Almost automorphic solutions for some partial functional differential equations ✩

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Abstract

In this work, we study the existence of almost automorphic solutions for some partial functional differential equations. We prove that the existence of a bounded solution on \( \mathbb{R}^+ \) implies the existence of an almost automorphic solution. Our results extend the classical known theorem by Bohr and Neugebauer on the existence of almost periodic solutions for inhomegeneous linear almost periodic differential equations. We give some applications to hyperbolic equations and Lotka–Volterra type equations used to describe the evolution of a single diffusive animal species.

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1. Introduction

The aim here is to study the existence of almost automorphic solutions for the following partial functional differential equation

\[
\begin{aligned}
\frac{du(t)}{dt} &= Au(t) + L(u_t(t)) + f(t), \quad \text{for } t \geq 0, \\
u_0 &= \varphi \in C := C([-r, 0]; X),
\end{aligned}
\]

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where $A$ is a linear operator on a Banach space $X$ not necessarily densely defined and satisfies the Hille–Yosida condition: there exist $M \geq 0$, $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$
(\lambda - A)^{-n} \leq \frac{M}{(\lambda - \omega)^n}, \quad \text{for } n \in \mathbb{N}, \lambda > \omega,
$$

where $\rho(A)$ is the resolvent set of $A$ and $C$ is the space of continuous functions from $[-r, 0]$ to $X$ endowed with the uniform norm topology. $L$ is a bounded linear operator from $C$ to $X$ and $f$ is an almost automorphic function from $\mathbb{R}$ to $X$, the history function $u_t \in C$ is defined by

$$
u_t(\theta) = u(t + \theta), \quad \text{for } \theta \in [-r, 0].$$

We prove that the existence of a bounded solution on $\mathbb{R}^+$ implies the existence of an almost automorphic solution of Eq. (1.1). The achievement of this goal will be done through several results. We employ the variation of constants formula obtained in [2], we develop new fundamental results about the spectral analysis of the solutions and we establish a new principle reduction which allows us to prove the existence of an almost automorphic solution.

Almost automorphic functions are more general than almost periodic functions and they were introduced by S. Bochner [4,5], for more details about this topics we refer to the recent book [11] where the author gave an important overview about the theory of almost automorphic functions and their applications to differential equations. In [11], the author proved the existence of almost automorphic solution for the following ordinary differential equation

$$
\frac{d}{dt} x(t) = Gx(t) + e(t), \quad t \in \mathbb{R},
$$

where $G$ is a constant $(n \times n)$-matrix and $e : \mathbb{R} \to \mathbb{R}^n$ is almost automorphic. He proved that the existence of a bounded solution on $\mathbb{R}^+$ implies the existence of an almost automorphic solution. The existence of almost automorphic solutions for differential equations in infinite dimensional space has been studied by several authors. Recently in [8], the authors established the existence of almost automorphic solutions for functional differential equations of neutral type, they proved that the existence of a bounded solution on $\mathbb{R}^+$ implies the existence of an almost automorphic solution. In [10], the authors studied the existence of almost automorphic solutions for the following partial functional differential equations with infinite delay

$$
\begin{cases}
\frac{dx}{dt}(t) = Dx(t) + \mathcal{L}(t)x_t + \mathcal{K}(t), & \text{for } t \geq 0, \\
x_0 = \varphi \in \mathcal{B},
\end{cases}
$$

where $\mathcal{D}$ is the generator of a strongly continuous semigroup of linear operators on a Banach space $E$ which is equivalent by Hille–Yosida’s theorem that $\mathcal{D}$ satisfies the Hille–Yosida condition and $\mathcal{D}(\mathcal{D}) = E$. The phase space $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into $E$ satisfying some axioms introduced by Hale and Kato [10], for all $t \geq 0$, $\mathcal{L}(t)$ is a bounded linear operator form $\mathcal{B}$ to $E$ and periodic in $t$. For all $t \geq 0$, the history function $x_t \in \mathcal{B}$ is defined by

$$
x_t(\theta) = x(t + \theta), \quad \text{for } \theta \leq 0.
$$

The function $\mathcal{K}$ is an almost automorphic function from $\mathbb{R}$ to $E$. The authors proved that the existence of a bounded mild solution on $\mathbb{R}^+$ of Eq. (1.3) is equivalent to the existence of an almost automorphic solution. In [12], the author studied the existence of almost automorphic solutions for the following semilinear abstract differential equation

$$
\frac{d}{dt} x(t) = Cx(t) + \theta(t, x(t)), \quad \text{for } t \geq 0,
$$
where $C$ generates an exponentially stable semigroup on a Banach space $Y$ and $\theta$ is an almost automorphic function from $\mathbb{R}$ to $Y$. The author proved that the only bounded mild solution of Eq. (1.4) on $\mathbb{R} \times Y$ is almost automorphic. In [13,14], the authors proved the existence of almost automorphic solutions for some non-autonomous inhomogeneous linear evolution equation in Banach spaces.

This work is organized as follows: In Section 2, we give the variation of constants formula that will be used in the whole of this work. In Section 3, we establish new fundamental results about the spectral decomposition of solutions of Eq. (1.1). In Section 4, we recall some results about almost automorphic functions. In Section 5, we prove the existence of an almost automorphic solution of Eq. (1.1). In hyperbolic case, we prove that Eq. (1.1) has a unique bounded solution which is almost automorphic. To illustrate our approach, we propose to study the Lotka–Volterra model with diffusion.

