Oscillation for Solutions of Nonlinear Neutral Differential Equations with Impulses

ZHIGUO LUO AND JIANHUA SHEN
Department of Mathematics
Hunan Normal University
Changsha 410081, P.R. China
luozg@mail.hunnu.edu.cn

(Received September 2000; accepted February 2001)

Abstract—This paper is concerned with nonlinear neutral differential equations with impulses of the form

\[
\left[ x(t) - \sum_{i=1}^{m} p_i(t)x(t-\tau_i) \right]' + Q(t) \prod_{j=1}^{n} (|f_j(x(t-\sigma_j))|^{\alpha_j} \text{sgn} x(t-\sigma_j)) = 0, \quad t \geq t_0,
\]

\[\quad x(t_k^+) = I_k(x(t_k)), \quad k = 1, 2, \ldots.\]

Some oscillation criteria for solutions of this equation are established. An interesting example is also given, which illustrates that impulses play an important role in giving rise to the oscillation of equations. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Oscillation, Neutral differential equation, Impulse.

1. INTRODUCTION

The theory of impulsive differential equations is now being recognized to be not only richer than the corresponding theory of differential equations without impulses but also provides a more adequate mathematical model for numerous processes and phenomena studied in physics, biology, engineering, etc. [1,2]. In recent years, there is increasing interest on the oscillatory and nonoscillatory properties of this class of equations (see [3–9] and the references contained therein), and many results are obtained. However, the theory of impulsive functional differential equations is developing comparatively slowly due to numerous theoretical and technical difficulties caused by their peculiarities. In particular, to the best of our knowledge, there is little in the way of results for the oscillation of impulsive delay differential equations of neutral type despite the extensive development of the oscillatory and nonoscillatory properties of neutral differential equations without impulses (for example see [10–13]).
In this paper, we consider the oscillation of all solutions of impulsive neutral delay differential equations with variable coefficients of the form
\[
\begin{align*}
\left[ x(t) - \sum_{i=1}^{m} P_i(t)x(t - \tau_i) \right]' + Q(t) \prod_{j=1}^{n} |f_j(x(t - \sigma_j))|^{\alpha_j} \sgn x(t - \sigma_j) &= 0, \quad t \geq t_0, \\
x(t_k^+) &= I_k(x(t_k)), \quad k = 1, 2, \ldots.
\end{align*}
\]

(1.1)

(1.2)

Our aim is to establish sufficient conditions for the oscillation of all solutions of equations (1.1) and (1.2). An example is also given which shows that the oscillation of all solutions of equations (1.1) and (1.2) can be caused by the impulsive perturbations though the corresponding equation without impulses admits a nonoscillatory solution.

The following assumptions will be used throughout this paper without further mention.

\begin{enumerate}
\item[(A1)] \( \tau_i > 0, \sigma_j > 0, \alpha_j \geq 0 \) \( (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n) \), \( \sum_{j=1}^{n} \alpha_j = 1 \), and \( 0 \leq t_0 < t_1 < \cdots < t_k < t_{k+1} \rightarrow \infty \) as \( k \rightarrow \infty \).
\item[(A2)] \( P_i, Q \in PC([t_0, \infty), R^+) \), \( Q(t) \not\equiv 0 \) on \( (t_{k-1}, t_k) \) \( (k \geq 1) \), \( f_j \in C(R, R), x f_j(x) > 0(x \neq 0) \) \( (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n) \), where \( R^+ = [0, \infty), PC([t_0, \infty), R^+) = \{ y : [t_0, \infty) \rightarrow R^+ : g(t) \) is continuous for \( t_0 \leq t \leq t_1, t_k < t < t_{k+1} \) and \( \lim_{t \rightarrow t_k^+} g(t) = g(t_k^+) \) exists \( (k = 1, 2, \ldots) \} \).
\item[(A3)] \( I_k(x) \) is continuous and there exist positive numbers \( b_k^*, b_k \) such that \( b_k^* \leq I_k(x)/x \leq b_k \) for \( x \neq 0 \) and \( k = 1, 2, \ldots \).
\end{enumerate}

With equations (1.1) and (1.2), one associates an initial condition of the form
\[
x_{t_0} = \phi(s), \quad s \in [-\rho, 0), \quad \rho = \max_{1 \leq i \leq m, 1 \leq j \leq n} \{ \tau_i, \sigma_j \},
\]

(1.3)

where \( x_{t_0} = x(t_0 + s) \) for \( -\rho \leq s \leq 0 \) and \( \phi \in PC([-\rho, 0], R) = \{ \phi : [-\rho, 0] \rightarrow R : \phi \) is continuous everywhere except at the finite number of points \( \bar{s} \) and \( \phi(\bar{s}^+) \) and \( \phi(\bar{s}^-) = \lim_{s \rightarrow \bar{s}^-} \phi(s) \) exist with \( \phi(\bar{s}^+) = \phi(\bar{s}^-) \} \).

