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Dynamics of the heat semigroup in Jacobi analysis ☆

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ABSTRACT

Let Δ be the Jacobi Laplacian. We study the chaotic and hypercyclic behaviour of the strongly continuous semigroups of operators generated by perturbations of Δ with a multiple of the identity on L^p spaces.

1. Introduction

Chaos in the context of strongly continuous semigroups of bounded linear operators in Banach spaces has been introduced by W. Desch, W. Schappacher and G. Webb [5] as a generalization to continuous time of the discrete time case. In their paper [5], the authors give a sufficient condition for a strongly continuous semigroup to be chaotic in terms of the spectral properties of its infinitesimal generator. Moreover they study in detail many examples, mainly the transport equation and second order differential operators with constant coefficients (see also [4]).

The purpose of this paper is to apply the criterion in [5] to study of the dynamics of the (modified) heat semigroup generated by the Jacobi operator (i.e., the generator is a perturbation of the Jacobi operator by a multiple of the identity), which is a second order differential operator with nonconstant coefficients.

Jacobi analysis can be developed as a generalization of the Fourier-cosine transform and has been studied by many authors (see [12]), the main interest being the interplay between the analytic and geometric properties of the Jacobi operator. Indeed, in certain cases, the Jacobi operator is the radial part of the Laplace–Beltrami operator on Damek–Ricci spaces [2], therefore Jacobi analysis includes radial analysis on symmetric spaces of real rank one as a special case.

The dynamics of the (modified) heat semigroup on noncompact symmetric spaces was already studied in [9,10]. We extend the results in [9] to the context of Jacobi analysis. Therefore, as particular cases, we cover Damek–Ricci spaces and Heckman–Opdam root spaces of rank one, which were not treated in [9].

The paper is organized as follows: in Section 2 we settle notation and recall some basic facts regarding chaotic semigroups, Jacobi analysis and Lorentz spaces. In Section 3 we establish some properties of spherical functions, the weak type $(1,1)$ boundedness of the heat maximal function and the $L^{p,q}$ inversion formula. In Section 4 we apply the results in the previous section to the study of the dynamics of (modified) heat semigroups.

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2. Notation and preliminaries

2.1. Chaotic semigroups

In this paper we follow R.L. Devaney [6], who has defined chaos in metric spaces in the following sense. A continuous map f on a metric space X is said to be chaotic if it is topologically transitive, i.e., some element has a dense orbit, and if the set of its periodic points is dense in X. These two conditions imply (see [3]) that f has sensitive dependence on initial conditions.

In [5] the authors have generalized this definition to strongly continuous semigroups as follows.

Definition 1. A strongly continuous semigroup $\{T(t)\colon t \geqslant 0\}$ on a Banach space $\mathcal{X}$ is said to be *hypercyclic* if there exists f in $\mathcal{X}$ such that its orbit $\{T(t)f\colon t \geqslant 0\}$ is dense in $\mathcal{X}$.

Definition 2. A strongly continuous semigroup $\{T(t)\colon t \geqslant 0\}$ on a Banach space $\mathcal{X}$ is said to be *chaotic* if it is hypercyclic and the set of periodic points

$$\big\{f \in \mathcal{X}\colon \exists t > 0 \text{ such that } T(t)f = f\big\}$$

is dense in $\mathcal{X}$.

We denote by $\sigma(B)$ and $\sigma_{pt}(B)$ respectively the spectrum and the point spectrum of a linear operator B on a Banach space $\mathcal{X}$.

A sufficient condition for a strongly continuous semigroup to be chaotic in terms of the spectral properties of its generator was given by Desch, Schappacher, and Webb [5, Theorem 3.1]:

Theorem 2.1. *Suppose that B is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\colon t \geqslant 0\}$ on a separable Banach space $\mathcal{X}$. Let Ω be an open connected subset of $\sigma_{pt}(B)$ which intersects the imaginary axis. For each z in Ω, let ϕ_z be a nonzero eigenvector, i.e., $B\phi_z = z\phi_z$. Suppose that for every f in the dual space $\mathcal{X}'$ of $\mathcal{X}$ the function $F_f : \Omega \to \mathbb{C}$, defined by*

$$F_f(z) = \langle f, \phi_z\rangle \quad \forall z \in \Omega,$$

is analytic and does not vanish identically unless $f = 0$. Then the semigroup $\{T(t)\colon t \geqslant 0\}$ is chaotic.