2. A variation of constants formula

Throughout this work, we suppose that

\[(H_0)\quad A \text{ satisfies the Hille–Yosida condition.}\]

We consider the following results which are taken from [1].

**Definition 2.1.** [1] We say that a continuous function $u$ from $[-r, \infty)$ into $X$ is an integral solution of Eq. (1.1), if the following conditions hold:

\[(i)\quad \int_0^t u(s) \, ds \in D(A), \quad \text{for} \ t \geq 0;\]

\[(ii)\quad u(t) = \varphi(0) + A \int_0^t u(s) \, ds + \int_0^t (L(u_s) + f(s)) \, ds, \quad \text{for} \ t \geq 0;\]

\[(iii)\quad u_0 = \varphi.\]

If $\overline{D(A)} = X$, the integral solutions coincide with the known mild solutions. From the closedness property of $A$, we can see that if $u$ is an integral solution of Eq. (1.1), then $u(t) \in \overline{D(A)}$ for all $t \geq 0$, in particular $\varphi(0) \in \overline{D(A)}$. Let us introduce the part $A_0$ of the operator $A$ in $\overline{D(A)}$ which defined by

\[
\left\{ D(A_0) = \{ x \in D(A): \ Ax \in \overline{D(A)} \} ,
A_0 x = Ax, \quad \text{for} \ x \in D(A_0). \right.\]

**Lemma 2.2.** [3] $A_0$ generates a strongly continuous semigroup $(T_0(t))_{t \geq 0}$ on $\overline{D(A)}$.

For the existence of the integral solutions, one has the following result.
Theorem 2.3. [1] Assume that $(H_0)$ holds, then for all $\varphi \in C$ such that $\varphi(0) \in \overline{D(A)}$, Eq. (1.1) has a unique integral solution $u$ on $[-r, +\infty)$. Moreover, $u$ is given by

$$u(t) = T_0(t)\varphi(0) + \lim_{\lambda \to +\infty} \int_0^t T_0(t-s)B_\lambda(L(u_s) + f(s))\,ds, \quad \text{for } t \geq 0,$$

where $B_\lambda = \lambda R(\lambda - A)^{-1}$, for $\lambda > \omega$.

In the sequel of this work, for simplicity, integral solutions are called solutions.

The phase space $C_0$ of Eq. (1.1) is defined by

$$C_0 = \{ \varphi \in C : \varphi(0) \in \overline{D(A)} \}.$$

For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on $C_0$ by

$$\mathcal{U}(t)\varphi = v_t(., \varphi),$$

where $v(., \varphi)$ is the solution of the following homogeneous equation

$$\frac{d}{dt}v(t) = Av(t) + L(v_t), \quad \text{for } t \geq 0,$$

$$v_0 = \varphi \in C.$$

Proposition 2.4. [1] $(\mathcal{U}(t))_{t \geq 0}$ is a strongly continuous semigroup of linear operators on $C_0$:

(i) for all $t \geq 0$, $\mathcal{U}(t)$ is a bounded linear operator on $C_0$;
(ii) $\mathcal{U}(0) = I$;
(iii) $\mathcal{U}(t+s) = \mathcal{U}(t)\mathcal{U}(s)$, for all $t, s \geq 0$;
(iv) for all $\varphi \in C_0$, $\mathcal{U}(t)\varphi$ is a continuous function of $t \geq 0$ with values in $C_0$.

Moreover,

(v) $(\mathcal{U}(t))_{t \geq 0}$ satisfies, for $t \geq 0$ and $\theta \in [-r, 0]$, the following translation property:

$$(\mathcal{U}(t)\varphi)(\theta) = \begin{cases} 
(\mathcal{U}(t+\theta)\varphi)(0), & \text{if } t + \theta \geq 0, \\
\varphi(t + \theta), & \text{if } t + \theta \leq 0.
\end{cases}$$

Theorem 2.5. [2, Theorem 3] Let $A_u$ be defined on $C_0$ by

$$D(A_u) = \{ \varphi \in C^1([-r, 0]; X) : \varphi(0) \in D(A), \varphi'(0) \in \overline{D(A)} \text{ and } \varphi'(0) = A\varphi(0) + L(\varphi) \},$$

$$A_u\varphi = \varphi', \quad \text{for } \varphi \in D(A_u).$$

Then, $A_u$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t \geq 0}$ on $C_0$.

In order to give the variation of constants formula, we need to recall some notations and results which are taken from [2]. Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{ X_0c : c \in X \},$$

where the function $X_0c$ is defined by

$$(X_0c)(\theta) = \begin{cases} 
0 & \text{if } \theta \in [-r, 0), \\
c & \text{if } \theta = 0.
\end{cases}$$
The space $C_0 \oplus \langle X_0 \rangle$ is equipped with the norm $\|\phi + X_0c\| = |\phi|_C + |c|$ for $(\phi, c) \in C_0 \times X$, is a Banach space and consider the extension $\tilde{A}_U$ of the operator $A_u$ defined on $C_0 \oplus \langle X_0 \rangle$ by

$$D(\tilde{A}_U) = \{ \phi \in C^1([-r, 0]; X): \phi(0) \in D(A) \text{ and } \phi'(0) \in D(A) \},$$

$$\tilde{A}_U \phi = \phi' + X_0(A\phi(0) + L\phi - \phi'(0)).$$

**Lemma 2.6.** [2, Theorem 13 and Lemma 15] Assume that $(H_0)$ holds. Then, $\tilde{A}_U$ satisfies the Hille–Yosida condition on $C_0 \oplus \langle X_0 \rangle$: there exist $\tilde{M} \geq 0, \tilde{\omega} \in \mathbb{R}$ such that $(\tilde{\omega}, +\infty) \subset \rho(\tilde{A}_U)$ and

$$\| (\lambda - \tilde{A}_U)^{-n} \| \leq \frac{\tilde{M}}{(\lambda - \tilde{\omega})^n}, \text{ for } n \in \mathbb{N}, \lambda > \tilde{\omega}.$$ 

Moreover, the part of $\tilde{A}_U$ on $D(\tilde{A}_U) = C_0$ is exactly the operator $A_u$.

