $(n-1)$-dimensional polyhedron (by definition, $(n+1)$-branched), and these metrics are generic in the class of metrics without conjugate points.

For a compact 3-manifold with boundary, some relations between the homology and the combinatorics of simple polyhedral spines have been studied by Ikeda [Ik] (see Remark 5.4 below).

1.2. Outline of Paper

Section 2 gives basic definitions and the proof of Theorem 1.2. Section 3 concerns holonomy in simple polyhedral spines. In particular, we point out a bundle equivalence that is important for later arguments. We then examine holonomy in 3-branched spines, and holonomy restrictions that apply when the boundary is simply connected. Section 4 examines 3-branched spines from a combinatorial viewpoint. We use bipartite graphs to represent the components of the strata of the spine and their preimages in the boundary, and we relate the topology of the spine to that of the graphs. Section 5 uses the work of the preceding sections to study the topological consequences of having a 3-branched simple polyhedron as spine. Theorems 1.1 and 1.3 are proved here.

2. PROOF OF THEOREM 1.2

2.1. Collapsing to Simple Polyhedra

A simple $(n-1)$-dimensional polyhedron $C$ is one locally modelled on the $(n-1)$-skeleton of an $(n+1)$-simplex [M, GMR]. This means that $C$
is stratified by manifolds $C_r$, of dimension $n + 1 - r$, where $C_r$ is the set of $r$-branchpoints of $C$ and $2 \leq r \leq n + 1$. That is, every $p \in C_r$ has a neighborhood that is PL homeomorphic to the product of $I^{n+1-r}$ with a cone over the $(r-3)$-sphere of an $(r-1)$-simplex (say over the barycenter $v$). Here $I = [-1, 1]$ and $p$ corresponds to $v \times \{0\}$. Below we shall use stratum to mean a component of any $C_r$. By a $k$-branched simple $(n-1)$-dimensional polyhedron, $2k \leq n + 1$, we mean one whose strata have dimension at least $n + 1 - k$, or equivalently, which consists only of $r$-branchpoints for $r \leq k$. (Our notion of “$k$-branched” is different from Benedetti and Petronio’s notion of “branched” [BP2], which assigns a bifurcation structure to our 3-branchpoints.)

When using the terms “spine” and “collapse,” we always refer to the category of polyhedra and piecewise-linear maps. Suppose a simple $(n-1)$-dimensional polyhedron $C$ is imbedded as a closed subset of the interior of an $n$-dimensional manifold $M$ with boundary $B$. In this case, we say $M$ collapses to $C$ if there is a (PL) map $\Phi : B \times [0, 1] \to M$ whose restriction to $B \times \{0, 1\}$ is a homeomorphism onto $M \setminus C$, and whose restriction to $B \times \{1\}$ has image $C$. Then since $\Phi$ is PL, $\Phi$ acts as an imbedding on the product of some neighborhood of each point in $B$ with $[0, 1]$. See Matveev’s discussion in [M]; in particular, it is equivalent in the case under consideration to say that $M$ collapses to $C$ in the standard sense of [RS], [Z], and the imbedding of $C$ in $M$ is nice in the sense that it is locally equivalent to the standard imbedding of the $(n-1)$-sphere of an $(n+1)$-simplex into the $n$-sphere. Moreover, if $p$ is an $r$-branchpoint of $C$, then $\Phi^{-1}(p)$ consists of exactly $r$ points of $B \times \{1\}$.

2.2. Cut Loci of Thin Riemannian Manifolds with Boundary

Now suppose $M$ is a complete Riemannian manifold with boundary $B$. Starting at any point $q$ in the boundary $B$ of $M$, there is a maximal geodesic segment perpendicular to $B$ that realizes the distance from each of its points to $B$. If that segment is not a ray, then its other end is called the cut point of $B$ with base point $q$, and its length, the cut distance of $q$. The cut locus $C$ of $B$ is the set of all such cut points. A similar definition of the cut locus of a point in a manifold without boundary has been discussed in basic texts on Riemannian geometry (e.g., [BC]), and the basic properties for the cut locus of a boundary do not differ essentially from the case of the cut locus of a point. In particular, $C$ is the union of the focal points of $B$ and the sets $C_r$, $r \geq 2$, of points having $r$ minimizers to $B$.

**Theorem 2.1** [AB1]. There exists a sequence $a_2 < a_3 < \cdots$ such that if any complete Riemannian manifold $M$ with boundary $B$ has curvature-normalized inradius less than $a_2$, then $B$ has no focal points and for any cut
point \( p \) of \( B \), the minimum angle between the initial directions of the minimizers from \( p \) to \( B \) is greater than \( \cos^{-1}(-1/k) \).

Let us denote by \( \sigma_k(k) \), the largest possible minimum angle between \( k \) unit vectors in \( E^n \). The following lemma is known, but a reference for part (b) is not easily found; we thank R. Alexander for informing us and providing this proof and one based on Helly’s theorem.

**Lemma 2.2.** (a) \( \sigma_k(k + 1) = \cos^{-1}(-1/k) = \sigma_n(k + 1), n \geq k \).

(b) \( \sigma_k(k + 2) = \pi/2 \) if \( k \geq 2 \).

**Proof.** (a) (cf. [LF1]) Let \( v_1, \ldots, v_{k+1} \) be unit vectors in \( E^n \). It follows from \( \langle \sum_{i=1}^{k+1} v_i, \sum_{i=1}^{k+1} v_i \rangle \geq 0 \) that \( v_i \cdot v_j \geq -1/k \) for some \( i \neq j \). If \( n \geq k \) and \( \sum_{i=1}^{k+1} v_i = 0 \), this inequality becomes equality for all \( i \neq j \).

(b) Let \( v_1, \ldots, v_{k+2} \) be nonzero vectors in \( E^k \). If \( k \geq 2 \), we may realize a minimum angle of \( \pi/2 \) by choosing the \( v_i \) to lie in positive and negative coordinate directions. Now we show that if \( k \geq 1 \), then \( v_i \cdot v_j \geq 0 \) always holds for some \( i \neq j \).

Clearly this assertion holds for \( k = 1 \). Suppose it is true for \( k - 1 \), but there are \( v_1, \ldots, v_{k+2} \) in \( E^k \) with all mutual inner products strictly negative. By an isometry we may assume \( v_{k+2} = (0, 0, \ldots, c) \) for \( c < 0 \). The negativity condition requires that the \((k+2)\)-component of \( v_i \) be positive for \( i < k + 2 \). Next, strike this \((k+2)\)-component from each of the \( v_i, i < k + 2 \). Now we are left with \( k + 1 \) vectors in \( E^k \) having all mutual inner products negative, a contradiction of the induction hypothesis.

2.3. **Proof of Theorem 1.2**

It follows from Theorem 2.1 and Lemma 2.2 that if \( M \) has curvature-normalized inradius less than \( \alpha_k \), then there are at most \( k \) minimizers \( \sigma_1, \ldots, \sigma_r, r \leq k \), from any cut point \( p \) to \( B \). Moreover, their initial unit tangent vectors \( u_1, \ldots, u_r \) (which are distinct) do not lie in a subspace of dimension \( r - 2 \). Equivalently, \( u_1 - u_2, \ldots, u_1 - u_r \) are linearly independent. This condition implies that the cut locus \( C \) is a \( k \)-branched simple polyhedron and \( M \) collapses to \( C \), as we now verify.

