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Non(anti)commutativity for open superstrings

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Abstract

Non(anti)commutativity in an open free superstring and also one moving in a background antisymmetric tensor field is investigated. In both cases, the non(anti)commutativity is shown to be a direct consequence of the nontrivial boundary conditions which, contrary to several approaches, are not treated as constraints. The above non(anti)commutative structures lead to new results in the algebra of superconstraints which still remain involutive, indicating the internal consistency of our analysis.

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1. Introduction

In the last few years, there has been a considerable interest in the study of open strings propagating in the presence of a background Neveu–Schwarz two-form field $B_{\mu\nu}$, leading to a noncommutative structure [1,2]. This structure manifests in the noncommutativity in the spacetime coordinates of D-branes, where the end points of the open strings are attached. Different approaches have been adopted to obtain this result in the case of both the bosonic as well as the fermionic superstring. A Hamiltonian operator treatment was provided in [4] and a world sheet approach in [5]. These studies have been done in the bosonic theory. An alternative Hamiltonian (Dirac [6]) approach based on regarding the Boundary Conditions (BC) as constraints was given in [9,10], investigations being

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carried out in both the bosonic and fermionic string theories. The interpretation of the BC as primary constraints usually lead to an infinite tower of second class constraints [11], in contrast to the usual Dirac formulation of constrained systems [6,12]. Besides, in this approach, where one tries to obtain noncommutativity through Dirac brackets between coordinates, one encounters ambiguous factor like $\delta(0)$. Furthermore, different results are obtained depending on the interpretations of these factors [9].

On the other hand, it has also been shown, by one of the authors, that noncommutativity can be obtained in a more transparent manner by modifying the canonical Poisson bracket structure, so that it is compatible with the boundary condition [7]. In this approach, the boundary conditions are *not* treated as constraints. This is similar in spirit to the treatment of Hanson, Regge and Teitelboim [12], where modified PBs were obtained for the free NG string, in the orthonormal gauge, which is the counterpart of the conformal gauge in the free Polyakov string. Those studies were, however, restricted to the case of the bosonic string and membrane only. We extend the same methodology to the superstring in this Letter.

Some other approaches to this problem have also been discussed in [13,14]. As has been stressed in [1], it is very important to understand this noncommutativity from different perspectives.

We find that the super-Virasoro constraints play a crucial role in revealing the non(anti)commutative structure. The Letter is organized as follows. In Section 2, the RNS superstring action in the conformal gauge is discussed. This also helps to fix the notations. In Section 3, the boundary conditions of the fermionic sector of the superstring is given and the non-anticommutativity of the theory is revealed in the conventional Hamiltonian framework. The results are also tied up with the bosonic theory. Section 4 discusses the non(anti)commutativity in the interacting superstring theory in the RNS formulation. The Letter ends with a conclusion in Section 5.

2. Free superstring

Let us consider the action for the free superstring, in conformal gauge [3],

$$S = \frac{i}{4}\int_\Sigma d^2\sigma\, d^2\theta\, \left(\bar{D}Y^\mu DY_\mu\right), \tag{1}$$

where the superfield

$$Y^\mu(\sigma,\theta) = X^\mu(\sigma) + \bar{\theta}\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma) \tag{2}$$

unites the bosonic ($X^\mu(\sigma)$) and fermionic ($\psi^\mu(\sigma)$) spacetime string coordinates with a new auxiliary bosonic field $B^\mu(\sigma)$.

In component form the action reads[1]

$$S = -\frac{1}{2}\int_\Sigma d^2\sigma\, \left(\eta_{\mu\nu}\partial_a X^\mu \partial^a X^\nu - i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu\right) = S_B + S_F, \tag{3}$$

where

$$S_B = -\frac{1}{2}\int_\Sigma d^2\sigma\, \eta_{\mu\nu}\partial_a X^\mu \partial^a X^\nu, \qquad S_F = \frac{1}{2}\int_\Sigma d^2\sigma\, i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu \tag{4}$$

[1] Our conventions are: $\rho^0 = \sigma^2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\rho^1 = i\sigma^1 = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. Our signature of the induced world-sheet metric and target spacetime metric are $\eta^{ab} = \{-,+\}$, $\eta^{\mu\nu} = \{-,+,+,\cdots,+\}$, respectively, and $\bar{\theta}$ is defined as $\bar{\theta} = \theta^{\mathrm{T}}\rho^0$.

represent the decoupled bosonic and fermionic actions, respectively. The fermions are taken to be Majorana and we refer to the component of ψ as $\psi_\pm$ (compatible with our conventions)

$$\psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix}. \tag{5}$$

The equal time canonical antibrackets read, in terms of the components of ψ,

$$\begin{aligned} &\{\psi_+^\mu(\sigma), \psi_+^\nu(\sigma')\}_{\mathrm{DB}} = \{\psi_-^\mu(\sigma), \psi_-^\nu(\sigma')\}_{\mathrm{DB}} = -i\eta^{\mu\nu}\delta(\sigma-\sigma'), \\ &\{\psi_+^\mu(\sigma), \psi_-^\nu(\sigma')\}_{\mathrm{DB}} = 0. \end{aligned} \tag{6}$$

This, along with the brackets

$$\{X^\mu(\sigma), \Pi^\nu(\sigma')\} = \eta^{\mu\nu}\delta(\sigma-\sigma') \tag{7}$$

from the bosonic sector, defines the preliminary symplectic structure of the theory (Π^μ is the canonically conjugate momentum to X^μ, defined in the usual way).