**Theorem 2.7.** [2, Theorem 16] Assume that $(H_0)$ holds. Then, for all $\phi \in C_0$, the solution $u$ of Eq. (1.1) is given by the following variation of constants formula

$$u_t = U(t)\phi + \lim_{\lambda \to +\infty} \int_0^t U(t-s)\tilde{B}_\lambda(X_0 f(s)) \, ds, \text{ for } t \geq 0,$$

where $\tilde{B}_\lambda = \lambda(\lambda - \tilde{A}_U)^{-1}$, for $\lambda > \tilde{\omega}$.

3. Spectral analysis and decomposition

In the following, we assume that

$(H_1)$ the operator $T_0(t)$ is compact on $D(A)$, for every $t > 0$.

**Theorem 3.1.** [2] Assume that $(H_0)$ and $(H_1)$ hold, then $U(t)$ is compact for $t > r$.

As a consequence from the compactness property of the operator $U(t)$, we have that the spectrum $\sigma(A_u)$ is the point spectrum and we can see that

$$\sigma(A_u) = \{ \lambda \in \mathbb{C}: \ker \Delta(\lambda) \neq \{0\} \},$$

where the linear operator $\Delta(\lambda) : D(A) \rightarrow E$ is defined by

$$\Delta(\lambda) = \lambda I - A - L(e^{\lambda \cdot}I),$$

and $e^{\lambda \cdot} : E \rightarrow C$, is defined by

$$(e^{\lambda \cdot} x)(\theta) = e^{\lambda \theta} x, \quad x \in E \text{ and } \theta \in [-r, 0].$$

From [7], we have the following spectral decomposition result.

**Corollary 3.2.** [2] $C_0$ is decomposed as follows:

$$C_0 = S \oplus V,$$

where $S$ is $U$-invariant and there are positive constants $\alpha$ and $N$ such that

$$\|U(t)\phi\|_C \leq Ne^{-\alpha t} |\phi|_C, \quad \text{for each } t \geq 0 \text{ and } \phi \in S. \quad (3.1)$$

$V$ is a finite dimensional space and the restriction of $U$ to $V$ is a group.
In the sequel, \( U_s(t) \) and \( U_v(t) \) denote the restriction of \( U(t) \) respectively on \( S \) and \( V \) which correspond to the above decomposition.

Let \( d = \dim V \) with a vector basis \( \Phi = \{\phi_1, \ldots, \phi_d\} \). Then, there exist \( d \)-elements \( \{\psi_1, \ldots, \psi_d\} \) in \( C_0^* \) such that

\[
\begin{cases}
\langle \psi_i, \phi_j \rangle = \delta_{ij}, \\
\langle \psi_i, \phi \rangle = 0,
\end{cases}
\]

for all \( \phi \in S \) and \( i \in \{1, \ldots, d\} \),

(3.2)

where \( \langle \ldots, \ldots \rangle \) denotes the duality pairing between \( C_0^* \) and \( C_0 \) and \( \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \)

Let \( \Psi = \text{col}\{\psi_1, \ldots, \psi_d\} \). \( \langle \Psi, \Phi \rangle \) is a \((d \times d)\)-matrix, where the \((i, j)\)-component is \( \langle \psi_i, \phi_j \rangle \).

Denote by \( \Pi^s \) and \( \Pi^v \) the projections respectively on \( S \) and \( V \). For each \( \phi \in C_0 \), we have

\[ \Pi^v \phi = \Phi \langle \Psi, \phi \rangle. \]

In fact, for \( \phi \in C_0 \), we have \( \phi = \Pi^s \phi + \Pi^v \phi \) with \( \Pi^v \phi = \sum_{i=1}^d \alpha_i \phi_i \) and \( \alpha_i \in \mathbb{R} \). By (3.2), we conclude that

\[ \alpha_i = \langle \psi_i, \phi \rangle. \]

Hence

\[ \Pi^v \phi = \sum_{i=1}^d \langle \psi_i, \phi \rangle \phi_i = \Phi \langle \Psi, \phi \rangle. \]

Since \( (U^v(t))_{t \geq 0} \) is a group on \( V \), then there exists a \((d \times d)\)-matrix \( G \) such that

\[ U^v(t) \Phi = \Phi e^{tG}, \quad \text{for } t \in \mathbb{R}. \]

Moreover, \( \sigma(G) = \{\lambda \in \sigma(A_u) : \text{Re}(\lambda) \geq 0\} \).