For each of the \( r \) minimizers \( \sigma_j \) from \( p \) to \( B \) we can define a **local distance function** \( f_j \), which measures the distance of a point in a neighborhood of \( p \) to \( B \) along a geodesic which is in a neighborhood of \( \sigma_j \), among the set of geodesics perpendicular to \( B \). The gradient vectors at \( p \) of the \( f_j \) are the unit vectors \( u_j \), so the differences \( f_1 - f_j \) are functionally independent. It is easy to see that the common zero set of the \( f_1 - f_j \), an \((n + 1 - r)\)-manifold, is precisely a neighborhood of \( p \) in \( C \).

To describe the structure of the cut locus around \( p \), extend coordinates at \( p \) in \( C \), to coordinates on \( M \) complementary to the \( f_1 - f_j \), in a
neighborhood that excludes cut points of order higher than \( r \). For an appropriate constant \( c < f_j(p) \), the inequalities \( f_j \geq c \) describe a differentiable \((r-1)\)-simplex in each of the coordinate slices obtained by setting the complementary coordinates equal to constants. We can take the following functions as barycentric coordinates in those simplexes:

\[
g_j = \frac{f_j - c}{\sum f_r - rc}.
\]

Then the equations and inequalities describing the cut locus strata within these simplex slices in terms of the \( g_j \) are exactly the same as the equations and inequalities in terms of barycentric coordinates describing the strata of the cone of the \((r-3)\)-skeleton of an \((r-1)\)-simplex over its barycenter. Thus in a neighborhood of \( p \), the pair \((M, C)\) is locally differentiably equivalent to the pair determined by the \( n \)-skeleton and the \((n-1)\)-skeleton of a standard \((n+1)\)-simplex, with \( C \) corresponding to a point of an \((n + 1 - r)\)-face. Therefore \( C \) is a \( k \)-branched simple polyhedron, with \( C_r \) agreeing with the set \( C_r \) of Subsection 2.1.

Define the map \( \Phi : B \times [0, 1] \to M \) by letting \( \Phi(q, t) \) lie on the minimizing geodesic perpendicular to \( B \) at \( q \), with distance from \( B \) equal to \( t \) times the cut distance. For \( p \) as above, let \( q \) be one of the \( r \) points in \( B \) satisfying \( \Phi(q, 1) = p \); that is, some \( s_j \) joins \( q \) to \( p \). It follows from the above analysis that \( (q, 1) \) has an open neighborhood \( U \times \{1\} \) in \( B \times \{1\} \) that is piecewise diffeomorphic under \( \Phi | B \times \{1\} \) to a neighborhood of \( p \) in the union of the faces of \( C \) satisfying \( f_j = f_{j'} \), \( j \neq j' \). More specifically, the restriction of \( \Phi \) to the product of \([0, 1]\) with the closure in \( U \) of the preimage of each \((n-1)\)-face is a diffeomorphism into \( M \). This shows that \( M \) collapses to \( C \).

3. HOLONOMY

3.1. Holonomy of Simple Polyhedral Spines

Suppose an \( n \)-dimensional manifold \( M \) with boundary \( B \) collapses to an \((n-1)\)-dimensional simple polyhedral spine \( C \), via the map \( \Phi : B \times [0, 1] \to C \). Let \( \Phi_1 : B \to C \) correspond to the restriction of \( \Phi \) to \( B \times \{1\} \). For \( 2 \leq r \leq n + 1 \), the points having exactly \( r \) preimages in \( B \) under \( \Phi_1^{-1} \) form the manifold \( C_r \), whose components \( C_{ri} \) are the \((n + 1 - r)\)-dimensional strata of \( C \). Furthermore, the boundary \( B \) is stratified by the manifolds \( B_r = \Phi_1^{-1}(C_r) \), where \( B_r \) is an \( r \)-fold cover of \( C_r \).

Corresponding to the \( r \)-fold covering of \( C_{ri} \), we have an \( r \)-od bundle \( T_{ri}^* \) over \( C_{ri} \), whose fiber at \( p \) consists of the cone over the \( r \) points of \( \Phi_1^{-1}(p) \). For \( r > 2 \), we define a second \( r \)-od bundle \( T_{ri}^* \) over \( C_{ri} \), as follows. The closure of each \((n + 2 - r)\)-dimensional stratum adjacent to \( C_{ri} \) is the result
of making boundary identifications on a manifold with boundary whose interior is the stratum. Collaring the boundaries of these manifolds with boundary yields an r-od bundle over \( C_n \).

In either bundle, moving around a loop in \( C_n \) yields a permutation of the fiber. Thus we obtain an induced homomorphism of the fundamental group \( \pi_1(C_n) \) to the permutation group \( S_r \), which we refer to as the holonomy map of the bundle. The fact that \( T_n \) and \( T^*_n \) have the same holonomy map, which is the following proposition, will be key in transferring topological information from \( B \) to \( M \):

**Proposition 3.1.** The bundles \( T_n \) and \( T^*_n \) are isomorphic.

**Proof.** Intuitively, the corresponding arms of the two r-ods at a point are opposite each other. More precisely, we proceed as follows. A neighborhood in \( C \) of \( p \in C_n \) is equivalent to the product of \( I^{n+1-r} \) and a cone over the \((r-3)\)-skeleton of an \((r-1)\)-simplex. Let \( U \) be the product of \( I^{n+1-r} \) and the cone over the vertices of the \((r-1)\)-simplex. Then \( U \setminus C_{ni} \) has \( r \) components, and an arm of the r-od over \( p \) in \( T_{ni} \) lies one of these components. By our definition of collapse, there is exactly one point \( q \in \Phi^{-1}(p) \) such that the image of the restriction of \( \Phi \) to the product of \( I \) and a sufficiently small neighborhood of \( q \) in \( B \) does not intersect this component. Thus we have an identification between the fibers of \( T_{ni} \) and \( T^*_{ni} \). It is immediate to verify that this identification commutes with the holonomy action around any loop based at \( p \).

**Remark 3.2.** In [BP1], it is shown that for a 3-manifold collapse, holonomy makes sense for loops in the complex \( C_3 \cup C_4 \), namely, the entire singular set of the spine. Here we have not considered the question of extending holonomy beyond a single stratum. This is because in this paper our main interest lies in 3-branched spines, that is, those whose highest branching number is 3.

### 3.2. Holonomy of \((n-2)\)-Dimensional Strata

Now we look more closely at the holonomy of the bundles \( T_{3i} \) and \( T^*_{3i} \) over a stratum \( C_{3i} \). Corresponding to a loop \( \gamma \) in \( C_{3i} \), consider the pullback \( T_\gamma \) to \( \gamma \) of \( T_{3i} \), where we regard \( T_\gamma \) as being immersed in \( C \). The points of \( M \) that collapse to points of this immersed triod bundle form an immersed 3-dimensional triod block in \( M \). This triod block is equivalent to the 3-dimensional Euclidean block in Fig. 1 with its two ends identified.

The holonomy around \( \gamma \) is an element of the permutation group \( S_3 \), either the identity, a cyclic permutation or a transposition. The two ends in Fig. 1 are identified by translation for identity holonomy; by translation and rotation by angle \( 2\pi/3 \) for cyclic holonomy; and by translation and a
reflection in the plane of one of the three central rectangles for transposition holonomy.

In the case of identity holonomy, the part of the triod block in $B$ consists of three cylinders. The boundary of these cylinders is three pairs of circles, each pair doubly covering one of the three circles that comprise the boundary $U$ of $T_\#$.

In the case of cyclic holonomy, the part in $B$ consists of a single cylinder. The boundary of this cylinder is a pair of circles, doubly covering the single circle that comprises the boundary $U$ of $T_\#$.