Many authors studied sufficient conditions for a strongly continuous semigroups to be hypercyclic (see [9] and the references therein).

2.2. Lorentz spaces

Let f be a measurable function on the measure space $(X, \mathcal{M}, \mu)$. The *nonincreasing rearrangement* of f is the function f^* on $\mathbb{R}^+$ defined by

$$f^*(t) = \inf\big\{s \in \mathbb{R}^+\colon \mu\big(\big\{x \in X\colon \big|f(x)\big| > s\big\}\big) \leqslant t\big\} \quad \forall t \in \mathbb{R}^+.$$

The function f^* is nonincreasing, nonnegative, equimeasurable with f and right-continuous. For any given measurable function f on X, we define

$$\|f\|_{L^{p,q}} = \left(\frac{q}{p}\int_0^\infty \big(s^{1/p} f^*(s)\big)^q \frac{ds}{s}\right)^{1/q}, \quad 1 \leqslant p < \infty,\ 1 \leqslant q < \infty,$$

and

$$\|f\|_{L^{p,\infty}} = \sup\big\{s^{1/p} f^*(s)\colon s \in \mathbb{R}^+\big\}, \quad 1 \leqslant p < \infty.$$

Definition 3. Let $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$. The Lorentz space $L^{p,q}(\mu)$ consists of those measurable functions f on X such that $\|f\|_{L^{p,q}}$ is finite.

It is easy to check that $L^{p,p}(\mu)$ coincides with the usual Lebesgue space $L^p(\mu)$, with equality of norms. Moreover, if $q_1 < q_2$, then $L^{p,q_1}(\mu)$ is contained in $L^{p,q_2}(\mu)$ and, if $1 < p, q < \infty$, the dual space of $L^{p,q}(\mu)$ is $L^{p',q'}(\mu)$. Here and elsewhere in this paper p' and q' are the conjugate exponents of p and q, i.e., $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

A good reference for Lorentz spaces is [8]. We recall the following multiplication theorem from that paper.

Lemma 2.2. *(See [8, p. 271].) Let p_0, p_1 and q_0, q_1 be in $[1, \infty]$. Then there exists a constant C such that for every f in $L^{p_0,q_0}(\mu)$ and m in $L^{p_1,q_1}(\mu)$*

$$\|mf\|_{L^{p,q}} \leqslant C\|m\|_{L^{p_1,q_1}}\|f\|_{L^{p_0,q_0}},$$

where $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$ and $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$.

2.3. Jacobi analysis

We recall some facts about Jacobi analysis which we shall need in the sequel. We follow Koornwinder [12] and the normalizations therein. Throughout this paper, α, β will be real numbers, with $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, $\alpha > -\frac{1}{2}$. We define

$$A(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1} \quad \forall x > 0.$$

For complex λ and $\rho = \alpha + \beta + 1$, let

$$\Delta = -\left(\frac{d}{dx}\right)^2 - \frac{A'(x)}{A(x)}\frac{d}{dx}$$

and consider the differential equation

$$\Delta u(x) = \left(\lambda^2 + \rho^2\right)u(x), \quad x > 0. \tag{2.1}$$

Using the substitution $z = -\sinh^2 x$, we can transform Eq. (2.1) in the well-known hypergeometric differential equation

$$z(1-z)u''(z) + \left(c - (a+b+1)z\right)u'(z) - abu(z) = 0$$

of parameters $a = \frac{1}{2}(\rho - i\lambda)$, $b = \frac{1}{2}(\rho + i\lambda)$, $c = \alpha + 1$.

Let ${}_2F_1$ denote the Gaussian hypergeometric function. The Jacobi function $\varphi_\lambda = \varphi_\lambda^{(\alpha,\beta)}$ of order (α, β)

$$\varphi_\lambda(x) = {}_2F_1\left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda); \alpha + 1, -\sinh^2 x\right), \quad x \in \mathbb{R}$$

is the unique even smooth function on $\mathbb{R}$ which satisfies $u(0) = 1$ and the differential equation (2.1).