Confining our attention to S_F (4), we vary the action (4)

$$\delta S_F = i\int_\Sigma d^2\sigma\,\big[\rho^a\partial_a\psi^\mu\delta\bar{\psi}_\mu - \partial_\sigma\big(\psi_-^\mu\delta\psi_{\mu-} - \psi_+^\mu\delta\psi_{\mu+}\big)\big] \tag{8}$$

to obtain the Euler–Lagrange equation for the fermionic field

$$i\rho^a\partial_a\psi^\mu = 0. \tag{9}$$

The total divergence term yields the necessary BC. We shall consider its consequences in the following sections where the preliminary (anti)brackets will be modified. Using the standard Noether procedure,[2] the forms of the supercurrent and the energy–momentum tensor (which are constraints themselves [3]) can be derived. The expressions are:

$$J_a = -\frac{1}{2}\rho^b\rho_a\psi^\mu\partial_b X_\mu = 0, \tag{10}$$

$$T_{ab} = \partial_a X^\mu\partial_b X_\mu - \frac{i}{4}\bar{\psi}^\mu\rho_a\partial_b\psi_\mu - \frac{i}{4}\bar{\psi}^\mu\rho_b\partial_a\psi_\mu - \frac{1}{2}\eta_{ab}\left(\partial^c X^\mu\partial_c X_\mu + \frac{i}{2}\bar{\psi}^\mu\rho^a\partial_a\psi_\mu\right) = 0. \tag{11}$$

All the components of T_{ab} are, however, not independent as the energy–momentum tensor is traceless

$$T^a{}_a = \eta^{ab}T_{ab} = 0, \tag{12}$$

leaving us with only two independent components of T_{ab}. These components, which are the constraints of the theory, are given by

$$\begin{aligned} &\chi_1(\sigma) = 2T_{00} = 2T_{11} = \Phi_1(\sigma) + \lambda_1(\sigma) = 0, \\ &\chi_2(\sigma) = T_{01} = \Phi_2(\sigma) + \lambda_2(\sigma) = 0, \end{aligned} \tag{13}$$

where

$$\begin{aligned} &\Phi_1(\sigma) = \Pi^2(\sigma) + \big(\partial_\sigma X(\sigma)\big)^2, \qquad \Phi_2(\sigma) = \Pi(\sigma)\partial_\sigma X(\sigma), \\ &\lambda_1(\sigma) = -i\bar{\psi}^\mu(\sigma)\rho_1\partial_\sigma\psi_\mu(\sigma) = -i\big(\psi_-^\mu(\sigma)\partial_\sigma\psi_{\mu-}(\sigma) - \psi_+^\mu(\sigma)\partial_\sigma\psi_{\mu+}(\sigma)\big), \\ &\lambda_2(\sigma) = -\frac{i}{2}\bar{\psi}^\mu(\sigma)\rho_0\partial_\sigma\psi_\mu(\sigma) = \frac{i}{2}\big(\psi_-^\mu(\sigma)\partial_\sigma\psi_{\mu-}(\sigma) + \psi_+^\mu(\sigma)\partial_\sigma\psi_{\mu+}(\sigma)\big). \end{aligned} \tag{14}$$

[2] We now use the supersymmetry transformations on-shell and hence we drop the auxiliary field B^μ henceforth.

The role of these constraints in generating those infinitesimal diffeomorphisms which do not lead out of the conformal gauge is well known [3] and we are not going to elaborate on this. Note that the constraints that we obtain in this Letter are on-shell, i.e., we have used the equation of motion (9) for the fermionic field ψ. This allows us to write them down in terms of the phase-space variables[3] and hence they look quite different from the standard results found in the literature [3] where they are written down in the light-cone coordinates which involves time derivatives.

From the basic brackets (6), it is easy to generate a closed (involutive) algebra:

$$\begin{aligned}
&\big\{\chi_1(\sigma), \chi_1(\sigma')\big\} = 4\big(\chi_2(\sigma) + \chi_2(\sigma')\big)\partial_\sigma\delta(\sigma - \sigma'),\\
&\big\{\chi_2(\sigma), \chi_2(\sigma')\big\} = \big(\chi_2(\sigma) + \chi_2(\sigma')\big)\partial_\sigma\delta(\sigma - \sigma'),\\
&\big\{\chi_2(\sigma), \chi_1(\sigma')\big\} = \big(\chi_1(\sigma) + \chi_1(\sigma')\big)\partial_\sigma\delta(\sigma - \sigma').
\end{aligned} \tag{15}$$

It is interesting to observe that the structure of the superconstraint algebra is exactly similar to the bosonic theory [7].