In the case of the transposition holonomy, the part in $B$ consists of a cylinder and a Möbius band. One boundary circle of the cylinder forms a double cover of a boundary circle of $T_\#$. The other boundary circle of the cylinder is paired with the boundary circle of the Möbius band, to doubly cover the second boundary circle of $T_\#$.

**Example 3.3.** Suppose $M$ is 3-dimensional and compact. The case where the dimensions of the strata of the spine are not restricted but the strata are assumed to be open cells is described in [BP1]. On the other hand, we are interested in 3-branched spines; that is, the strata do not have restricted topology but there are no 0-dimensional strata. Then the 1-dimensional strata are circles. $M$ consists of imbedded triod blocks, glued on connecting surfaces to *diod blocks*. For a triod block with identity holonomy, the connecting surfaces are three annuli along which the block is attached to the rest of $M$. For cyclic holonomy, the connecting surface is an annulus. For transposition holonomy, the connecting surfaces are a Möbius band and an annulus.
The dioid blocks are easier to describe. The intersection of a dioid block with $B$ forms a double cover, either trivial or nontrivial, of a base surface in a 2-dimensional stratum. The base surface has as interior, the intersection of the stratum with the complement of the $T_k$’s, and has one or more boundary circles. The part of the dioid block over such a circle is either an annulus or a Möbius band. These annuli and Möbius bands are the connecting surfaces which are glued to those in the triod blocks to make up $M$.

3.3. Restrictions on Holonomy

Suppose the $n$-dimensional manifold $M$ with boundary $B$ collapses to a 3-branched simple polyhedral spine $C$, via the map $\Phi: B \times [0, 1] \rightarrow M$. Then $C = C_2 \cup C_3$ where the singular set $C_3$ is an $(n - 2)$-manifold without boundary, triply covered by the hypersurface of $B$ that is its preimage under $\Phi_1$.

Now we show that in the setting of our main theorems, transposition holonomy does not occur.

**Proposition 3.4.** Suppose a manifold $M$ with boundary $B$ collapses to a 3-branched simple polyhedral spine. If $H_1(B, \mathbb{Z}) = 0$, then each $(n - 2)$-dimensional stratum of the spine is orientable, and its holonomy group is either the identity or the alternating subgroup of $S_3$.

**Proof.** Since $H_1(B, \mathbb{Z})$ has no subgroup of index two, $B$ is orientable. Moreover, each component $\tilde{A}$ of the preimage in $B$ of a stratum $C_3$ separates $B$ into two components, each with boundary $\tilde{A}$. Indeed, since $\tilde{A}$ is closed in $B$, by Alexander duality [S, Section 6.9], $H_1(B, B\setminus \tilde{A}; \mathbb{Z})$ is isomorphic to $H_{n-2}(\tilde{A}, \mathbb{Z})$, hence it has dimension 1 since $\tilde{A}$ is connected. Then the long exact sequence for the pair $(B, \tilde{A})$ shows that $H_0(B\setminus \tilde{A})$ has dimension 2. It is clear from the definition of collapse that each component of $B\setminus \tilde{A}$ has boundary $\tilde{A}$. Therefore $\tilde{A}$ is orientable.

Suppose $C_3$ contains a loop with transposition holonomy. Following the discussion in Section 3.3, $\gamma$ lifts to two loops in $B$, one covering $\gamma$ doubly and one singly. Let $\tilde{\gamma}$ be the latter loop, and $\tilde{A}$ be the covering hypersurface of $C_3$ in $B$ that contains $\tilde{\gamma}$. Then $\tilde{A}$ is orientable. The triod bundle $T_3$ over $C_3$ determines a collar of $\tilde{A}$ in each of the two components into which $\tilde{A}$ separates $B$, hence determines a dioid bundle over $\tilde{A}$. As in Section 3.2, the pullback of this dioid bundle to $\tilde{\gamma}$ is a Möbius band. This contradicts the orientability of $B$.

**Remark 3.5.** The first paragraph of the preceding proof corresponds to [LF1, Lemma 15]. It is easy to see that transposition holonomy is ruled out if both $M$ and $C_3$ are orientable [LF1, Lemma 12]. The point of Proposition 3.4 is that orientability of $M$ is unnecessary.
4. GRAPHICAL CODING OF THE STRATIFICATIONS OF THE SPINE AND BOUNDARY

For the rest of this paper, unless otherwise specified, we continue to consider a manifold $M$ with boundary $B$ that collapses to a 3-branched simple polyhedral spine $C = C_2 \cup C_3$.

4.1. The Spine Graph and the Boundary Graph

The spine graph $c$ is a bipartite graph whose vertices $c_{3i}$ and $c_{2j}$ represent the $(n-2)$- and $(n-1)$-dimensional strata $C_{3i}$ and $C_{2j}$, respectively, of the spine $C$. The vertices $c_{3i}$ and $c_{2j}$ will be referred to as $C_{3i}$ and $C_{2j}$ vertices, respectively. Adjacent to a given vertex $c_{3i}$ are three, two or one edges, representing the components into which $C_{3i}$ separates its triod bundle $T_{3i}$. (Thus the valence of $c_{3i}$ is 3, 2, or 1 respectively according as the holonomy group of $C_{3i}$ is the identity, contains a transposition but no cyclic permutation, or contains a cyclic permutation.) Each such component lies in some $C_{2j}$, and the edge representing that component joins $c_{3i}$ to $c_{2j}$. Thus the edges adjacent to a $C_{2j}$ vertex are in one-one correspondence with the boundary components of the corresponding $C_{2j}$ stratum.

The boundary graph $b$ is a bipartite graph whose vertices $b_{3u}$ and $b_{2v}$ represent the strata $B_{3u}$ and $B_{2v}$ of $B$. Each $b_{3u}$ is adjacent to one or two edges, representing the components into which $B_{3u}$ separates a regular neighborhood of itself in $B$. Each such component lies in some $B_{2v}$, and the edge representing that component joins $b_{3u}$ to $b_{2v}$. The pair $(b, c)$ is equipped with a simplicial map $\varphi: b \to c$ corresponding to the action of the map $\Phi_1: B \to C$, which maps each stratum of the boundary onto a stratum of the spine.

By Proposition 3.1, for any $(n-2)$-dimensional stratum $C_{3i}$, the number of components of its complement in $T_{3i}$ equals the number of components of its preimage under $\Phi_1$. In terms of the graph pair:

**Proposition 4.1.** At a $C_{3i}$ vertex, the valence equals the multiplicity of $\varphi$.

**Remark 4.2.** This spine graph was introduced by Fet and Lagunov [LF1] under the assumption of identity holonomy. We have modified the definition to include information on the holonomy group of each stratum in $C_{3i}$, and have introduced the graph pair to encode further information.

**Example 4.3.** Figure 2 illustrates two graph pairs. We use the convention that $C_{2j}$ vertices and their preimages are represented by circles, and $C_{3i}$ vertices and their preimages, by filled disks. Figure 2a may be realized, for
example, by taking $M$ to be punctured $RP^n$, so that $B = S^{n-1}$ and $C = RP^{n-1}$.

Figure 2b may be realized, for example, by taking $M$ to be the punctured 3-dimensional lens space $L_{3,1}$. Here $C$ is a 2-cell with boundary circle identified in thirds, and $B = S^2$. Starting with Fig. 1 with the cyclic holonomy identification, we obtain $M$ by attaching a thickened disk along the annular connecting surface. Thus $C$ is obtained by gluing a disk to the portion of $C$ shown in Fig. 1, and $B$ by gluing two disks to the cylinder which is the portion of $B$ shown there. We remark that such a construction can always be done in the context of Riemannian geometry, so that $M$ carries a Riemannian metric for which $C$ is the cut locus of $B$ [AB2].