A function \( x(t) \) is said to be a solution of equations (1.1) and (1.2) satisfying the initial value condition (1.3) if

\begin{enumerate}
\item[(i)] \( x(t) = \phi(t - t_0) \) for \( t_0 - \rho \leq t \leq t_0 \), \( x(t) \) is continuous for \( t \geq t_0 \) and \( t \neq t_k \) \( (k = 1, 2, \ldots) \);
\item[(ii)] \( x(t) - \sum_{i=1}^{m} P_i(t)x(t - \tau_i) \) is continuously differentiable for \( t > t_0, t \neq t_k, t \neq t_k + \tau_i, t \neq t_k + \sigma_j \) \( (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, k = 1, 2, \ldots) \) and satisfies (1.1);
\item[(iii)] \( x(l_k^+) \) and \( x(l_k^-) \) exist with \( x(l_k^+) = x(l_k^-) \) and satisfy (1.2).
\end{enumerate}

Using the method of steps as in the case without impulses, one can show the global existence and uniqueness of the solution of the initial value problem (1.1)–(1.3).

As is customary, a solution of equations (1.1) and (1.2) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

\section*{2. MAIN RESULTS}

\textbf{Lemma 2.1.} Assume that the following two conditions hold.

\begin{enumerate}
\item[(L1)] There exist a \( \tau > 0 \), natural numbers \( l_i \) \( (i = 1, 2, \ldots, m) \), and a \( t^* \geq t_0 \) such that
\[
\tau_i = l_i \tau \quad (i = 1, 2, \ldots, m), \quad \sum_{i=1}^{m} P_i(t^* + l \tau) \leq 1, \quad l = 1, 2, \ldots.
\]
(2.1)
\item[(L2)] \( b_0 = 1 \), and for \( k = 1, 2, \ldots, b_k \leq 1 \),
\[
P_i(t_k^+) \geq P_i(t_k), \quad \text{for } i \in E_{k}, \quad \text{for } i \in E_{k}, \quad \text{for } i \in E_{k}.
\]
(2.2)
\[
b_k P_i(t_k^+) > P_i(t_k), \quad \text{for } i \in E_{k}, \quad \text{for } i \in E_{k}.
\]
(2.3)
\end{enumerate}
where $b_k = b^*_k$, when $t_k - \tau_i = t_k, (k_i < k)$. Let $x(t)$ be a solution of (1.1) and (1.2) such that $x(t - \rho) > 0$ for $t \geq t_0$, and let

$$z(t) = x(t) - \sum_{i=1}^{m} P_i(t)x(t - \tau_i). \quad (2.4)$$

Then $z(t) > 0$ for $t \geq t_0$ and $z(t^+_k) \leq b_k z(t_k)$ for $k = 1, 2, \ldots$.

**Proof.** By (1.1) and (2.4), we get

$$z'(t) = -Q(t) \prod_{j=1}^{n} |f_j(x(t - \sigma_j))|^{\alpha_j} \leq 0, \quad t_k < t \leq t_{k+1}, \quad k \geq 0. \quad (2.5)$$

From (2.4) and in view of $b_k \leq 1$, we have

$$z(t^+_k) = x(t^+_k) - \sum_{i=1}^{m} P_i(t^+_k)x(t^+_k - \tau_i) - \sum_{i=1}^{m} P_i(t^+_k)x(t^+_k - \tau_i)$$
$$= b_k x(t_k) - \sum_{i \in E_1} P_i(t_k)b_k x(t_k - \tau_i) - \sum_{i \in E_2} P_i(t_k)b_k x(t_k - \tau_i)$$
$$\leq b_k x(t_k) - \sum_{i \in E_1} P_i(t_k)b_k x(t_k - \tau_i) - \sum_{i \in E_2} P_i(t_k)b_k x(t_k - \tau_i)$$
$$= b_k z(t_k).$$

Thus,

$$z(t^+_k) \leq b_k z(t_k) \leq z(t_k), \quad k = 1, 2, \ldots, \quad (2.6)$$

and $z(t)$ is nonincreasing on $[t_0, \infty)$.