Coming to the supercurrent J_{aA},[4] note that it is a two component spinor. Further, since J_a obeys the relation $\rho^a J_a = 0$, the components of J_{0A} and J_{1A} are related to each other. Hence, we only deal with the components of J_{0A} or simply J_1 and J_2. These are[5]:

$$\begin{aligned}
&\tilde{J}_1(\sigma) = 2J_1(\sigma) = \psi_-^\mu(\sigma)\Pi_\mu(\sigma) - \psi_-^\mu(\sigma)\partial_\sigma X_\mu = 0,\\
&\tilde{J}_2(\sigma) = 2J_2(\sigma) = \psi_+^\mu(\sigma)\Pi_\mu(\sigma) + \psi_+^\mu(\sigma)\partial_\sigma X_\mu = 0.
\end{aligned} \tag{16}$$

The algebra between the above constraints read:

$$\begin{aligned}
&\big\{\tilde{J}_1(\sigma), \tilde{J}_1(\sigma')\big\} = -i\big(\chi_1(\sigma) - 2\chi_2(\sigma)\big)\delta(\sigma - \sigma'),\\
&\big\{\tilde{J}_2(\sigma), \tilde{J}_2(\sigma')\big\} = -i\big(\chi_1(\sigma) + 2\chi_2(\sigma)\big)\delta(\sigma - \sigma'),\\
&\big\{\tilde{J}_1(\sigma), \tilde{J}_2(\sigma')\big\} = 0.
\end{aligned} \tag{17}$$

The algebra between $\tilde{J}(\sigma)$ and $\chi(\sigma)$ is also given by

$$\begin{aligned}
&\big\{\chi_1(\sigma), \tilde{J}_1(\sigma')\big\} = -\big(2\tilde{J}_1(\sigma) + \tilde{J}_1(\sigma')\big)\partial_\sigma\delta(\sigma - \sigma'),\\
&\big\{\chi_1(\sigma), \tilde{J}_2(\sigma')\big\} = \big(2\tilde{J}_2(\sigma) + \tilde{J}_2(\sigma')\big)\partial_\sigma\delta(\sigma - \sigma'),\\
&\big\{\chi_2(\sigma), \tilde{J}_1(\sigma')\big\} = \left(\tilde{J}_1(\sigma) + \frac{1}{2}\tilde{J}_1(\sigma')\right)\partial_\sigma\delta(\sigma - \sigma'),\\
&\big\{\chi_2(\sigma), \tilde{J}_2(\sigma')\big\} = \left(\tilde{J}_2(\sigma) + \frac{1}{2}\tilde{J}_2(\sigma')\right)\partial_\sigma\delta(\sigma - \sigma').
\end{aligned} \tag{18}$$

3. Boundary conditions, super-Virasoro algebra and non(anti)commutativity

As in the case of bosonic variables [7], fermionic coordinates also require careful consideration of the surface terms arising in the variation of the action (8).[6] Vanishing of these surface terms requires that $(\psi_+\delta\psi_+ - \psi_-\delta\psi_-)$ should vanish at each end point of the open string. This is satisfied by making $\psi_+ = \pm\psi_-$ at each end. Without

[3] This is in the true spirit of Dirac's classic analysis of constrained Hamiltonian dynamics [6].

[4] $A = 1, 2$ being the spinor index.

[5] J_1, J_2 along with $\chi_1(\sigma)$ and $\chi_2(\sigma)$ constitutes the full set of super-Virasoro constraints.

[6] A detailed treatment of the boundary conditions is given in [3].

loss of generality we set

$$\psi_+^\mu(0,\tau)=\psi_-^\mu(0,\tau). \tag{19}$$

The relative sign at the other end now becomes meaningful and there are two cases to be considered. In the first case (Ramond (R) boundary conditions)

$$\psi_+^\mu(\pi,\tau)=\psi_-^\mu(\pi,\tau), \tag{20}$$

and in the second case (Neveu–Schwarz (NS) boundary conditions)

$$\psi_+^\mu(\pi,\tau)=-\psi_-^\mu(\pi,\tau). \tag{21}$$

Here we will work with Ramond boundary conditions. Combining (19) and (20) we can write

$$\left(\psi_+^\mu(\tau,\sigma)-\psi_-^\mu(\tau,\sigma)\right)\Big|_{\sigma=0,\pi}=0. \tag{22}$$

The mode expansion of the components of Majorana fermion takes the form [3]

$$\psi_-^\mu(\sigma,\tau)=\frac{1}{\sqrt{2}}\sum_{n\in Z}d_n^\mu e^{-in(\tau-\sigma)}, \qquad \psi_+^\mu(\sigma,\tau)=\frac{1}{\sqrt{2}}\sum_{n\in Z}d_n^\mu e^{-in(\tau+\sigma)}. \tag{23}$$