Remark 4.4. It can be proved that for all compact manifolds that collapse to 3-branched spines and have connected, simply connected boundaries, their graph pairs are recursively generated from the two pairs shown in Fig. 2 [AB2]. One of the 78 such graph pairs having at most three $C_3$ vertices is shown in Fig. 3.

Example 4.5. Figure 4 shows graph pairs in which the cut graph $\gamma$ has two $C_3$ vertices of valence 2; such vertices correspond to $(n-2)$-dimensional strata of the spine with transpositions but no cyclic permutations in their holonomy groups. Figure 4a may be realized by a 3-dimensional manifold whose boundary is the connected sum of two projective planes,
namely the Klein bottle. In this case, the three $B_2$ vertices of the boundary graph represent annuli, two of which have a double identification along one of their boundary circles. The resulting 3-manifold is the result of taking two triod blocks with transposition holonomy (as in Example 3.3), gluing them to each other along their respective Möbius band connecting surfaces, and gluing to each of their respective annular connecting surfaces a nontrivial diod block over a Möbius band.

On the other hand, if we replace one or both of these nontrivial diod blocks with a trivial diod block over a disk, then the boundary graph splits into two or three components respectively. These will consist of two projective planes in the former case (illustrated in Fig. 4b), plus, in the latter case, a 2-sphere.

**Remark 4.6.** It follows from Subsection 3.2 and Proposition 4.1 that for a $C_3$ vertex of valence 2 in $c$, the restriction of $\varphi$ to the two preimage vertices in $b$ and their adjacent edges is as shown in Fig. 4. For a $C_1$ vertex of valence 3 or 1, the restriction of $\varphi$ to the preimage vertices in $b$ and their adjacent edges is as shown in Fig. 3. (Hence if all $C_1$ vertices have valence 3, then every edge of $c$ is covered by exactly two edges of $b$, a fact we shall use later.)

Note that by Proposition 3.4, transposition holonomy cannot occur if $H_1(B, \mathbb{Z}_2) = 0$. Thus $C_1$ vertices of valence 2 will be ruled out in Section 5, but not in the present section.

4.2. **Topology of the Spine and Boundary Graphs**

Whenever we make specific arguments about the homology of a 3-branched spine $C$, we will refer to a decomposition of $C$ of the following type into closed cells. Start with a neighborhood in $C$ of each stratum $C_{3i}$, that corresponds to the triod bundle $T_{3i}$. Thus we have a 3-fold covering of $C_{3i}$ by $U_i = \partial T_{3i}$. Let $C_{2j}$ be $C_{2j}$ with the interiors of all the tubular
neighborhoods $T_j$, that it intersects removed. Hence $C'_j$ is a manifold whose boundary consists of components of $U_j$'s.

Choose a simplicial decomposition of each $C'_j$, and lift that decomposition to $U_j$. Decompose $T_j$, in the natural way into product cells $I \times \Delta$ with the simplices $\Delta$ of $U_j$. Here the boundary operator is given by $\partial(I \times \Delta) = \Delta^* - \Delta - I \times \partial \Delta$, where $\Delta^*$ is the projection of $\Delta$ to $C'_j$. Finally, extend the simplicial decompositions of the $U_j$ to simplicial decompositions of the $C'_j$. Figure 5 illustrates the case $n = 3$ and is a schematic for the general case.

In homology arguments on $B$, we use a similar decomposition, consisting of all preimages under $\Phi_1$ of cells in a decomposition of $C$.

**Theorem 4.7.** Suppose a manifold $M$ with boundary $B$ collapses to a 3-branched simple polyhedral spine $C$.

(a) There is an imbedding of the spine graph $c$ in the spine $C$ that induces a monomorphism of $H_1(c, \mathbb{Z})$ into $H_1(C, \mathbb{Z})$.

(b) If $H_1(B, \mathbb{Z})$ is finite, then the image of this monomorphism consists of all of the free part of $H_1(C, \mathbb{Z})$. More specifically, every element of $H_1(C, \mathbb{Z})$ can be decomposed into a sum of elements, each supported by the image of $c$ or a stratum of $C$, and the latter summands all have finite order.

**Proof.** (a) We imbed $c$ in $C$ as follows. Choose a base point in each $C'_j$ and each $C''_j$. Connect each base point in $C'_j$ to each component $U_i$ of the boundary $U_i$ of $T_j$ by a fiber. Then connect each of these fiber endpoints to the base point of the stratum $C''_j$ in which it lies by a simple curve in $C''_j$. The connecting curves in $C''_j$ should not meet each other except at the base point. Note that there is one connecting curve from the

![Figure 5](image-url)
base point in $C_{3i}$ for each component of $T_{3i}\setminus C_{3i}$. Thus, by definition of the spine graph $c$, there is one base point for every vertex of $c$ and one connecting curve for every edge.

Now choose a cell decomposition of $C$ as above; we may assume the base points are 0-cells and the connecting curves are in the 1-skeleton. Let us specify representatives for $H_1(c, \mathbb{Z})$. We assume that $c$ is connected, since the components of $c$ are in one–one correspondence with the components of $M$. Let $t$ be a maximal tree of $c$ with base vertex $v$. Then for each edge $e \notin t$ we get a corresponding loop based at $v$ by chaining together the unique simple path from $v$ to one end of $e$, $e$ itself, and the unique simple path from the other end of $e$ to $v$. Generally the two paths will have a common part starting at $v$, and removing this common part leaves a simple closed curve containing $e$. Choosing an orientation for this simple closed curve gives us an oriented 1-cycle $z_e$, and these form a basis for $H_1(c, \mathbb{Z})$.

By means of the imbedding of $c$ into $C$ we identify the $z_e$ with 1-cycles in $C$. What we have to prove is that if for some 2-chain $f$ in $C$ we have $\partial f = \sum e \oplus I \cdot p$ (where $\sum$ indicates a sum with finitely many nonvanishing coefficients), then all the coefficients $n_e$ are 0. Any particular $e \notin t$ goes from the basepoint of a stratum $C_{3i}$, along a fiber in a component $T_{3i}$ of $T_{3i}$ to $U_{4i}$, and then to the basepoint of the $C_2$ stratum that contains $T_{3i}$. Since $e$ is disjoint from every 1-cycle in our basis except $z_e$, then $z_e$ is the only such 1-cycle that contains a term entering the interior of $T_{3i}$. Denoting that term by $\pm I \cdot p$ for $p \in U_{4i}$, we have $\partial f = \pm n_e I \cdot p$ + terms not entering the interior of $T_{3i}$.

We may write $f = \sum_{\Delta \in c} m_{\Delta} (I \times \Delta) + \text{terms not entering the interior of } T_{3i}$, where $\Delta$ runs over the 1-cells in $U_{4i}$. Then $\partial f = -I \times \partial \sum_{\Delta \in c} m_{\Delta} \cdot \Delta = \pm n_e p$. But for any 1-chain the sum of the coefficients of its boundary is 0. Hence $n_e = 0$.

(b) If $H_1(B, \mathbb{Z})$ is finite, then every 1-cycle $z$ supported in a stratum of $C$ must represent an element of finite order in $H_1(c, \mathbb{Z})$, because $z$ is covered, at most triply, by a cycle in $B$. A finite multiple of this cycle in $B$ will bound a 2-cycle, which projects to $C$ as a 2-cycle whose boundary is a finite multiple of $z$. The proof will be completed by showing that any 1-cycle in $C$ is homologous in $C$ to a sum of cycles in $c$ and the strata of $C_3$ and $C_2$.