Therefore the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic for all $x \in \mathbb{R}$. Moreover $\{\varphi_\lambda\}$ is a continuous orthogonal system on $\mathbb{R}^+$ with respect to the weight function A.

We consider on $\mathbb{R}^+ = (0, \infty)$ the measure μ which is absolutely continuous with respect to the ordinary Lebesgue measure and has density A.

Note that

$$\left|A(x)\right| \leqslant C \begin{cases} x^{2\alpha+1}, & 0 < x < 1, \\ e^{2\rho x}, & x \geqslant 1. \end{cases} \tag{2.2}$$

The Jacobi transform $f \mapsto \hat{f}$ is defined by

$$\hat{f}(\lambda) = \int_{\mathbb{R}^+} f(x)\varphi_\lambda(x)\,d\mu(x),$$

for all functions f on $\mathbb{R}^+$ and complex numbers λ for which the right-hand side is well defined.

Let $\mathcal{D}^\sharp(\mathbb{R})$ be the space of smooth even functions on $\mathbb{R}$ with compact support and denote by $\mathcal{D}^\sharp(\mathbb{R}^+)$ the space of the restrictions to $\mathbb{R}^+$ of functions in $\mathcal{D}^\sharp(\mathbb{R})$. Note that

$$(\Delta f)\hat{} = \left(\lambda^2 + \rho^2\right)\hat{f} \quad \forall f \in \mathcal{D}^\sharp\left(\mathbb{R}^+\right).$$

The following inversion formula holds for functions in $\mathcal{D}^\sharp(\mathbb{R})$ [12, p. 9]

$$f(x) = \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\varphi_\lambda(x)\left|\mathbf{c}(\lambda)\right|^{-2}d\lambda, \quad \forall x \in \mathbb{R}$$

where $\mathbf{c}(\lambda)$ is a multiple of the meromorphic Harish–Chandra function given by the formula

$$\mathbf{c}(\lambda) = \frac{2^{\rho - i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\rho + i\lambda))\Gamma(\frac{1}{2}(\rho + i\lambda) - \beta)}.$$

Moreover the Jacobi transform $f \mapsto \hat{f}$ extends to an isometry from $L^2(\mu)$ onto the space $L^2(\mathbb{R}^+, \frac{1}{2\pi}|\mathbf{c}(\lambda)|^{-2}\,d\lambda)$.

The following formula holds

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(u)W(x, y, u)\, d\mu(u), \tag{2.3}$$

where the kernel W is explicitly known (see [12, p. 58]). Thus one can define the (generalized) translation operator

$$\tau_x f(y) = \int_0^\infty f(u)W(x, y, u)\, d\mu(u) \quad \forall f \in \mathcal{D}^\sharp(\mathbb{R}^+)$$

and the (generalized) convolution of functions f, g in $\mathcal{D}^\sharp(\mathbb{R}^+)$, say, by

$$f \star g(x) = \int_0^\infty \tau_x f(y)g(y)\, d\mu(y),$$

so that

$$(f \star g)\hat{} = \hat{f}\hat{g}.$$

The function W is nonnegative, supported in $|x - y| \leqslant u \leqslant x + y$ and symmetric in its three variables, so that $f \star g = g \star f$. From (2.3) with $\lambda = i\rho$ it follows

$$\int_0^\infty W(x, y, u)\, d\mu(u) = 1.$$

Moreover $\int_0^\infty \tau_x f\, d\mu = \int_0^\infty f\, d\mu$ and $\tau_0 f = f$. Finally, the Young inequality holds [12, p. 61]

$$\|f \star g\|_r \leqslant C\|f\|_p\|g\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{2.4}$$

2.4. Heat kernel

Let f be in $\mathcal{D}^\sharp(\mathbb{R}^+)$ and consider the initial value problem for the heat equation

$$\begin{cases} \partial_t u(t, x) = -\Delta u(t, x), & \\ \lim_{t \to 0^+} u(t, x) = f(x), & t > 0,\ x \in \mathbb{R}. \end{cases} \tag{2.5}$$

One can show that the solution u to the problem (2.5) can be written as $u(t, x) = f \star h_t(x)$, where the function h_t is called *heat kernel* and it is defined on the Jacobi transform side by

$$\hat{h}_t(\lambda) = e^{-t(\lambda^2+\rho^2)} \quad \forall t > 0,\ \lambda \in \mathbb{C}.$$

The heat kernel h_t is a nonnegative function in $L^p(\mu)$ for all $p \geqslant 1$ and $\|h_t\|_1 = 1$ (see [1]).