The above mode expansions immediately leads to

$$\psi_-^\mu(-\sigma,\tau)=\psi_+^\mu(\sigma,\tau), \tag{24}$$

which further yields, using (20),

$$\psi_\pm^\mu(\sigma=-\pi,\tau)=\psi_\pm^\mu(\sigma=\pi,\tau) \tag{25}$$

in the R-sector.[7]

In the bosonic sector, on the other hand, we have to enlarge the domain of definition of the bosonic field X^μ as

$$X^\mu(\tau,-\sigma)=X^\mu(\tau,\sigma), \tag{26}$$

so that it is an even function and satisfies Neumann BC [7]. This is in contrast to (24). Consistent with this, we must have

$$\Pi^\mu(\tau,-\sigma)=\Pi^\mu(\tau,\sigma), \qquad \left(X^\mu\right)'(\tau,-\sigma)=-X^{\mu\prime}(\tau,\sigma). \tag{27}$$

Now, from (24), (26), (27), we note that the constraints $\chi_1(\sigma)=0$ and $\chi_2(\sigma)=0$ are even and odd, respectively, under $\sigma\to-\sigma$. This also enables us to increase the domain of definition of the length of the string from $(0\leqslant\sigma\leqslant\pi)$ to $(-\pi\leqslant\sigma\leqslant\pi)$.

We may then write the generator of all τ and σ reparametrization as the functional [12]

$$L[f]=\frac{1}{2}\int_0^\pi d\sigma\,\left\{f_+(\sigma)\chi_1(\sigma)+2f_-(\sigma)\chi_2(\sigma)\right\}, \tag{28}$$

where $f_\pm(\sigma)=\frac{1}{2}(f(\sigma)\pm f(-\sigma))$ are by construction even/odd function and $f(\sigma)$ is an arbitrary differentiable function defined in the extended interval $[-\pi,\pi]$. The above expression can be simplified to

$$L[f]=\frac{1}{4}\int_{-\pi}^{\pi} d\sigma\, f(\sigma)\left[\left\{\Pi(\sigma)+\partial_\sigma X(\sigma)\right\}^2+2i\psi_+^\mu\partial_\sigma\psi_{\mu+}\right]. \tag{29}$$

[7] In the NS sector, we obtain a antiperiodic boundary condition $\psi_-^\mu(-\sigma,\tau)=-\psi_-^\mu(\sigma,\tau)$ at $\sigma=\pi$.

Coming to the generators J_1 and J_2, note that $J_1(-\sigma) = J_2(\sigma)$ (16). This enables us to write down the functional $G[g]$

$$G[g] = \int_0^\pi d\sigma\,\big(g(\sigma)J_1(\sigma) + g(-\sigma)J_2(\sigma)\big) = \int_{-\pi}^{\pi} d\sigma\, g(\sigma)J_1(\sigma) = \int_{-\pi}^{\pi} d\sigma\, g(-\sigma)J_2(\sigma) \tag{30}$$

for any differentiable function $g(\sigma)$, defined again in the extended interval $[-\pi,\pi]$. These functionals (29), (30) generate the following super-Virasoro algebra:

$$\begin{aligned}
&\big\{L\big[f(\sigma)\big], L\big[g(\sigma)\big]\big\} = L\big[f(\sigma)g'(\sigma) - f'(\sigma)g(\sigma)\big],\\
&\big\{G\big[g(\sigma)\big], G\big[h(\sigma)\big]\big\} = -iL\big[g(-\sigma)h(-\sigma)\big],\\
&\big\{L\big[f(\sigma)\big], G\big[g(\sigma)\big]\big\} = G\left[f(\sigma)g'(-\sigma) - \frac{1}{2}f'(\sigma)g(-\sigma)\right].
\end{aligned} \tag{31}$$

Defining

$$L_m = L\big[e^{-im\sigma}\big] \quad \text{and} \quad G_n = G\big[e^{in\sigma}\big], \tag{32}$$

one can write down an equivalent form of the super-Virasoro algebra

$$\{L_m, L_n\} = i(m-n)L_{m+n}, \qquad \{G_m, G_n\} = -iL_{m+n}, \qquad \{L_m, G_n\} = i\left(\frac{m}{2} - n\right)G_{m+n}. \tag{33}$$

Note that we do not have a central extension here, as the analysis is entirely classical.

Coming back to the preliminary symplectic structure, given in (6), we note that the boundary conditions (22) are not compatible with the brackets, although one could get the super-Virasoro algebra (31) or (33) just by using (6) and (7). Hence, the last of the brackets in (6) should be altered suitably. A simple inspection suggests that

$$\big\{\psi_+^\mu(\sigma), \psi_-^\nu(\sigma')\big\} = -i\eta^{\mu\nu}\delta(\sigma - \sigma'). \tag{34}$$