For \( n, n_0 \in N \) such that \( n \geq n_0 \geq \tilde{\omega} \) and \( i \in \{1, \ldots, d\} \), we define the linear operator \( x_{i,n}^* \) by

\[ x_{i,n}^*(a) = \langle \psi_i, \tilde{B}_n X_0 a \rangle, \quad \text{for } a \in X. \]

Since \( |\tilde{B}_n| \leq \frac{n}{n-n_0} \tilde{M} \), for any \( n \geq n_0 \), then \( x_{i,n}^* \) is a bounded linear operator from \( X \) to \( \mathbb{R} \) such that

\[ |x_{i,n}^*| \leq \frac{n}{n-n_0} \tilde{M} |\psi_i|, \quad \text{for any } n \geq n_0. \]

Define the \( d \)-column vector \( x_n^* = \text{col}(x_{1,n}^*, \ldots, x_{d,n}^*) \), then one can see that

\[ \langle x_n^*, a \rangle = \langle \Psi, \tilde{B}_n X_0 a \rangle, \quad \text{for } a \in X, \]

with

\[ \langle x_n^*, a \rangle_i = \langle \psi_i, \tilde{B}_n X_0 a \rangle, \quad \text{for } i = 1, \ldots, d, \quad \text{and } a \in X. \]

Consequently,

\[ \sup_{n \geq n_0} |x_n^*| < \infty, \]

which implies that \( (x_n^*)_{n \geq n_0} \) is a bounded sequence in \( L(X, \mathbb{R}^d) \). Then, we get the following important result of this work.
Theorem 3.3. There exists $x^* \in \mathcal{L}(X, \mathbb{R}^d)$, such that $(x^*_n)_{n \geq n_0}$ converges weakly to $x^*$ in the sense that

$$\langle x^*_n, x \rangle \to \langle x^*, x \rangle, \quad \text{as } n \to \infty, \text{ for all } x \in X.$$  

For the proof, we need the following fundamental theorem:

Theorem 3.4. [15, p. 776] Let $Y$ be any separable Banach space and $(z^*_n)_{n \geq 0}$ any bounded sequence in $Y^*$. Then, there exists a subsequence $(z^*_{n_k})_{k \geq 0}$ of $(z^*_n)_{n \geq 0}$ which converges weakly in $Y^*$ in the sense that there exists $z^* \in Y^*$ such that

$$\langle z^*_{n_k}, x \rangle \to \langle z^*, x \rangle, \quad \text{as } n \to \infty, \text{ for all } x \in Y.$$  

Proof. Let $Z_0$ be any closed separable subspace of $X$. Since $(x^*_n)_{n \geq n_0}$ is a bounded sequence, then by Theorem 3.4 we get that the sequence $(x^*_n)_{n \geq n_0}$ has a subsequence $(x^*_{n_{k_p}})_{p \in \mathbb{N}}$ which converges weakly to some $x^*_Z$ in $Z_0$. We claim that all the sequence $(x^*_n)_{n \geq n_0}$ converges weakly to $x^*_Z$ in $Z_0$. In fact, we proceed by contradiction and suppose that there exists a subsequence $(x^*_{n_p})_{p \in \mathbb{N}}$ of $(x^*_n)_{n \geq n_0}$ which converges weakly to some $\tilde{x}^*_Z$ with $\tilde{x}^*_Z \neq x^*_Z$. Let $u_t(., \sigma, \varphi, f)$ denote the solution of Eq. (1.1). Then,

$$\Pi^u u_t(., \sigma, 0, f) = \lim_{n \to +\infty} \Phi \int_{t}^{t} e(t - \xi) G \langle x^*_n, f(\xi) \rangle d\xi,$$

and

$$\Pi^u (\widetilde{B}_n X_0 f(\xi)) = \Phi \langle \psi, \widetilde{B}_n X_0 f(\xi) \rangle = \Phi \langle x^*_n, f(\xi) \rangle.$$  

It follows that

$$\Pi^u u_t(., \sigma, 0, f) = \lim_{n \to +\infty} \Phi \int_{t}^{t} e(t - \xi) G \langle x^*_n, f(\xi) \rangle d\xi,$$

and

$$= \lim_{n \to +\infty} \Phi \int_{t}^{t} e(t - \xi) G \langle x^*_n, f(\xi) \rangle d\xi.$$  

For any $a \in Z_0$, set $f(.) = a$, then

$$\lim_{k \to +\infty} \int_{\sigma}^{t} e(t - \xi) G \langle x^*_{n_k}, a \rangle d\xi = \lim_{p \to +\infty} \int_{\sigma}^{t} e(t - \xi) G \langle x^*_{n_p}, a \rangle d\xi, \quad \text{for } a \in Z_0,$$

which implies that

$$\int_{\sigma}^{t} e(t - \xi) G \langle x^*_Z, a \rangle d\xi = \int_{\sigma}^{t} e(t - \xi) G \langle \tilde{x}^*_Z, a \rangle d\xi, \quad \text{for } a \in Z_0,$$

consequently $x^*_Z = \tilde{x}^*_Z$, which gives a contradiction. We conclude that the whole sequence $(x^*_n)_{n \geq n_0}$ converges weakly to $x^*_Z$ in $Z_0$. 


Let $Z_1$ be another closed separable subspace of $X$, by using the same argument as above, we get that $(x^*_n)_{n\geq n_0}$ converges weakly to $x^*_1$ in $Z_1$. Since $Z_0 \cap Z_1$ is a closed separable subspace of $X$, we get that $x^*_1 \equiv x^*_0$ in $Z_0 \cap Z_1$. For any $x \in X$, we define $x^*$ by
\[ \langle x^*, x \rangle = \langle x^*_{Z_1}, x \rangle, \]
where $Z$ is any closed separable subspace of $X$ such that $x \in Z$. Then $x^*$ is well defined on $X$ and $x^*$ is a bounded linear operator from $X$ to $\mathbb{R}^d$ such that
\[ |x^*| \leq \sup_{n \geq n_0} |x^*_n| < \infty, \]
and $(x^*_n)_{n \geq n_0}$ converges weakly to $x^*$ in $X$. \qed

As a consequence, we conclude that

**Corollary 3.5.** For any continuous function $h : \mathbb{R} \to X$, we have
\[ \lim_{n \to +\infty} \int_{-\infty}^{t} U^v(t - \xi) \Pi^v \left( \tilde{B}_n X_0 h(\xi) \right) d\xi = \Phi \int_{-\infty}^{t} e^{(t - \xi)^G} \langle x^*, h(\xi) \rangle d\xi, \quad \text{for all } t, \sigma \in \mathbb{R}. \]

As a consequence of the above, we establish the following fundamental reduction principle which allows us to prove the existence of an almost automorphic solution.