Now choose a cell decomposition of $C$ as in part (a), that is, with a copy of $c$ in its 1-skeleton. Let $z$ be any 1-cycle in $C$, represented as a formal finite sum of 1-cells. Since $z$ can be written as a sum of 1-cycles corresponding to simple loops, we assume $z$ corresponds to a simple loop. Suppose $z$ does not lie entirely in $C_3$. Then $z$ is obtained by chaining together segments $\sigma_1, \tau_1, \sigma_2, \ldots, \tau_k$, where each $\sigma_i$ is in some $C_3$ stratum $C_{3i}$ and has
initial vertex $v_i$ and final vertex $w_i$, and each $\tau_i$ lies in some $C_{2i}$. We allow $\sigma_i$ to be trivial, in which case $v_i = w_i$. Let $p_i$ be the vertex of $c$ in $C_{3i}$ and $q_i$ be the vertex of $c$ in the interior of $C_{2i}$. Denote by $p_i q_i$ the 1-chain that corresponds to an edge of $c$ from $p_i$ to $q_i$, and intersects the same component of $T_i \setminus C_{3i}$ as $\tau_i$. Similarly, $q_i p_{i+1}$ corresponds to an edge of $c$ from $q_i$ to $p_{i+1}$, and intersects the same component of $T_{i+1} \setminus C_{3i+1}$ as $\tau_i$. Denote the negative 1-chains by $q_i p_i - p_{i+1} q_i$.

Choose a simple 1-chain $p_i v_i$ from $p_i$ to $v_i$ in $C_{3i}$. Then $z$ is the sum of cycles in the closures of the $C_2$ strata, namely $z_i = (p_i v_i) \sigma_i \tau_i (v_{i+1} p_{i+1}) (p_{i+1} q_i) (q_i p_i)$, and 1-chains in $c$, namely $(p_i q_i) (q_i p_{i+1})$. (Here, since the right-hand endpoint of $\tau_i$ is $v_i$, we may define $v_{i+1} = v_i$ and $p_{i+1} = p_i$.) The sum of the latter is a 1-cycle in $c$. It only remains to show that each $z_i$ is homologous to a cycle in $C_{2i}$ itself. But our choice of cell decomposition makes it clear that $z_i$ is homologous to the 1-cycle obtained from $z_i$ by replacing $(p_i v_i) \sigma_i$ by its lift to the boundary $U_i$ of $T_i$, replacing $(v_{i+1} p_{i+1})$ by its lift to $U_{i+1}$, and removing the four 1-cells in which $\tau_i$ and $(p_{i+1} q_i) (q_i p_i)$ intersect $T_i$ and $T_{i+1}$.

Remark 4.8. The main idea of the proof of part (a) of Theorem 4.7 is contained in [LF1, Lemmas 17 and 19]. Our motivation for proving part (b) is Theorem 1.3, for which it is a key ingredient.

There is a companion to the first part of Theorem 4.7 for the boundary graph, with no essential change in the proof. Part (b) of the following theorem is an immediate corollary of (a).

**Theorem 4.9.** (a) An imb Students $b$ in $B$ induces a monomorphism of $H_1(b, \mathbb{Z})$ into $H_1(B, \mathbb{Z})$. (b) If $H_1(B, \mathbb{Z})$ is finite, then the components of $b$ are trees.

4.3. Submanifold Graphs

Now we show that $H^\infty_{n-1}(M, \mathbb{Z})$ is encoded by the graph pair. Recall that $H^\infty_n(M)$ may be defined in terms of our (locally finite) cell decompositions just as $H_n(M)$ is defined, except that chains need not be finite (i.e., we do not require all but finitely many coefficients to vanish). Of course, if $M$ is compact, $H^\infty_n(M) = H_n(M)$. In the noncompact case, $H^\infty_n$ is usually discussed in the guise of cohomology with compact support, from which it can be extracted. This subsection formalizes and extends a notion introduced in [LF1] in their discussion of proper trees.

We are interested in how to glue aggregates of $(n-1)$-dimensional strata of $C$ along $(n-2)$-dimensional strata to form $(n-1)$-dimensional manifolds without boundary that are closed in $C$. Suppose the holonomy group of an $(n-2)$-dimensional stratum $C_h$ contains a cyclic permutation (i.e., the valence of the corresponding vertex $c_h$ in the spine graph is 1).
Then \( T_b \setminus C_b \) is connected, and lies in an \( (n-1) \)-dimensional stratum \( C_{2j} \). The closure of \( C_{2j} \) makes a 3-fold identification along \( C_{3i} \) of a boundary component of the manifold with boundary whose interior is \( C_{2j} \). Therefore \( C_{2j} \) cannot be part of any aggregate of \( C_2 \) strata and their adjacent \( C_3 \) strata that forms a manifold without boundary that is closed in \( C \). Similarly, suppose the holonomy group of \( C_{3i} \) contains a transposition but no cyclic permutation (i.e., the valence of the corresponding vertex in the cut graph is 2). Then \( T_b \setminus C_b \) has two components. The closures of the \( C_2 \) strata containing these components make, respectively, a 1- and 2-fold identification along \( C_{3i} \). If \( C_{2j} \) is the stratum with the 1-fold identification, \( C_{2j} \) cannot be part of such an aggregate, since extension to a manifold across \( C_{3i} \) would include the second sheet and again force a triple identification. However, the stratum with the 2-fold identification might be included, since that identification yields a manifold structure in a neighborhood of \( C_{3i} \).

The conditions for an aggregate of \( C_2 \) strata in the spine and their adjacent \( C_3 \) strata to form a manifold without boundary that is closed in \( C \) may be formulated in terms of the graph pair. A subgraph of \( c \) is associated to the aggregate, consisting of the vertices representing its strata and all the edges joining them. This subgraph must be of the following type:

**Definition 4.10.** A subgraph \( s \) of \( c \) is a submanifold graph if

1. every \( C_2 \) vertex of \( s \) has the same valence in \( s \) as in \( c \).
2. every \( C_3 \) vertex of \( s \) has valence 3 or 2 in \( c \) and one less in \( s \); if the valence in \( c \) is 2, then the adjacent edge chosen for \( s \) is that covered by two edges in the boundary graph (see Fig. 4).

**Remark 4.11.** Suppose there are no \( C_3 \) vertices of valence 2 in \( c \) (as happens, by Proposition 3.4, if \( H_1(B, \mathbb{Z}_2) = 0 \)). Then condition (2) of the definition simplifies to:

(2') every \( C_3 \) vertex of \( s \) has valence 2 in \( s \).

In this case, the submanifold graphs are determined by the spine graph \( c \) alone.

The submanifold graphs are in one-one correspondence with the imbedded \( (n-1) \)-dimensional submanifolds without boundary that are closed in \( C \). Even when all the \( C_2 \) strata in one of these submanifolds are orientable, the induced orientations on the adjoining \( C_3 \) strata may not be invariably opposite for any specific choice of orientations of the \( C_2 \) strata. Thus the submanifold itself may not be orientable.
It is clear that if an \((n-1)\)-cycle in \(C\) with coefficients in \(\mathbb{Z}_2\) includes as a summand some \((n-1)\)-cell in a given \(C_2\) stratum, then it includes every \((n-1)\)-cell in that stratum; and further that the \((n-1)\)-cycles are in one-one correspondence with the submanifold graphs. The operation on submanifold graphs that corresponds to the sum of \((n-1)\)-cycles with coefficients in \(\mathbb{Z}_2\) is as follows: \(s_1 + s_2\) contains all the edges that occur in exactly one of the \(s_i\), and all the vertices that are adjacent to those edges.