Although the bracket structures (6) and (34) agree with [8] (in the free case), they can, however, not be regarded as the final ones. This is because the presence of the usual Dirac delta function $\delta(\sigma-\sigma')$ implicitly implies that the finite physical range of $\sigma \in [0,\pi]$ for the string has not been taken into account. Besides, it is also not compatible with (24). In [7], the equal time commutators were given in terms of certain combinations $(\Delta_+(\sigma,\sigma'))$ of periodic delta function[8]

$$\big\{X^\mu(\tau,\sigma), \Pi_\nu(\tau,\sigma')\big\} = \delta^\mu_\nu \Delta_+(\sigma,\sigma'), \tag{35}$$

where

$$\Delta_\pm(\sigma,\sigma') = \delta_P(\sigma-\sigma') \pm \delta_P(\sigma+\sigma'), \tag{36}$$

rather than an ordinary delta function to ensure compatibility with Neumann BC in the bosonic sector. Basically, there one has to identify the appropriate "delta function" for the physical range $[0,\pi]$ of σ starting from the periodic delta function $\delta_P(\sigma-\sigma')$ for the extended (but finite) range $[-\pi,\pi]$ and make use of the even nature of the bosonic variables X^μ (26) in the extended interval. Furthermore, the occurrence of $\delta_P(\sigma-\sigma')$ itself was justified by the fact that a scalar field, subjected to periodic BC in a one-dimensional box of length 2π has $\delta_P(\sigma-\sigma')$, rather than the usual delta function, occurring in the basic Poisson-bracket between the scalar field and its conjugate momentum Π.

[8] The form of the periodic delta function is given by $\delta_P(x-y) = \delta_P(x-y+2\pi) = \frac{1}{2\pi}\sum_{n\in Z} e^{in(x-y)}$ and is related to the usual Dirac δ-function as $\delta_P(x-y) = \sum_{n\in Z}\delta(x-y+2\pi n)$ [18].

We can essentially follow the same methodology here in the fermionic sector as $\psi^\mu_\pm(\tau,\sigma)$ also satisfy periodic BC of period 2π (25). The only difference with the bosonic case, apart from the Grassmanian nature of the latter, is that, instead of their even property (26), the components of Majorana fermions satisfy (24). As we shall show now that this condition is quite adequate to identify the appropriate delta-functions for the "physical interval" $[0,\pi]$.

We start by noting that the usual properties of a delta function is also satisfied by $\delta_P(x)$

$$\int_{-\pi}^{\pi} dx'\,\delta_P(x'-x)f(x') = f(x) \tag{37}$$

for any periodic function $f(x)=f(x+2\pi)$ defined in the interval $[-\pi,\pi]$. Hence, one can immediately write down the following expressions for ψ^μ_- and ψ^μ_+:

$$\int_0^{\pi} d\sigma'\,\big[\delta_P(\sigma'+\sigma)\psi^\mu_+(\sigma')+\delta_P(\sigma'-\sigma)\psi^\mu_-(\sigma')\big] = \psi^\mu_-(\sigma), \tag{38}$$

$$\int_0^{\pi} d\sigma'\,\big[\delta_P(\sigma'+\sigma)\psi^\mu_-(\sigma')+\delta_P(\sigma'-\sigma)\psi^\mu_+(\sigma')\big] = \psi^\mu_+(\sigma). \tag{39}$$

Combining the above equations and writing them in a matrix form, we get,

$$\int_0^{\pi} d\sigma'\,\Lambda_{AB}(\sigma,\sigma')\psi^\mu_B(\sigma') = \psi^\mu_A(\sigma) \quad (A=-,+), \tag{40}$$

where $\Lambda_{AB}(\sigma,\sigma')$, defined by

$$\Lambda_{AB}(\sigma,\sigma') = \begin{pmatrix} \delta_P(\sigma'-\sigma) & \delta_P(\sigma'+\sigma) \\ \delta_P(\sigma'+\sigma) & \delta_P(\sigma'-\sigma) \end{pmatrix}, \tag{41}$$

acts like a matrix valued "delta function" for the two component Majorana spinor in the reduced physical interval $[0,\pi]$ of the string. We therefore propose the following antibrackets in the fermionic sector:

$$\big\{\psi^\mu_A(\sigma),\psi^\nu_B(\sigma')\big\} = -i\eta^{\mu\nu}\Lambda_{AB}(\sigma,\sigma'), \tag{42}$$

instead of (6) which, when written down explicitly in terms of components, reads

$$\begin{aligned} &\big\{\psi^\mu_+(\sigma),\psi^\nu_+(\sigma')\big\} = \big\{\psi^\mu_-(\sigma),\psi^\nu_-(\sigma')\big\} = -i\eta^{\mu\nu}\delta_P(\sigma-\sigma'), \\ &\big\{\psi^\mu_-(\sigma),\psi^\nu_+(\sigma')\big\} = -i\eta^{\mu\nu}\delta_P(\sigma+\sigma'). \end{aligned} \tag{43}$$