For every p in $[1, \infty]$ denote by Δ_p the closure in $L^p(\mu)$ of the operator Δ with domain $\mathcal{D}^\sharp(\mathbb{R}^+)$. By the Young inequality (2.4), the family of operators $\{H_p(t)\}_{t>0}$ defined by

$$H_p(t)f = h_t \star f \quad \forall f \in L^p(\mu)$$

is a strongly continuous symmetric semigroup on $L^p(\mu)$ whose generator is $-\Delta_p$.

The following sharp estimate for the heat kernel was established in [11]

$$h_t(x) \simeq t^{-\alpha-1}(1 + t + x)^{\alpha-\frac{1}{2}}(1 + x)e^{-\rho x-\rho^2 t}e^{-x^2/4t} \quad \forall x > 0,\ t > 0. \tag{2.6}$$

Here $f(x) \simeq g(x)$ stands for $C_1 g(x) \leqslant f(x) \leqslant C_2 g(x)$, for some positive constants C_1 and C_2.

3. The L^p inversion formula

Since for real values of λ, we have $|\varphi_\lambda| \leqslant \varphi_0 \leqslant 1$ (see [12, p. 53]), the function space

$$L^1_0(\mu) = \{f : \mathbb{R}^+ \to \mathbb{C}\colon f \text{ measurable and } f\varphi_0 \in L^1(\mu)\}$$

contains $L^1(\mu)$ and the Jacobi transform $\hat{f}$ is well defined as a function on $\mathbb{R}$ when f is in $L^1_0(\mu)$.

In the next results we show that the Lorentz spaces $L^{p,q}(\mu)$, $1 < p < 2$, $1 \leqslant q \leqslant \infty$ are subspaces of $L^1_0(\mu)$.

When $1 \leqslant p < \infty$ we define

$$S_p = \left\{\lambda \in \mathbb{C}\colon \left|\operatorname{Im}(\lambda)\right| \leqslant \left|1 - \frac{2}{p}\right|\rho\right\}.$$

Note that $S_p = S_{p'}$, when $p > 1$. By S_p° and ∂S_p we denote respectively the interior and the boundary of S_p. It is well known that from the estimate [12, p. 53]

$$\left|\varphi_\lambda(x)\right| \leqslant C(1+x)e^{(|\operatorname{Im}(\lambda)|-\rho)x} \quad \forall x \geqslant 0,\ \lambda \in \mathbb{C}, \tag{3.1}$$

it follows that if $1 < p < 2$ and λ is in S_p° then φ_λ is in $L^{p'}(\mu)$. A more precise result holds (see [14] for the case of Damek–Ricci spaces).

Lemma 3.1. *The following estimates hold.*

(i) *Let $1 < p < 2$. If λ is in S_p° then φ_λ is in $L^{p',q}(\mu)$ for any q in $[1,\infty]$.*
(ii) *If $1 \leqslant p < 2$ and λ is in ∂S_p then φ_λ is in $L^{p',\infty}(\mu)$.*
(iii) *If $p = 2$ and λ is real, then $\varphi_\lambda/(1+x)$ is in $L^{2,\infty}(\mu)$.*