**Proposition 4.12.** The collection of submanifold graphs, including the empty one, can be viewed as a vector space over \(\mathbb{Z}_2\), which is isomorphic to \(H^{mf}_{n-1}(M, \mathbb{Z}_2)\).

**Proof.** The preceding discussion shows that the submanifold graphs form a vector space isomorphic to \(H^{mf}_{n-1}(C, \mathbb{Z}_2)\). The conclusion follows because \(C\) is a deformation retract of \(M\) under a proper homotopy, by the definition of collapse, and \(H^{mf}_{s}\) satisfies the homotopy axiom for proper homotopies [S, pp. 320, 341].

For example, the dimension of the vector space of submanifold graphs in the spine graphs of Figs. 1–4, respectively, is 1, 0, 1, 2.

**Remark 4.13.** There is an analogous but more involved description of oriented submanifold graphs, depending on the graph pair with certain marks, and corresponding to \(H^{mf}_{n-1}(M, \mathbb{Z})\).

### 4.4. Invariants of the Graph Pair

We shall use the following notation for invariants of \(M\) determined by the graph pair \((b, c)\) and its projection map \(\varphi\). If \(M\) is compact, these invariants are nonnegative integers. In the noncompact case, they are cardinal numbers (see [R, p. 61] for the rank of an abelian group); \(\ell, p, t, \beta\) are either finite or \(\infty\). The equation below for \(t\) is by Proposition 4.1.

\[
\begin{align*}
  m &= \dim_{\mathbb{Z}_2}\{\text{submanifold graphs in } c\}, \\
  \ell &= \text{rank } H_1(c, \mathbb{Z}), \\
  p &= \text{the number of } C_2 \text{ vertices in } c \text{ of } \varphi\text{-multiplicity 1}, \\
  t &= \text{the number of } C_3 \text{ vertices in } c \text{ of } \varphi\text{-multiplicity 1}, \\
  \beta &= \text{the number of components of } b.
\end{align*}
\]

**Lemma 4.14** [LF1, Lemma 16]. If all the \(C_3\) vertices of \(c\) have valence 3, then \(\ell + m > 0\).
Proof. By assumption, \( t = 0 \) and there are no \( C_3 \) vertices of valence 2 in \( c \). Suppose \( \ell = 0 \), so that \( c \) is a tree. In this case, we can construct a nonempty submanifold graph in \( c \) by starting at any \( C_2 \) vertex, and sequentially fulfilling first condition (1) and then (2') of Definition 4.10 and Remark 4.11. No violation of (2') is forced retroactively because there are no loops in \( c \). Therefore \( m > 0 \).

5. HOMOLOGY OF MANIFOLDS WITH SIMPLY CONNECTED BOUNDARY

5.1. Proof of Theorems 1.1 and 1.3

Again in this section, \( M \) is a manifold with boundary \( B \) and 3-branched simple polyhedral spine \( C \). Since \( C \) is a deformation retract of \( M \), then \( H_\bullet(M, \mathbb{Z}) = H_\bullet(C, \mathbb{Z}) \).

Now we assume \( H_1(B, \mathbb{Z}) = 0 \). By Proposition 3.4, this rules out transposition holonomy. In particular, every \((n-2)\)-dimensional stratum is covered by either one or three connected hypersurfaces in \( B \) under the map \( \Phi_1 : B \to C \).

Recall the invariants \( m, \ell, p, t, \beta \), defined in Subsection 4.4 in terms of the graph pair of \( M \). By Proposition 4.12 and Theorem 4.7, they may also be described as follows:

\[
m = \dim H_\infty^{mf}(M, \mathbb{Z}_2),
\]
\[
\ell = \text{rank } H_1(M, \mathbb{Z}),
\]
\[
p = \text{the number of } C_2 \text{ strata of the spine with connected preimage in } B,
\]
\[
t = \text{the number of } C_3 \text{ strata of the spine with connected preimage in } B,
\]
\[
\beta = \text{the number of components of } B.
\]

Note that by definition, \( \ell \) and \( t \) depend only on the spine graph. When \( H_1(B, \mathbb{Z}) = 0 \), the same is true of \( m \), by Remark 4.11. If moreover \( M \) is compact, the following theorem shows that the same is true of \( p + \beta \).

For \( k \) finite or \( \mathbb{R}_0 \), we use the symbol “\( \oplus_k \)” for the direct sum of \( k \) copies, and “\( H^k \)” for the direct product of \( k \) copies (of course these are the same if \( k \) is finite).

**Theorem 5.1.** Let \( M \) be a connected manifold with boundary \( B \), that collapses to a 3-branched simple polyhedral spine. If \( H_1(B, \mathbb{Z}) = 0 \), then \( M \) has the following homology groups:

\[
H_1(M, \mathbb{Z}) = \mathbb{Z}_{\oplus p} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3,
\]
\[
H_\infty^{mf}(M, \mathbb{Z}_2) = \mathbb{Z}_\ell \oplus p + \beta \oplus \mathbb{Z}_2.
\]
Thus if $M$ is compact:

$$H_{n-1}(M, Z_2) = \bigoplus_a Z_2,$$

where $m = \ell + p + \beta - 1$.

If $M$ is compact and orientable:

$$H_{n-2}(M, Z) \text{ has torsion subgroup } \bigoplus_p Z_2 \bigoplus_i Z_3,$$

$$H_{n-1}(M, Z) = \bigoplus_{\ell + \beta - 1} Z_2.$$

If $M$ is compact, all these groups depend only on the spine graph of $M$ and the number of components of $B$.

**Proof.** For a stratum $A$ in $C_k$ ($k = 2, 3$), denote by $v(A)$ the number of times $A$ is covered under $\Phi_1$ by a single component, say $A$, of its preimage. Thus $v(A) = k$ if the preimage of $A$ is connected, and otherwise $v(A) = 1$.

Let $\overline{A}$ be a stratum of $B$ that is a $(v(A))$-fold covering of $A$ under $\Phi_1$.

(a) First we show that the image of the induced map $t_\ast: H_1(A, Z) \to H_1(C, Z)$ of the inclusion map $r: A \to C$ has order 1 or $v(A)$.

Suppose $A$ is a stratum in $C_3$. Then $(\Phi_1)_\ast \pi_1(A)$ is the kernel of the holonomy map of the bundle $T_\Lambda$ from $\pi_1(A)$ to $S_3$, by Proposition 3.1. Therefore $\pi_1(A)/(\Phi_1)_\ast \pi_1(A)$ is the identity or alternating subgroup of $S_3$, and in particular is abelian. Since the normalizers of $(\Phi_1)_\ast \pi_1(A)$ and $\pi_1(A)$ coincide, we have $H_1(A, Z)/(\Phi_1)_\ast H_1(C, Z) = \pi_1(A)/(\Phi_1)_\ast \pi_1(A)$, which is of order $v(A)$.

The map $t_\ast = (\Phi_1)_\ast$ of $H_1(\overline{A}, Z)$ into $H_1(C, Z)$ factors through $H_1(B, Z)$, which vanishes by assumption. Therefore the kernel of $t_\ast$ contains the subgroup $(\Phi_1)_\ast H_1(C, Z)$, which has index $v(A)$.

Suppose $A$ is a stratum in $C_2$. Since $v(A) = 1$ or 2, $\Phi_1|\overline{A}$ is a regular covering. It follows that $\pi_1(A)/(\Phi_1)_\ast \pi_1(A)$ has order $v(A)$, and the argument proceeds as before.