We shall now investigate the consistency of this structure. Firstly, this structure of the antibracket relations is completely consistent with the boundary condition (22). To see this explicitly, we compute the anticommutator of $\psi_+(\sigma')$ with (22), the left-hand side of which gives

$$-i\big(\delta_P(\sigma-\sigma')-\delta_P(\sigma+\sigma')\big)\big|_{\sigma=0,\,\pi} = -i\Delta_-(\sigma,\sigma')\big|_{\sigma=0,\pi} = \frac{1}{\pi}\sum_{n\neq 0}\sin(n\sigma')\sin(n\sigma)\big|_{\sigma=0,\,\pi} = 0, \tag{44}$$

where the form of the periodic delta function has been used. Not only that, as a bonus, we reproduce the modified form of (34). Observe the occurrence of $\delta_P(\sigma+\sigma')$ rather than $\delta_P(\sigma-\sigma')$ in the mixed bracket $\{\psi_+,\psi_-\}$, which plays a crucial role in obtaining the following involutive algebra in the fermionic sector. Indeed, using (42), one can show that

$$\big\{\lambda_1(\sigma),\lambda_1(\sigma')\big\} = 4\big(\lambda_2(\sigma)\partial_\sigma\Delta_+(\sigma,\sigma')+\lambda_2(\sigma')\partial_\sigma\Delta_-(\sigma,\sigma')\big),$$

$$\{\lambda_2(\sigma), \lambda_2(\sigma')\} = \lambda_2(\sigma')\partial_\sigma \Delta_+(\sigma, \sigma') + \lambda_2(\sigma)\partial_\sigma \Delta_-(\sigma, \sigma'),$$
$$\{\lambda_2(\sigma), \lambda_1(\sigma')\} = (\lambda_1(\sigma) + \lambda_1(\sigma'))\partial_\sigma \Delta_+(\sigma, \sigma') \tag{45}$$

hold for the fermionic sector.

In order to write down the complete algebra of the super-Virasoro constraints $\chi_1(\sigma)$ and $\chi_2(\sigma)$, one must take into account the algebra of constraints between the bosonic variables. Interestingly, these have exactly the same structure as the fermionic algebra (45) with the λ's being replaced by Φ's,[9] so that the complete algebra of the super-Virasoro constraints also have identical structures:

$$\{\chi_1(\sigma), \chi_1(\sigma')\} = 4(\chi_2(\sigma)\partial_\sigma \Delta_+(\sigma, \sigma') + \chi_2(\sigma')\partial_\sigma \Delta_-(\sigma, \sigma')),$$
$$\{\chi_2(\sigma), \chi_2(\sigma')\} = \chi_2(\sigma')\partial_\sigma \Delta_+(\sigma, \sigma') + \chi_2(\sigma)\partial_\sigma \Delta_-(\sigma, \sigma'),$$
$$\{\chi_2(\sigma), \chi_1(\sigma')\} = (\chi_1(\sigma) + \chi_1(\sigma'))\partial_\sigma \Delta_+(\sigma, \sigma'). \tag{46}$$

The algebra between the constraints (16) now gets modified to

$$\{\tilde{J}_1(\sigma), \tilde{J}_1(\sigma')\} = -i(\chi_1(\sigma) - 2\chi_2(\sigma))\delta_P(\sigma - \sigma'),$$
$$\{\tilde{J}_2(\sigma), \tilde{J}_2(\sigma')\} = -i(\chi_1(\sigma) + 2\chi_2(\sigma))\delta_P(\sigma - \sigma'),$$
$$\{\tilde{J}_1(\sigma), \tilde{J}_2(\sigma')\} = -i(\chi_1(\sigma) - 2\chi_2(\sigma))\delta_P(\sigma + \sigma'). \tag{47}$$

The algebra between $\tilde{J}(\sigma)$ and $\chi(\sigma)$ can now be computed by using the modified bracket (35) to get

$$\{\chi_1(\sigma), \tilde{J}_1(\sigma')\} = -(2\tilde{J}_1(\sigma) + \tilde{J}_1(\sigma'))\partial_\sigma \delta_P(\sigma - \sigma') + (2\tilde{J}_2(\sigma) + \tilde{J}_1(\sigma'))\partial_\sigma \delta_P(\sigma + \sigma'),$$
$$\{\chi_1(\sigma), \tilde{J}_2(\sigma')\} = (2\tilde{J}_2(\sigma) + \tilde{J}_2(\sigma'))\partial_\sigma \delta_P(\sigma - \sigma') - (2\tilde{J}_1(\sigma) + \tilde{J}_2(\sigma'))\partial_\sigma \delta_P(\sigma + \sigma'),$$
$$\{\chi_2(\sigma), \tilde{J}_1(\sigma')\} = \left(\tilde{J}_1(\sigma) + \frac{1}{2}\tilde{J}_1(\sigma')\right)\partial_\sigma \delta_P(\sigma - \sigma') + \left(\tilde{J}_2(\sigma) + \frac{1}{2}\tilde{J}_1(\sigma')\right)\partial_\sigma \delta_P(\sigma + \sigma'),$$
$$\{\chi_2(\sigma), \tilde{J}_2(\sigma')\} = \left(\tilde{J}_2(\sigma) + \frac{1}{2}\tilde{J}_2(\sigma')\right)\partial_\sigma \delta_P(\sigma - \sigma') + \left(\tilde{J}_1(\sigma) + \frac{1}{2}\tilde{J}_2(\sigma')\right)\partial_\sigma \delta_P(\sigma + \sigma'), \tag{48}$$

which clearly displays a new structure for the super-Virasoro algebra.