Proof. Let $1 < p < 2$. It suffices to show that (i) holds with $q = 1$. By (3.1), when λ is in S_p° for some $\varepsilon > 0$,

$$\left|\varphi_\lambda(x)\right| \leqslant Ce^{-(\frac{2}{p'}+2\varepsilon)\rho x} \quad \forall x \in \mathbb{R}^+,$$

therefore

$$\varphi_\lambda^*(t) \leqslant C\psi^*(t) \quad \forall t \in \mathbb{R}^+,$$

where $\psi(x) = e^{-(\frac{2}{p'}+2\varepsilon)\rho x}$. We now compute ψ^*. By Eq. (2.2) we have

$$\psi^*(t) = \inf\left\{s \in \mathbb{R}^+\colon \int_0^{\log(s^{-1/(2/p'+2\varepsilon)\rho})} A(x)\,dx \leqslant t\right\}$$
$$\leqslant C\begin{cases} t^{-(\frac{1}{p'}+\varepsilon)}, & t > 1, \\ \text{const}, & t < 1. \end{cases}$$

We conclude that ψ^* belongs to $L^{p',q}(\mu)$ for any q in $[1,\infty]$.

The proofs of (ii) and (iii) are similar. If $p = 2$ and λ is real we apply again (3.1).

If $|\operatorname{Im}(\lambda)| = (2/p - 1)\rho$ and $1 \leqslant p < 2$, we use the inequality [7]

$$\left|\varphi_\lambda(x)\right| \leqslant C_\delta e^{-(|\operatorname{Im}(\lambda)|+\rho)x} \quad \forall x \geqslant \delta > 0, \tag{3.2}$$

and the proof is complete. □

Denote by P_p, $1 \leqslant p < \infty$, the parabolic region

$$P_p = \left\{\lambda^2 + \rho^2\colon \lambda \in S_p^\circ\right\}.$$

Corollary 3.2. *Let $2 < p < \infty$. For any z in the parabolic region P_p there exists a nonzero ϕ_z in L^p such that $\Delta\phi_z = z\phi_z$.*

Note that Corollary 3.2 implies that the parabolic region P_p is contained the point spectrum $\sigma_{pt}(\Delta_p)$, when $2 < p < \infty$. Indeed, by standard arguments, it is easy to show that $H_p(t)\varphi_\lambda = e^{-t(\lambda^2+\rho^2)}\varphi_\lambda$, so that if λ is in S_p°, then φ_λ is in the domain of Δ_p and $\Delta_p\varphi_\lambda = (\lambda^2+\rho^2)\varphi_\lambda$.

Corollary 3.3. $L^{p,q}(\mu) \subset L_0^1(\mu)$, $1 < p < 2$, $1 \leqslant q \leqslant \infty$.

Proof. Use the multiplication theorem with f in $L^{p,q}$ and $m = \varphi_0$ which is in $L^{p',q'}$. □

So, when f is in $L^{p,q}$, we can define the Jacobi transform $\hat{f}$ as a function on $\mathbb{R}$ and actually $\hat{f}$ turns out to be holomorphic in S_p°.

Corollary 3.4. *If f is in $L^{p,q}(\mu)$, $1 < p < 2$, $1 \leqslant q \leqslant \infty$, then $\hat{f}$ is a bounded function in the strip S_p and holomorphic in S_p°.*

Proof. Use Lemma 3.1 and the Morera Theorem. □

We include the next result, which we were unable to find in the literature (see [2] in the case of Damek–Ricci spaces).

Proposition 3.5. *The heat maximal operator H^* defined by*

$$H^* f(x) = \sup_{t>0} \left| h_t \star f(x) \right|, \quad f \in \mathcal{D}^\sharp\left(\mathbb{R}^+\right),\ x > 0$$

is of weak type $(1,1)$ *and of strong type* (p,p).

Proof. Since $\|h_t\|_1 = 1$ for every $t > 0$ and by (2.4), the heat maximal operator is trivially bounded on $L^\infty(\mu)$. We now prove the weak L^1-estimate. From this estimate and the Marcinkiewicz Interpolation Theorem the thesis follows.