(b) Next we show that the image of the induced map $t_\ast: H_1(A, Z) \to H_1(C, Z)$ has order exactly $v(A)$. This follows from (a) if $v(A) = 1$, so we suppose $v(A) = k$ for $k = 2, 3$.

Let $z$ be a 1-cycle in $A$ which in $H_1(A, Z)/(\Phi_1)_\ast H_1(C, Z)$ represents a nonzero element. That is, $z$ is not homologous in $A$ to the $(\Phi_1)_\ast$-image of a 1-cycle in $B$. We claim that the injection of $z$ into $H_1(C, Z)$ is nonzero. Suppose, to the contrary, $z = \partial f$ for some 2-chain $f$ in $C$.

Suppose $A = C_3$. We refer to a cell decomposition of $C$ as described at the start of Subsection 4.2. Then $f = \text{terms in } C_3 + \sum_{\Delta \in U} m_\Delta \partial \Delta$ + \text{terms not entering the interior of } T_\Lambda \text{ in } C_3 = f_1 + f_2 + f_3$, where $\Delta$ runs over the 1-cells in $U_i$. Thus $z = \partial f_1 + \partial f_2 + \partial f_3$, which we write as $(z - w) + \partial f_2 + \partial f_3$. Therefore $\partial f_2 = w - \partial f_3$, where $w \subset C_3$ and $\partial f_3 \subset U_i = \partial T_\Lambda$. Since $\partial f_2 = -I \sum_{\Delta \in \partial U} m_\Delta \partial \Delta + \sum_{\Delta \in \partial U} m_\Delta \Delta^* - \sum_{\Delta \in \partial U} m_\Delta$, it follows
that \(w\) is the projection, in the triod bundle \(T_3\), of the 1-cycle \(\partial f_3\) in \(U_i\) to the base \(C_3\). Since \(z - \partial f_1 = w\), then \(z\) is homologous in \(C_3\) to the projection in \(T_\gamma\) of a 1-cycle in \(U_i\). By Proposition 3.1, \(z\) is therefore homologous to the \((\Phi_1)_*\)-image of a 1-cycle in \(B\), a contradiction.

Suppose \(A = C_3\), and let \(U\) be the boundary of \(C_3\). Then each component of \(U\) is trivially doubly covered by \(\Phi_1\). Otherwise, there would be a loop \(\gamma'\) in \(U\) whose lift under \(\Phi_1\) is not a loop. The loop \(\gamma'\) lies in \(U_i = \partial T_{\gamma_3}\) for some stratum \(C_3\); let \(\gamma\) be the projection of \(\gamma'\) to \(C_{3\gamma}\). Then the discussion in Subsection 3.2 of the triod block over \(T_{\gamma}\) shows that \(\gamma\) has transposition holonomy, contrary to assumption.

Now write \(f = \text{terms in } C_3 + \text{terms in } C_2 = f_1 + f_2\). Without loss of generality, we may suppose \(z\) lies in the interior of \(C_3\). Therefore \(f_2\) lies in the boundary \(U\) of \(C_3\), and \(z = f_1 + f_2\) is homologous in \(C_2\) to a 1-cycle in \(U\). This is impossible, since we have just seen that any 1-cycle in \(U\) is the \((\Phi_1)_*\)-image of a 1-cycle in \(B\).

(c) Now we prove that the images \(t_\delta(H_1(A, Z))\) for all strata \(A\) are independent, in the following sense: If \(z_A\) is a 1-cycle supported by \(A\) and \(z = \sum z_A = \partial f\) for some 2-chain \(f\), then each \(z_A\) is also a boundary.

Suppose \(A\) is a \(C_3\) stratum. Let \(f_A\) be the sum of the terms of \(f\) entering \(A\). Then \(\partial f_A = z_A + w\), where \(w\) is a 1-cycle supported by \(C_3\). Then by part (a), the homology classes in \(C_3\) represented by \(z_A\) and \(w\) are 0 or of orders 2 and 3 respectively. But a nonzero element cannot be of both orders 2 and 3, so both classes must be 0.

Suppose \(A = C_2\). Let \(f_A\) be the sum of the terms of \(f\) entering the interior of \(T_3\). Then \(\partial f_A = z_A + w\) where \(w\) is supported by \(U_i \subset C_2\). The same argument as in the first case, with 2 and 3 reversed, shows that again \(z_A\) and \(w\) are boundaries.

(d) By Theorem 4.7, the rank of \(H_1(M, Z)\) is \(\ell\), and the torsion comes from stratum-supported summands. By (b) and (c), the torsion is \(\otimes Z_2 \oplus Z_3\), proving the first equation of the theorem.

(e) Let \(N\) be the double of \(M\); i.e., \(N\) is obtained by gluing to \(M\) a copy of itself along the common boundary \(B\). Then we have the Meyer–Vietoris sequence:

\[
H_\infty^m(N, Z_2) = Z_2 \rightarrow H_\infty^m(B, Z_2) = \Pi_{\ell-1} Z_2
\]

\[
\rightarrow \otimes Z_2 H_\infty^m(M, Z_2)
\]

\[
\rightarrow H_\infty^m(N, Z_2) = H^1(N, Z_2)
\]

\[
\rightarrow H_\infty^m(B, Z_2) = H^1(B, Z_2).
\]
Here we have used Poincaré duality for the manifolds $N$ and $B$. We get

$$H^1(B, \mathbb{Z}_2) = \text{Hom}(H_1(B, \mathbb{Z}), \mathbb{Z}_2) \oplus \text{Ext}(H_0(B, \mathbb{Z}), \mathbb{Z}_2) = 0 \oplus 0$$

from the cohomology universal coefficient theorem, the assumption that $H_1(B, \mathbb{Z}) = 0$, and the fact that $H_0(B, \mathbb{Z})$ is free. Similarly, $H^1(N, \mathbb{Z}_2) = \text{Hom}(H_1(N, \mathbb{Z}), \mathbb{Z}_2) \oplus 0$. We calculate $H^1(N, \mathbb{Z})$ from the Mayer–Vietoris sequence:

$$H_1(B, \mathbb{Z}) = 0 \rightarrow \bigoplus_2 H_1(M, \mathbb{Z}) \rightarrow H_1(N, \mathbb{Z})$$

$$H_0(B, \mathbb{Z}) = \bigoplus_2 H_0(M, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}.$$

The kernel of the last map is isomorphic to $\bigoplus_{\beta+1} \mathbb{Z}$. We have already calculated $H_1(M, \mathbb{Z})$, so we get a short exact sequence

$$0 \rightarrow \bigoplus_{2^\ell} \mathbb{Z} \oplus \bigoplus_{2^p} \mathbb{Z}_2 \oplus \bigoplus_{2^t} \mathbb{Z}_3 \rightarrow H_1(N, \mathbb{Z}) \rightarrow \bigoplus_{\beta+1} \mathbb{Z} \rightarrow 0,$$

which splits because $\bigoplus_{\beta+1} \mathbb{Z}$ is free [R, p. 234]. Hence

$$H_1(N, \mathbb{Z}) = \bigoplus_{2^\ell+\beta+1} \mathbb{Z} \oplus \bigoplus_{2^p} \mathbb{Z}_2 \oplus \bigoplus_{2^t} \mathbb{Z}_3$$

and so $H^1(N, \mathbb{Z}_2) = \Pi_{2^\ell+\beta+1} \mathbb{Z}_2$. Thus our original sequence reduces to a short exact sequence

$$0 \rightarrow \Pi_{\beta+1} \mathbb{Z}_2 \rightarrow \bigoplus_2 H^1_{\text{inf}}(M, \mathbb{Z}_2) \rightarrow \Pi_{2^\ell+2^p+\beta+1} \mathbb{Z}_2 \rightarrow 0,$$

from which follows the desired equation, $H^1_{\text{inf}}(M, \mathbb{Z}_2) = \Pi_{2^\ell+2^p+\beta+1} \mathbb{Z}_2$.