**Theorem 3.6.** Assume that $(H_0)$ and $(H_1)$ hold. Let $u$ be a solution of Eq. (1.1) on $\mathbb{R}$. Then, $z(t) = \langle \Psi, u_t \rangle$ is a solution of the ordinary differential equation
\[ \frac{d}{dt} z(t) = Gz(t) + \langle x^*, f(t) \rangle, \quad t \in \mathbb{R}. \] (3.3)

Conversely, if $f$ is a bounded function on $\mathbb{R}$ and $z$ is a solution of Eq. (3.3) on $\mathbb{R}$, then the function $u$ given by
\[ u(t) = \left[ \Phi z(t) + \lim_{n \to +\infty} \int_{-\infty}^{t} U^v(t - \xi) \Pi^v \left( \tilde{B}_n X_0 f(\xi) \right) d\xi \right](0), \quad \text{for } t \in \mathbb{R}, \]
is a solution of Eq. (1.1) on $\mathbb{R}$.

Let $u$ be a solution of Eq. (1.1) on $\mathbb{R}$. Then,
\[ u_t = \Pi^x u_t + \Pi^v u_t, \quad \text{for all } t \in \mathbb{R}, \]
and
\[ \Pi^v u_t \equiv \mathcal{U}^v(t - \sigma) \Pi^v u_{t\sigma} + \lim_{n \to +\infty} \int_{\sigma}^{t} \mathcal{U}^v(t - \xi) \Pi^v \left( \tilde{B}_n X_0 f(\xi) \right) d\xi, \quad \text{for } t, \sigma \in \mathbb{R}. \]

Since $\Pi^v u_t = \Phi \langle \Psi, u_t \rangle$ and by Corollary 3.5, we get that
\[ \Phi(\Psi, u_t) = \mathcal{U}^v(t - \sigma) \Phi(\Psi, u_\sigma) + \Phi \int_{\sigma}^{t} e^{(t-\xi)G} \{x^*, f(\xi)\} d\xi \]

\[ = \Phi e^{(t-\sigma)G} \langle \Psi, u_\sigma \rangle + \Phi \int_{\sigma}^{t} e^{(t-\xi)G} \{x^*, f(\xi)\} d\xi, \quad \text{for } t, \sigma \in \mathbb{R}. \]

Let \( z(t) = \langle \Psi, u_t \rangle \). Then,

\[ z(t) = e^{(t-\sigma)G} z(\sigma) + \int_{\sigma}^{t} e^{(t-\xi)G} \{x^*, f(\xi)\} d\xi, \quad \text{for } t, \sigma \in \mathbb{R}. \]

Consequently, \( z \) is a solution of the ordinary differential equation (3.3) on \( \mathbb{R} \).

Conversely, assume that \( f \) is bounded on \( \mathbb{R} \), then \( \int_{-\infty}^{t} U^\delta(t - \xi) \Pi^\delta(\widetilde{B}_n X_0 f(\xi)) d\xi \) is well defined on \( \mathbb{R} \). Let \( z \) be a solution of (3.3) on \( \mathbb{R} \) and \( v \) be defined by

\[ v(t) = \Phi z(t) + \lim_{n \to +\infty} \int_{-\infty}^{t} U^\delta(t - \xi) \Pi^\delta(\widetilde{B}_n X_0 f(\xi)) d\xi, \quad \text{for } t \in \mathbb{R}. \]

Since

\[ z(t) = e^{(t-\sigma)G} z(\sigma) + \int_{\sigma}^{t} e^{(t-\xi)G} \{x^*, f(\xi)\} d\xi, \quad \text{for } t, \sigma \in \mathbb{R}. \]

Using Corollary 3.5, the function \( v_1 \) given by

\[ v_1(t) = \Phi z(t), \quad \text{for } t \in \mathbb{R}, \]

satisfies

\[ v_1(t) = \mathcal{U}^v(t - \sigma) v_1(\sigma) + \lim_{n \to +\infty} \int_{\sigma}^{t} \mathcal{U}^v(t - \xi) \Pi^v(\widetilde{B}_n X_0 f(\xi)) d\xi, \quad \text{for } t, \sigma \in \mathbb{R}. \]

Moreover, the function \( v_2 \) given by

\[ v_2(t) = \lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^\delta(t - \xi) \Pi^\delta(\widetilde{B}_n X_0 f(\xi)) d\xi, \quad \text{for } t \in \mathbb{R}, \]

satisfies

\[ v_2(t) = \mathcal{U}^\delta(t - \sigma) v_2(\sigma) + \lim_{n \to +\infty} \int_{\sigma}^{t} \mathcal{U}^\delta(t - \xi) \Pi^\delta(\widetilde{B}_n X_0 f(\xi)) d\xi, \quad \text{for } t \geq \sigma. \]