As usual, we shall deal with the small time maximal function

$$H_0^* f(x) = \sup_{0<t\leqslant 1} \left| h_t \star f(x) \right|, \quad f \in \mathcal{D}^\sharp\left(\mathbb{R}^+\right),\ x > 0$$

and the large time maximal function

$$H_\infty^* f(x) = \sup_{t>1} \left| h_t \star f(x) \right|, \quad f \in \mathcal{D}^\sharp\left(\mathbb{R}^+\right),\ x > 0$$

separately. Note that

$$H_\infty^* f \leqslant |f| \star \Big[\sup_{t>1} h_t\Big].$$

From the heat kernel estimate (2.6) it follows that $\sup_{t>1} h_t(x) = O((1+x)^{-1/2} e^{-2\rho x})$ when x is large so that H_∞^* is of weak type $(1,1)$. Indeed, if $k(x) = \cosh(x)^{-2\rho}$, Liu [13, Lemma 3.2] proved that

$$\tau_x k(y) \leqslant C \cosh(x)^{-2\rho} \quad \forall y > 0,$$

so that

$$|f| \star k(x) = \int_0^\infty \left| f(y) \right| \tau_x k(y) A(y)\, dy \leqslant \cosh(x)^{-2\rho} \|f\|_1.$$

Therefore

$$\begin{aligned} \mu\left\{x\colon |f| \star k(x) > \lambda\right\} &\leqslant \mu\left\{x\colon (\cosh x)^{2\rho} < C\|f\|_1/\lambda\right\} \\ &= \int_0^{x_0} A(u)\, du \\ &\leqslant C(\cosh x_0)^{2\rho} = C\|f\|_1/\lambda, \end{aligned}$$

where $x_0 > 0$ is such that $(\cosh x_0)^{2\rho} = C\|f\|_1/\lambda$ (if any, otherwise $x_0 = 0$ and the weak type inequality is trivial).

In order to prove that H_0^* is of weak type $(1,1)$, we first note that the estimates of the heat kernel imply that when $0 < t \leqslant 1$,

$$\begin{aligned} &0 \leqslant h_t(x) \leqslant C e^{-x^2/4} = k_1(x), \quad x > 1, \\ &h_t(x) \simeq k_t(x), \quad 0 < x \leqslant 1, \end{aligned}$$

where

$$k_t(x) = t^{-(\alpha+1)} e^{-x^2/4t}, \quad t, x > 0.$$

Let χ denote the characteristic function of the interval $[-1,1]$, and write

$$H_0^* f(x) \leqslant \sup_{0<t\leqslant 1} \left| \big((1-\chi)h_t\big) \star f(x) \right| + \sup_{0<t\leqslant 1} \left| (\chi h_t) \star f(x) \right| \leqslant k_1 \star |f|(x) + C \sup_{0<t\leqslant 1} (\chi k_t) \star |f|(x).$$

Since the kernel k_1 is integrable, the operator $f \mapsto k_1 \star f$ is bounded on L^1.

For the estimate of the other term we will use the weak type $(1,1)$ boundedness of a Hardy–Littlewood maximal function. Let X_r denotes the characteristic function of $[-r,r]$, normalized so that $\int X_r\, d\mu = 1$ and define

$$Mf = \sup_{r>0} X_r \star |f|.$$

Liu [13] proved that the operator M is of weak type $(1,1)$.

Let $\nu(y) = \int_0^y A(x)\,dx$ and $G(y) = \int_0^y \tau_x|f|(u)A(u)\,du$. Note that $G(y) \leqslant \nu(y)M\tau_x|f|(0)$. Applying the size estimate (2.2), we obtain

$$\begin{aligned}(\chi k_t) \star |f|(x) &= \int_0^1 \tau_x|f|(y)k_t(y)\,d\mu(y)\\ &= -\int_0^1 G(y)k_t'(y)\,dy + G(1)k_t(1)\\ &\leqslant M\tau_x|f|(0)\int_0^1 \nu(y)\bigl(-k_t'(y)\bigr)\,dy + G(1)k_t(1)\\ &\leqslant M\tau_x|f|(0)\int_0^1 \nu'(y)k_t(y)\,dy\\ &= Mf(x)\int_0^1 k_t(y)A(y)\,dy \leqslant Mf(x).\end{aligned}$$

The thesis follows. □

The standard method of approximation using the heat kernel gives the following inversion formula for functions in the Lorentz space $L^{p,q}(\mu)$, with $1 < p < 2$, $q \geqslant 1$ (see [14] for the case of Damek–Ricci spaces).