The last two equations follow from Poincaré duality and the universal coefficient theorem. The final statement follows from the equation $m = \ell + p + \beta - 1$ and the fact that $m$, $\ell$, and $t$ depend only on the spine graph. (As has already been discussed, these dependencies follow from Proposition 4.12 and Remark 4.11, Theorem 4.7, and Proposition 4.1, respectively.)

Theorem 1.1 of the Introduction combines Theorem 5.1 and the following corollary of Theorem 5.1:

**Corollary 5.2.** Suppose a manifold $M$ with connected boundary $B$ has a $3$-branched simple polyhedral spine. Then:

(a) $H^1(B, \mathbb{Z})$ and $H^1(M, \mathbb{Z})$ do not both vanish.  

(b) $\pi_1(B)$ and $\pi_1(M)$ are not isomorphic under the inclusion map.

**Proof.** (a) Suppose $H^1(B, \mathbb{Z}) = 0$ and $H^1(M, \mathbb{Z}) = 0$. By the first equation of Theorem 5.1, $\ell$, $p$, and $t$ vanish. Since $\beta = 1$, the dimension $m$ of $H^1_{\text{inf}}(M, \mathbb{Z}_2)$ also vanishes, by the second equation of Theorem 5.1. But
$t=0$ implies that all $C_3$ strata are trivially triply covered, and so Lemma 4.14 gives $\ell + m > 0$, a contradiction. (It is equally easy to argue directly, without Lemma 4.14, that the two preimages in $b$ of any $C_2$ vertex cannot be joined by a path in $b$ and so $B$ cannot be connected.)

(b) Let $\hat{B}$ be the boundary of the simply connected cover $\hat{M}$ of $M$. By (a), $\hat{B}$ cannot be both connected and simply connected. However, it is straightforward to verify from the homotopy lifting property that if the inclusion map of $\pi_1(B)$ into $\pi_1(M)$ is onto then $\hat{B}$ is connected and if it is one-one then $\hat{B}$ is simply connected. 

Remark 5.3. Fet and Lagunov prove a version of Corollary 5.2(a) stating, for a manifold $M$ with connected boundary $B$ and 3-branched simple polyhedral spine, that if $M$ is orientable and satisfies $H_1(B, \mathbb{Z}_2) = 0$ and $H^1_{\text{eff}}(M, \mathbb{Z}_2) = 0$, then $H_1(M, \mathbb{Z}) \neq 0$ [LF1, Lemma 19]. Their version of Corollary 5.2b (which is stated for the Euclidean case) has the added hypothesis that $\pi_2(M) = 0$ [LF1, Theorem 2]. Fet and Lagunov require added hypotheses because their work does not allow the actual calculation of homology as we have done in this paper.

For 3-manifolds, the situation is significantly simpler, because the least-dimensional strata are curves. The proof of [LF1, Theorem 3] shows that if a 3-manifold $M$ with connected boundary $B$ has a 3-branched simple polyhedral spine, then $\pi_1(B)$ is not mapped onto $\pi_1(M)$ by the inclusion map.

Remark 5.4. In the case of a compact 3-manifold $M$ with 3-branched simple polyhedral spine, Ikeda's paper [Ik, Lemma 4] contains a proof that if $M$ satisfies $H_1(M, \mathbb{Z}) = 0$ and $H_2(M, \mathbb{Z}) = \mathbb{Z}$, then $M$ is the product of a 2-sphere with an interval. Here is how our methods yield this result:

Since the hypotheses imply $H_2(M, \mathbb{Z}_2) = \mathbb{Z}_2$ and $H_1(B, \mathbb{Z}) = 0$, then Theorem 5.1 implies $\ell = p = t = 0$ and $\beta = 2$. Then every $C_k$ vertex ($k=2, 3$) of the spine graph $c$ is covered by $k$ vertices of the boundary graph $b$, and by Theorems 4.7 and 4.9, $c$ and $b$ are trees. Thus we have Euler characteristics $\chi(c) = 1$, $\chi(b) = 2$. Since every edge of $c$ is covered by two edges of $c$, by Remark 4.6, and every $C_2$ vertex of $c$ is covered by two vertices of $b$, there can be no $C_3$ vertices in $c$, from which the claim follows.

5.2. Euler Characteristic and Intermediate Homology

Suppose $M$ is a compact manifold with boundary $B$, that collapses to a $k$-branched simple polyhedral spine $C$. Since $B$ is the disjoint union of
covering spaces of the strata of $C$, we have the following relation on Euler characteristics, where $\tilde{C}_r$ is the closure of $C_r$:

$$\chi(B) = 2\chi(C_2) + 3\chi(C_3) + \cdots + k\chi(C_k)$$

$$= 2\chi(\tilde{C}_2) + \chi(\tilde{C}_3) + \cdots + \chi(\tilde{C}_{k-1}) + \chi(C_k)$$

$$= 2\chi(M) + \chi(\tilde{C}_3) + \cdots + \chi(\tilde{C}_{k-1}) + \chi(C_k).$$

In the case of 3-branched spines, this yields:

**Proposition 5.5.** Suppose $M$ is an even-dimensional compact manifold with boundary, that collapses to a 3-branched simple polyhedral spine. Then $\chi(M) = -\chi(C_3)/2$.

By analogy with Theorem 5.1, one might suppose that $B$ is a homology sphere and ask whether the intermediate homology of $M$ also suppresses the topology of the individual strata of the spine. The following proposition shows that this is not the case.

**Proposition 5.6.** Suppose $M$ is an orientable, compact, connected 4-manifold with $\beta$ simply connected boundary components, and $M$ collapses to a 3-branched simple polyhedral spine. Then

$$H_2(M, \mathbb{Z}) = \oplus_d \mathbb{Z} \oplus_p \mathbb{Z}_2 \oplus_1 \mathbb{Z}_3,$$

where $d = (\sum (g_i - 1)) - 2 + 2\ell + \beta$, and the $g_i$ are the genera of the 2-dimensional strata of the spine.

**Proof.** By Theorem 5.1,

$$\chi(M) = 1 - \ell + b_2(M) - (\ell + \beta - 1),$$

and the torsion subgroup of $H_2(M, \mathbb{Z})$ is $\oplus_p \mathbb{Z}_2 \oplus_1 \mathbb{Z}_3$. $C_3$ consists of finitely many compact surfaces $C_{3i}$, of genus $g_i$ respectively. Therefore by Proposition 5.5,

$$b_2(M) = \left(\sum (g_i - 1)\right) - 2 + 2\ell + \beta.$$  

5.3 Further work

We plan to develop these ideas further in a subsequent paper [AB2]. As was mentioned in Remark 4.4, we have a recursive generation procedure for the graph pairs of compact manifolds $M$ that collapse to 3-branched spines and have connected, simply connected boundaries. This fact and Theorem 5.1 allow us to prove that the number of $(n-1)$-dimensional and
For $3$-manifolds $M$ with spherical boundaries and 3-branched spines, we show that the homeomorphism class is “almost” determined by $H_1(M; \mathbb{Z})$. Specifically, $M$ is the connected sum of $p$ copies of $\mathbb{RP}^3$, $t$ copies of $L_{3,1}^3$, and $l$ copies of $S^2 \times S^1$. (Since $L_{3,1}^3$ is chiral, this specifies $M$ uniquely if $M$ is not orientable, and up to finitely many choices otherwise.)

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