Then, for all \( t \geq \sigma \) with \( t, \sigma \in \mathbb{R} \), one has

\[ \mathcal{U}(t - \sigma) v(\sigma) = \mathcal{U}^v(t - \sigma) v_1(\sigma) + \mathcal{U}^\delta(t - \sigma) v_2(\sigma), \]

\[ = v_1(t) - \lim_{n \to +\infty} \int_{\sigma}^{t} \mathcal{U}^v(t - \xi) \Pi^v(\widetilde{B}_n X_0 f(\xi)) d\xi + v_2(t) \]
\[
- \lim_{n \to +\infty} \int_{\sigma}^{t} \mathcal{U}(t - \xi) \Pi s(\tilde{B}_n X_0 f(\xi)) \, d\xi
= v(t) - \lim_{n \to +\infty} \int_{\sigma}^{t} \mathcal{U}(t - \xi) \Pi s(\tilde{B}_n X_0 f(\xi)) \, d\xi.
\]

Therefore
\[
v(t) = \mathcal{U}(t - \sigma) v(\sigma) + \lim_{n \to +\infty} \int_{\sigma}^{t} \mathcal{U}(t - \xi) \Pi s(\tilde{B}_n X_0 f(\xi)) \, d\xi,
\]
for \( t \geq \sigma \).

By Theorem 2.7, we obtain that the function \( u \) defined by \( u(t) = v(t)(0) \) is a solution of Eq. (1.1) on \( \mathbb{R} \).

4. Almost periodic and almost automorphic functions

We recall some properties about almost automorphic functions. Let \( BC(\mathbb{R}, X) \) be the space of bounded continuous functions from \( \mathbb{R} \) to \( X \), provided with the uniform norm topology. Let \( h \in BC(\mathbb{R}, X) \) and \( \tau \in \mathbb{R} \), we define the function \( h_\tau \) by
\[
h_\tau(s) = h(\tau + s), \quad \text{for } s \in \mathbb{R}.
\]

**Definition 4.1.** [9] A bounded continuous function \( h : \mathbb{R} \to X \) is said to be almost periodic if
\[
\{h_\tau : \tau \in \mathbb{R}\}
\]
is relatively compact in \( BC(\mathbb{R}, X) \).

**Definition 4.2.** (Bochner [11]) A continuous function \( h : \mathbb{R} \to X \) is said to be almost automorphic if for every sequence of real numbers \((s'_n)_n\) there exists a subsequence \((s_n)_n\) such that
\[
\lim_{n \to \infty} h(t + s_n) = k(t) \quad \text{exists for all } t \text{ in } \mathbb{R}
\]
and
\[
\lim_{n \to \infty} k(t - s_n) = h(t), \quad \text{for all } t \text{ in } \mathbb{R}.
\]

**Remark.** By the pointwise convergence, the function \( k \) is just measurable and not necessarily continuous. If the convergence in both limits is uniform, then \( h \) is almost periodic.

If \( h \) is almost automorphic, then its range is relatively compact. Let \( p(t) = 2 + \cos t + \cos \sqrt{2}t \) and \( h : \mathbb{R} \to \mathbb{R} \) such that \( h = \sin \frac{1}{t} \). Then \( h \) is almost automorphic, but \( h \) is not uniformly continuous on \( \mathbb{R} \). It follows that \( h \notin AP(X) \), the Banach space of all almost periodic \( X \)-valued functions. The concept of almost automorphy is more larger than almost periodicity.

**Definition 4.3.** (Bochner [11]) A continuous function \( h : \mathbb{R} \to X \) is said to be compact almost automorphic if for every sequence of real numbers \((s'_n)_n\), there exists a subsequence \((s_n)_n\) such that
\[
\lim_{m \to \infty} \lim_{n \to \infty} h(t + s_n - s_m) = h(t) \quad \text{exists uniformly on any compact set in } \mathbb{R}.
\]
Theorem 4.4. [11] If we equip $AA(X)$, the space of almost automorphic $X$-valued functions with the sup norm, then $AA(X)$ turns out to be a Banach space.

The following theorem provides sufficient and necessary condition for the existence of almost automorphic solutions of Eq. (1.2).

Theorem 4.5. [11, Theorem 5.8, p. 86] Assume that $e$ is an almost automorphic function. If Eq. (1.2) has a bounded solution on $\mathbb{R}^+$, then it an almost automorphic solution. Moreover, every bounded solution of Eq. (1.2) on $\mathbb{R}$ is almost automorphic.

5. Almost automorphic solutions of Eq. (1.1)

In the following, we assume that

(H2) $f$ is an almost automorphic function.

Theorem 5.1. Assume that (H0)–(H2) hold. If Eq. (1.1) has a bounded solution on $\mathbb{R}^+$, then it has an almost automorphic solution.

Proof. Let $u$ be a bounded solution of Eq. (1.1) on $\mathbb{R}^+$. By Theorem 3.6, the function $z(t) = \langle \Psi, u_t \rangle$, for $t \geq 0$, is a solution of the ordinary differential equation (3.3) and $z$ is bounded on $\mathbb{R}^+$. Moreover, the function

$$\varrho(t) = \langle x^*, f(t) \rangle,$$

for $t \in \mathbb{R}$,

is almost automorphic from $\mathbb{R}$ to $\mathbb{R}^d$. By Theorem 4.5, we get that the reduced system (3.3) has an almost automorphic solution $\tilde{z}$ and $\Phi_{\tilde{z}}(.)$ is an almost automorphic function on $\mathbb{R}$. From Theorem 3.6, we know that the function $u(t) = v(t)(0)$, where

$$v(t) = \Phi_{\tilde{z}}(t) + \lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t - \xi)\Pi^s(\tilde{B}_nX_0f(\xi))d\xi,$$

is a solution of Eq. (1.1) on $\mathbb{R}$. We claim that $v$ is almost automorphic. In fact, let $y$ be defined by

$$y(t) = \lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t - \xi)\Pi^s(\tilde{B}_nX_0f(\xi))d\xi,$$

for $t \in \mathbb{R}$.