As a matter of consistency, we write down the Hamiltonian of the superstring and then study the time evolution of the $\psi_\pm$ modes. This follows easily from the Virasoro functional $L[f]$ (29) by setting $f(\sigma) = e^{im\sigma}$, which gives

$$L_m = \frac{1}{4}\int_{-\pi}^{\pi} d\sigma\, e^{-im\sigma}\left[\{\Pi(\sigma) + \partial_\sigma X(\sigma)\}^2 + 2i\psi_+^\mu \partial_\sigma \psi_{\mu+}\right]. \tag{49}$$

Setting $m = 0$, gives the Hamiltonian

$$H = L_0 = \frac{1}{4}\int_{-\pi}^{\pi} d\sigma\left[\{\Pi(\sigma) + \partial_\sigma X(\sigma)\}^2 + 2i\psi_+^\mu \partial_\sigma \psi_{\mu+}\right]$$
$$= \frac{1}{2}\int_0^{\pi} d\sigma\left[\Pi^2(\sigma) + \partial_\sigma X(\sigma)^2 + i(\psi_+^\mu(\sigma)\partial_\sigma \psi_{\mu+}(\sigma) - \psi_-^\mu(\sigma)\partial_\sigma \psi_{\mu-}(\sigma))\right]. \tag{50}$$

This immediately leads to

$$\dot{\psi}_-(\sigma) = \{\psi_-(\sigma), H\} = -\partial_\sigma \psi_-(\sigma), \qquad \dot{\psi}_+(\sigma) = \{\psi_+(\sigma), H\} = \partial_\sigma \psi_+(\sigma), \tag{51}$$

[9] Note that there were some errors in [7].

which are precisely the equations of motion for the fermionic fields. One can therefore regard (35) and (43) as the final symplectic structure of the free superstring theory.

4. The interacting theory

The action for a superstring moving in the presence of a constant background Neveu–Schwarz two form field $\mathcal{F}_{\mu\nu}$ is given by

$$S=-\frac{1}{2}\int\limits_{\Sigma} d^2\sigma\,\left(\eta_{\mu\nu}\partial_a X^\mu\partial^a X^\nu+\epsilon^{ab}\mathcal{F}_{\mu\nu}\partial_a X^\mu\partial_b X^\nu-i\bar{\psi}^\mu\rho^a\partial_a\psi_\mu+i\mathcal{F}_{\mu\nu}\bar{\psi}^\mu\rho_b\epsilon^{ab}\partial_a\psi^\nu\right). \tag{52}$$

The bosonic and fermionic sectors decouple. We consider just the fermionic sector since the bosonic sector was already discussed [7]. In component the fermionic sector reads

$$S_F=\frac{i}{2}\int\limits_{\Sigma} d\tau\,d\sigma\,\left(\psi_-^\mu\partial_+\psi_{-\mu}+\psi_+^\mu\partial_-\psi_{+\mu}-\mathcal{F}_{\mu\nu}\psi_-^\mu\partial_+\psi_-^\nu+\mathcal{F}_{\mu\nu}\psi_+^\mu\partial_-\psi_+^\nu\right). \tag{53}$$

The minimum action principle $\delta S=0$ leads to a volume term that vanishes when the equations of motion hold, and also to a surface term

$$\left(\psi_-^\mu(\eta_{\mu\nu}-\mathcal{F}_{\mu\nu})\delta\psi_-^\nu-\psi_+^\mu(\eta_{\mu\nu}+\mathcal{F}_{\mu\nu})\delta\psi_+^\nu\right)\Big|_0^\pi=0. \tag{54}$$

It is not possible to find nontrivial boundary conditions involving ψ_-^μ and ψ_+^μ that makes the above surface term vanish. However, the addition of a boundary term [15,16]

$$S_{\text{bound}}=\frac{i}{2\pi\alpha'}\int\limits_{\Sigma} d\tau\,d\sigma\,\left(\mathcal{F}_{\mu\nu}\psi_+^\mu\partial_-\psi_+^\nu\right) \tag{55}$$

makes it possible to find a solution to the boundary condition. Addition of this term to S_F leads to the total action:

$$S=\frac{-i}{4\pi\alpha'}\int\limits_{\Sigma} d\tau\,d\sigma\,\left(\psi_-^\mu E_{\nu\mu}\partial_+\psi_-^\nu+\psi_+^\mu E_{\nu\mu}\partial_-\psi_+^\nu\right), \tag{56}$$

where $E^{\mu\nu}=\eta^{\mu\nu}+\mathcal{F}^{\mu\nu}$. The corresponding boundary term coming from $\delta S=0$ is given by

$$\left(\psi_-^\mu E_{\nu\mu}\delta\psi_-^\nu-\psi_+^\mu E_{\nu\mu}\delta\psi_+^\nu\right)\Big|_0^\pi=0. \tag{57}$$