Proposition 3.6. *Let f be in $L^{p,q}(\mu)$, with $1 < p < 2$, $q \geqslant 1$ or f in $L^1 \cup L^2(\mu)$. If $\hat{f}$ is in $L^1(\mathbb{R}, |\mathbf{c}(\lambda)|^{-2}\,d\lambda)$, then for almost every x in $\mathbb{R}^+$,*

$$f(x) = \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\varphi_\lambda(x)\bigl|\mathbf{c}(\lambda)\bigr|^{-2}\,d\lambda. \tag{3.3}$$

Proof. We can write $f = f_1 + f_2$, where f_1 is in $L^1(\mu)$ and f_2 is in $L^2(\mu)$ [8]. Then for every $t > 0$, we have $f \star h_t = f_1 \star h_t + f_2 \star h_t$. Since the heat maximal operator is of weak type $(1,1)$ and $g \star h_t(x) \to g(x)$, when $t \to 0$, whenever the function g is smooth and compactly supported, it follows that there exist measurable sets E_1, E_2 of null measure such that $f_1 \star h_t(x) \to f_1(x)$ for all x in ${}^cE_1$ and $f_2 \star h_t(x) \to f_2(x)$ for all x in ${}^cE_2$, as $t \to 0$.

This implies that if x is not in $E_1 \cup E_2$ then $f \star h_t(x) \to f(x)$ as $t \to 0$. Moreover $\mu(E) \leqslant \mu(E_1) + \mu(E_2) = 0$.

Since $\hat{f}$ is in $L^1(\mathbb{R}, |\mathbf{c}(\lambda)|^{-2}\,d\lambda)$, for every test function ψ we have

$$\langle f \star h_t, \psi\rangle = \frac{1}{2\pi}\int_0^\infty\int_0^\infty \hat{f}(\lambda)e^{-t(\lambda^2+\rho^2)}\varphi_\lambda(x)\bigl|\mathbf{c}(\lambda)\bigr|^{-2}\,d\lambda\,\psi(x)\,d\mu(x).$$

Using the Dominated Convergence Theorem we now get the result. □

4. Dynamics of the heat semigroup

We recall that for every p in $[1,\infty]$ we denote by Δ_p the closure in $L^p(\mu)$ of the operator Δ with domain $\mathcal{D}^\sharp(\mathbb{R}^+)$. As previously observed, the family $\{H_p(t)\}_{t>0}$, where

$$H_p(t)f = e^{-t\Delta_p}f = h_t \star f \quad \forall f \in L^p(\mu)$$

is a strongly continuous semigroup on $L^p(\mu)$.

In this section we study the dynamics of the shifted semigroup

$$e^{-t(\Delta_p-\theta)} : L^p(\mu) \to L^p(\mu),$$

where θ is real.

Theorem 4.1. *Let $2 < p < \infty$. Then for all $\theta > \theta_p = \rho^2 - \rho^2(\frac{2}{p} - 1)^2$ the semigroup*

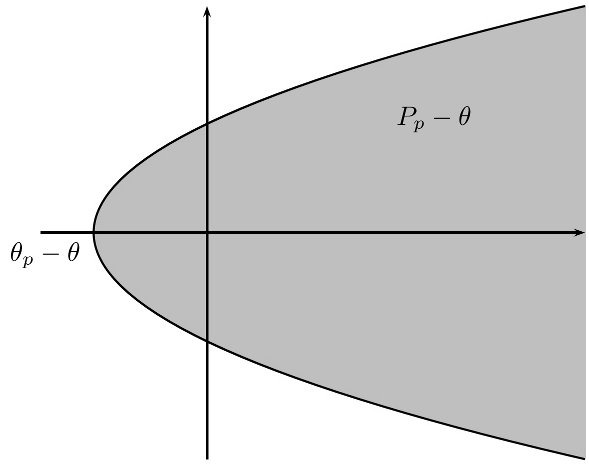
$$e^{-t(\Delta_p - \theta)} : L^p(\mu) \to L^p(\mu)$$

is chaotic.