Since $f$ is almost automorphic, then for any sequence of real numbers $(\alpha'_p)_{p \geq 0}$ there exists a subsequence $(\alpha_p)_{p \geq 0}$ of $(\alpha'_p)_{p \geq 0}$ such that

$$\lim_{p \to \infty} f(t + \alpha_p) = \tilde{f}(t), \quad \text{for } t \in \mathbb{R},$$

and

$$\lim_{p \to \infty} \tilde{f}(t - \alpha_p) = f(t), \quad \text{for } t \in \mathbb{R}.$$
Now
\[ y(t + \alpha_p) = \lim_{n \to +\infty} \int_{-\infty}^{t+\alpha_p} \mathcal{U}^s(t + \alpha_p - \xi) \Pi^s(\tilde{B}_n X_0 f(\xi)) \, d\xi, \quad \text{for } t \in \mathbb{R}, \]
which gives that
\[ y(t + \alpha_p) = \lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t - \xi) \Pi^s(\tilde{B}_n X_0 f(\xi + \alpha_p)) \, d\xi, \quad \text{for } t \in \mathbb{R}, \]
By the Lebesgue’s dominated convergence theorem, we get that
\[ y(t + \alpha_p) \to w(t) \quad \text{as } p \to \infty, \]
where \( w \) is given by
\[ w(t) = \lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t - \xi) \Pi^s(\tilde{B}_n X_0 f(\xi)) \, d\xi, \quad \text{for } t \in \mathbb{R}. \]
Using same argument as above, we can prove that
\[ w(t - \alpha_p) \to \lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t - \xi) \Pi^s(\tilde{B}_n X_0 f(\xi)) \, d\xi \quad \text{as } p \to \infty. \]
This holds for any sequence \((\alpha'_p)_{p \geq 0}\), which implies that \( y \) is almost automorphic. Consequently, \( v \) is also an almost automorphic solution of Eq. (1.1).

6. Hyperbolic case

**Definition 6.1.** We say that the semigroup \((\mathcal{U}(t))_{t \geq 0}\) is hyperbolic if
\[ \sigma(A_U) \cap i\mathbb{R} = \emptyset. \]

From the compactness of the semigroup \((\mathcal{U}(t))_{t \geq 0}\) and from [7], we get the following result on the spectral decomposition of the phase space \( C_0 \).

**Theorem 6.2.** Assume that \((H_1)\) holds. If the semigroup \((\mathcal{U}(t))_{t \geq 0}\) is hyperbolic, then the space \( C_0 \) is decomposed as a direct sum \( C_0 = S \oplus U \) of two \( \mathcal{U}(t) \) invariant closed subspaces \( S \) and \( U \) such that the restricted semigroup on \( U \) is a group and there exist positive constants \( M \) and \( \nu \) such that
\[
|\mathcal{U}(t)\varphi| \leq \frac{M}{\nu} e^{-\epsilon \nu} |\varphi|, \quad t \geq 0, \ \varphi \in S,
\]
\[
|\mathcal{U}(t)\varphi| \leq \frac{M}{\nu} e^{\epsilon \nu} |\varphi|, \quad t \leq 0, \ \varphi \in U.
\]
As a consequence of the hyperbolicity we get the uniqueness of the bounded solution of Eq. (1.1).

**Theorem 6.3.** Assume that \((H_1)\) holds and the semigroup \((\mathcal{U}(t))_{t \geq 0}\) is hyperbolic. If \( f \) is bounded on \( \mathbb{R} \), then Eq. (1.1) has a unique bounded solution on \( \mathbb{R} \) which is almost automorphic if \( f \) is almost automorphic.
Proof. Since the semigroup \((U(t))_{t \geq 0}\) is hyperbolic, then Eq. (1.1) has one and only bounded solution on \(\mathbb{R}\) which is given by the following formula

\[
\left( \lim_{n \to +\infty} \int_{-\infty}^{t} U^s(t - \xi) \Pi^s(\tilde{B}_n X_0 f(\xi)) \, d\xi + \lim_{n \to +\infty} \int_{+\infty}^{t} U^u(t - \xi) \Pi^u(\tilde{B}_n X_0 f(\xi)) \, d\xi \right)(0),
\]

\(t \in \mathbb{R}\).

By Theorem 3.6, we obtain this solution is almost automorphic when \(f\) is almost automorphic.

7. Lotka–Volterra equation

To illustrate the above results, we consider the following partial functional differential equations with diffusion which describes the evolution of a single diffusive animal species with population density \(v\). For more details, about this model, we refer to [16].

\[
\begin{cases}
\frac{\partial}{\partial t} v(t, x) = \frac{\partial^2}{\partial x^2} v(t, x) + \int_{-r}^{0} g(\theta)v(t + \theta, x) \, d\theta + h(t, x), & \text{for } t \geq 0 \text{ and } x \in [0, \pi], \\
v(t, 0) = v(t, \pi) = 0, & \text{for } t \geq 0, \\
v(\theta, x) = \varphi(\theta, x), & \text{for } \theta \in [-r, 0] \text{ and } x \in [0, \pi],
\end{cases}
\]

(7.1)

where \(g: [-r, 0] \to \mathbb{R}, \varphi: [-r, 0] \times [0, \pi] \to \mathbb{R}\) and \(h: \mathbb{R} \times [0, \pi] \to \mathbb{R}\) are continuous functions.