The above condition is satisfied by the following conditions that preserve supersymmetry [17] at the string endpoints $\sigma=0$ and $\sigma=\pi$:

$$E_{\nu\mu}\psi_+^\nu(0,\tau)=E_{\mu\nu}\psi_-^\nu(0,\tau),\qquad E_{\nu\mu}\psi_+^\nu(\pi,\tau)=\lambda E_{\mu\nu}\psi_-^\nu(\pi,\tau), \tag{58}$$

where $\lambda=\pm1$ with the plus sign corresponding to Ramond boundary condition and the minus corresponding to the Neveu–Schwarz case. Here too we work with Ramond boundary conditions. Now the BCs are recast as

$$\left(E_{\nu\mu}\psi_{(+)}^\nu(\sigma,\tau)-E_{\mu\nu}\psi_{(-)}^\nu(\sigma,\tau)\right)\Big|_{\sigma=0,\pi}=0. \tag{59}$$

This nontrivial BC leads to a modification in the original (naive) (6) DBs. The $\{\psi_{(+)}^\mu(\sigma,\tau),\psi_{(+)}^\nu(\sigma',\tau)\}_{\text{DB}}$ is the same as that of the free string (6). We therefore make an ansatz

$$\left\{\psi_+^\mu(\sigma,\tau),\psi_-^\nu(\sigma',\tau)\right\}_{\text{DB}}=C^{\mu\nu}\delta_P(\sigma+\sigma'). \tag{60}$$

Taking brackets between the BCs (59) and $\psi_-^\gamma(\sigma')$ we get

$$E_{\nu\mu}C^{\nu\gamma} = -iE_{\mu\gamma}. \tag{61}$$

Solving this, we find

$$C^{\mu\nu} = -i\big[\big(1-\mathcal{F}^2\big)^{-1}\big]^{\mu\rho}E_{\rho\gamma}E^{\gamma\nu}. \tag{62}$$

One can also take brackets between the BCs (59) and $\psi_+^\gamma(\sigma')$, which yields

$$C^{\nu\mu} = -i\big[\big(1-\mathcal{F}^2\big)^{-1}\big]^{\mu\rho}E_{\gamma\rho}E^{\nu\gamma}. \tag{63}$$

Although the expressions (62) and (63) look different, they are actually the same as can be easily verified. Finally we can write the matrix $C = \{C^{\mu\nu}\}$ more compactly as

$$C = -i\big[\big(1-\mathcal{F}^2\big)^{-1}(1+\mathcal{F})^2\big]. \tag{64}$$

We therefore get the following modification:

$$\big\{\psi_+^\mu(\sigma,\tau), \psi_-^\nu(\sigma',\tau)\big\}_{\mathrm{DB}} = -i\big[\big(1-\mathcal{F}^2\big)^{-1}\big]^{\mu\rho}E_{\rho\gamma}E^{\gamma\nu}\delta_P(\sigma+\sigma'), \tag{65}$$

which also reduces to those of [8], upto the $\delta_P\,(\sigma+\sigma')$ factor. Finally, note that in the limit $\mathcal{F}_{\mu\nu}\to 0$ (65), the last of (43) is reproduced.

5. Conclusions

In this Letter, we have derived the expressions for a non(anti)commutative algebra for an open superstring. The interesting thing to note that, unlike the bosonic case, we get an anticommutative structure in the fermionic sector even for the free superstring. Our results differ from those in [8] and are mathematically consistent which is reflected from the closure of the constraint algebras. The analysis of this Letter is a direct generalisation of [7], where only bosonic string was considered.

The origin of any modification in the usual canonical algebra is the presence of boundary conditions. This phenomenon is quite well known for a free scalar field subjected to periodic boundary conditions. Besides this method was also used earlier by [12] in the context of Nambu–Goto formulation of the bosonic string. We show that the same also holds true in the fermionic sector of the conformal gauge fixed free superstring, where the boundary conditions become periodic once we extend the domain of definition of the length of the string from $[0,\pi]$ to $[-\pi,\pi]$. This mathematical trick leads to a modification where the usual Dirac delta function gets replaced by a periodic delta function. Eventually one constructs the appropriate "delta function" for the physical interval $[0,\pi]$ of the string to write down the basic symplectic structure. Interestingly, here we get a 2×2 matrix valued "delta function" appropriate for the two component Majorana spinor which is in contrast to the bosonic case, where one has a single component "delta function" $\Delta_+(\sigma,\sigma')$ satisfying Neumann boundary condition [7,12]. This symplectic structure, interestingly, leads to a new involutive structure for the super-Virasoro algebra at the classical level. The corresponding quantum version and its implications are being investigated.

The same technique is adopted for the interacting case also. Here, the boundary condition is more involved and leads to a more general type of non(anti)commutativity that has been observed before. However, our results are once again different from the existing results since we get a periodic delta function instead of the usual delta function, apart from the relative sign of σ,σ'. This change of relative sign indeed plays a crucial role in the internal consistency of our analysis. Further, the interacting results go over smoothly to the free case once the interaction is switched off.

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