Proof. We apply Theorem 2.1 with $B = -(\Delta_p - \theta)$ as infinitesimal generator of the strongly continuous semigroup $T(t) = e^{-t(\Delta_p - \theta)}$ on the separable Banach space $\mathcal{X} = L^p(\mu)$. By Corollary 3.2, the parabolic region P_p is contained in the point spectrum $\sigma_{pt}(\Delta_p)$ and the corresponding eigenfunctions are given by the appropriate Jacobi functions. The vertex of the parabolic region P_p is at the point

$$\theta_p = \rho^2 - \rho^2 \left|\frac{2}{p} - 1\right|^2 = 4\rho^2 \frac{1}{pp'}.$$

Hence the point spectrum of $(\Delta_p - \theta)$ intersects the imaginary axis for all $\theta > \theta_p$.



Suppose now that $\theta > \theta_p$ is fixed and let Ω_θ denote the set $(P_p - \theta) \setminus \{z \in \mathbb{R}\colon z \leqslant \rho^2 - \theta\}$, i.e.,

$$\Omega_\theta = \left\{ z \in \mathbb{C}\colon z = \lambda^2 + \rho^2 - \theta,\ \left|\operatorname{Im}(\lambda)\right| < \left(1 - \frac{2}{p}\right)\rho \right\} \setminus \left\{ z \in \mathbb{R}\colon z \leqslant \rho^2 - \theta \right\}.$$

Then $\Omega = -\Omega_\theta$ is an open, connected subset of the point spectrum of $B = -(\Delta_p - \theta)$ that intersects the imaginary axis.

Since $(\Omega_\theta + \theta - \rho^2) \cap \{x \in \mathbb{R}\colon x < 0\} = \emptyset$ we choose an analytic branch of the square root so that $\operatorname{Re}\sqrt{z + \theta - \rho^2} > 0$, for every z in Ω_θ. Note that $z \mapsto \sqrt{z + \theta - \rho^2}$ maps Ω_θ onto the open strip

$$\left\{ \lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > 0,\ \left|\operatorname{Im}(\lambda)\right| < \rho\left(1 - \frac{2}{p}\right) \right\}.$$

For every z in Ω_θ we choose the eigenfunction ϕ_z defined by

$$\phi_z = \varphi_\lambda \quad \text{where } z = \lambda^2 + \rho^2 - \theta, \tag{4.1}$$

so that $(\Delta_p - \theta)\phi_z = z\phi_z$. As in Theorem 2.1, for every f in $\mathcal{X}' = L^{p'}(\mu)$, we define the function $F_f : \Omega = -\Omega_\theta \to \mathbb{C}$ by

$$F_f(z) = \langle f, \phi_{-z} \rangle = \hat{f}\left(\sqrt{-z + \theta - \rho^2}\right).$$

Then by Corollary 3.4 the function F_f is holomorphic since it is the composition of two holomorphic functions. Moreover $F_f = 0$ implies $f = 0$ by the inversion formula (3.3). □

In the following theorem we prove that the semigroup

$$e^{-t(\Delta_p - \theta)} : L^p(\mu) \to L^p(\mu)$$

is not chaotic when $1 < p \leqslant 2$ for every θ in $\mathbb{R}$.

Theorem 4.2. *Let $1 < p \leqslant 2$ and θ in $\mathbb{R}$. Then the semigroup*

$$e^{-t(\Delta_p - \theta)} : L^p(\mu) \to L^p(\mu)$$

does not have periodic elements. Moreover when $1 < p < 2$ it is not hypercyclic.

Proof. Let $1 < p \leqslant 2$, θ in $\mathbb{R}$ and f a periodic point in $L^p(\mu)$ for $e^{-t(\Delta_p - \theta)}$. Then there exists $t > 0$ such that $e^{t\theta} h_t \star f = f$ or equivalently $(e^{-t(\lambda^2+\rho^2-\theta)} - 1)\hat{f}(\lambda) = 0$ for every λ in S_p^o when $1 < p < 2$ and for almost every real λ when $p = 2$. By the inversion formula (3.3), we obtain that $f = 0$ and the first part follows.

Let $1 < p < 2$ and assume that the semigroup $e^{-t(\Delta_p - \theta)}$ is hypercyclic. Then, the dual operator $(\Delta_p - \theta)' = \Delta_{p'} - \theta$ of its generator would have empty point spectrum [5, Theorem 3.3]. This a contradiction and the thesis follows. □

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