Let \(X = C([0, \pi]; \mathbb{R})\) be the space of continuous functions from \([0, \pi]\) to \(\mathbb{R}\) endowed with the uniform norm topology. Define the operator \(A: D(A) \subset X \to X\) by

\[
D(A) = \{ y \in C^2([0, \pi]; \mathbb{R}) : y(0) = y(\pi) = 0 \},
\]

\[\text{Ay} = y''.\]

Lemma 7.1. [6]

\((0, +\infty) \subset \rho(A)\) and \(|(\lambda - A)^{-1}| \leq \frac{1}{\lambda}, \text{ for } \lambda > 0.\)

Moreover,

\[\overline{D(A)} = \{ y \in X : y(0) = y(\pi) = 0 \}.\]

Consequently, condition \((H_0)\) is satisfied.

In order to rewrite Eq. (7.1) in the abstract form (1.1), we introduce the following functions \(L: C \to X\) by

\[L(\phi)(x) = \int_{-r}^{0} g(\theta)\phi(\theta)(x) \, d\theta, \text{ for } x \in [0, \pi] \text{ and } \phi \in C,\]

and \(f : \mathbb{R} \to X\) is defined by

\[f(t)(x) = h(t, x), \text{ for } t \in \mathbb{R} \text{ and } x \in [0, \pi].\]

Then, \(L\) is a bounded linear operator from \(C\) to \(X\) and from continuity of \(h\) we get that \(f\) is a continuous function from \(\mathbb{R}\) to \(X\). With the above changes, Eq. (7.1) takes the abstract form (1.1).
Let $A_0$ be the part of $A$ in $\overline{D(A)}$. Then, $A_0$ is given by

$$D(A_0) = \left\{ y \in C^2([0, \pi]; \mathbb{R}) : y(0) = y(\pi) = y''(0) = y''(\pi) = 0 \right\},$$

$$A_0 y = Ay, \quad \text{for } y \in D(A_0).$$

It is well known from [7, Example 1.4.34, p. 123], that $A_0$ generates a strongly continuous compact semigroup $(T_0(t))_{t \geq 0}$ on $\overline{D(A)}$. Let $\varphi \in C([-r, 0] \times [0, \pi]; \mathbb{R})$ be such that $\varphi(0, 0) = \varphi(0, \pi) = 0$, then by Theorem 2.3, we get that Eq. (1.1) has a unique solution on $[-r, +\infty)$.

In order to study the existence of almost automorphic solution of Eq. (1.1), we suppose that

(H3) $h$ is bounded and almost automorphic in $t$ uniformly for $x \in [0, \pi]$, which means that there exists a measurable function $k : \mathbb{R} \times [0, \pi] \to \mathbb{R}$ such that

$$\lim_{n \to \infty} h(t + sn, x) = k(t, x) \quad \text{exists for all } t \in \mathbb{R} \text{ and uniformly in } x \in [0, \pi]$$

and

$$\lim_{n \to \infty} k(t - sn, x) = h(t, x), \quad \text{for all } t \in \mathbb{R} \text{ and uniformly in } x \in [0, \pi].$$

Moreover, we suppose that

(H4) $\int_{-r}^{0} |g(\theta)| d\theta < 1$.

Proposition 7.2. Assume that (H3) and (H4) hold. Then, Eq. (1.1) has a unique bounded solution on $\mathbb{R}$ which is almost automorphic.

Proof. We claim that the solution semigroup $(U(t))_{t \geq 0}$ is hyperbolic. In fact, let $\lambda \in \sigma(A_U)$, then there exists $x \in D(A), x \neq 0$ such that $\Delta(\lambda)x = 0$. Which implies that

$$\lambda x - Ax - \left( \int_{-r}^{0} g(\theta)e^{\lambda \theta} d\theta \right)x = 0$$

and

$$\lambda - \int_{-r}^{0} g(\theta)e^{\lambda \theta} d\theta \in \sigma_p(A).$$

On the other hand, the point spectrum $\sigma_p(A)$ of $A$ is given by

$$\sigma_p(A) = \{-n^2 : n \in \mathbb{N}^*\}.$$ 

Consequently, $\lambda \in \sigma(A_U)$ if and only if

$$\lambda - \int_{-r}^{0} g(\theta)e^{\lambda \theta} d\theta = -n^2, \quad \text{for some } n \geq 1. \quad (7.2)$$
Taking the real part in formula (7.2), we obtain that
\[ \text{Re}(\lambda) = \int_{-r}^{0} g(\theta) e^{\text{Re}(\lambda \theta)} \cos(\text{Im}(\lambda \theta)) \, d\theta - n^2, \quad n \geq 1. \]

Assume that \( \text{Re}(\lambda) \geq 0 \), then
\[ \text{Re}(\lambda) \leq \int_{-r}^{0} |g(\theta)| \, d\theta - 1 < 0, \]
this gives a contradiction. Consequently, \( \sigma(A_U) \subset \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) < 0 \} \), which implies that the semigroup \( (U(t))_{t \geq 0} \) is hyperbolic. By Theorem 6.3, we deduce that Eq. (1.1) has a unique bounded solution which is almost automorphic. \( \square \)

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