We first claim that $z(t_k) \geq 0$ for $k \geq 1$. Otherwise, suppose that there exists some $k \geq 1$ such that $z(t_k) = -\mu < 0$. Then $z(t) \leq -\mu < 0$ for $t > t_k$. From (2.4), we have

$$x(t) \leq -\mu + \sum_{i=1}^{m} P_i(t)x(t - \tau_i), \quad t \geq t_k. \quad (2.7)$$

By Assumption L1, there exist a $\tau > 0$, natural numbers $l_i (i = 1, 2, \ldots, m)$, and a $t^* \geq t_0$ such that (2.1) holds. We find a natural number $l_0$ such that $t^* + l_0 \tau \geq t_k$. Without loss of generality, we suppose that $\tau_1 < \tau_2 < \cdots < \tau_m$. Then for every number $l \geq l_0 + l_m$, from (2.7) we have

$$x(t^* + l\tau) \leq -\mu + \sum_{i=1}^{m} P_i(t^* + l\tau)x(t^* + l\tau - \tau_i)$$
$$\leq -\mu + \max_{1 \leq i \leq m} \{ x(t^* + l\tau - \tau_i) \}. \quad (2.8)$$

Set

$$\max_{1 \leq i \leq m} \{ x(t^* + l\tau - \tau_i) \} := x[t^* + (l - r_1)\tau]$$

and

$$\max_{1 \leq i \leq m} \left\{ x \left[ t^* + \left( l - \sum_{j=1}^{r} r_j \right) \tau - \tau_i \right] \right\} := x \left[ t^* + \left( l - \sum_{j=1}^{r+1} r_j \right) \tau \right], \quad (2.9)$$
where \( r = 1, 2, \ldots, N(l) - 1 \), \( r_j \in \{ l_1, l_2, \ldots, l_m \} \), and \( N(l) \) satisfies that \( l_0 \leq l - \sum_{j=1}^{N(l)} r_j \leq l_0 + l_m \). Clearly, \( N(l) > (l - l_0)/l_m - 1 \to \infty (l \to \infty) \). As

\[
x \left[ t^* + \left( l - \sum_{j=1}^{r} r_j \right) \tau \right] \leq -\mu + \max_{1 \leq i \leq m} \left\{ x \left[ t^* + \left( l - \sum_{j=1}^{r} r_j \right) \tau - \tau_i \right] \right\}
\]

\[
= -\mu + x \left[ t^* + \left( l - \sum_{j=1}^{r+1} r_j \right) \tau \right],
\]

we have

\[
x(t^* + l\tau) \leq -N(l)\mu + x \left[ t^* + \left( l - \sum_{j=1}^{N(l)} r_j \right) \tau \right]
\]

\[
< -N(l)\mu + M \to -\infty (l \to \infty).
\]

where \( M = \max\{x(t) : t^* + l_0\tau \leq t \leq t^* + (l_0 + l_m)\tau \} \). Which is a contradiction, and so \( z(t_k) \geq 0 \) \( (k \geq 1) \). From (2.3), \( z(t_0) \geq 0 \).

To prove \( z(t) > 0 \) for \( t \geq t_0 \), we first prove that \( z(t_k) > 0 \) \( (k \geq 0) \). If it is not true, then there exists some \( k \geq 0 \) such that \( z(t_k) = 0 \). Thus, from (2.5), we have

\[
z(t_{k+1}) = z(t_k) - \int_{t_k}^{t_{k+1}} Q(s) \prod_{j=1}^{n} |f_j(x(s - \sigma_j))|^{\alpha_j} ds
\]

\[
\leq z(t_k) - \int_{t_k}^{t_{k+1}} Q(s) \prod_{j=1}^{n} |f_j(x(s - \sigma_j))|^{\alpha_j} ds < 0.
\]

This contradiction shows that \( z(t_k) > 0 \) \( (k \geq 0) \). Therefore, from (2.5), \( z(t) \geq z(t_{k+1}) > 0 \) for \( t \in (t_k, t_{k+1}] \) \( (k \geq 0) \). The proof is complete.

**Lemma 2.2.** Let (L2) hold. Assume that \( x f_j(x) \geq A_j x^2 \), \( A_j > 0 \), and

(L3) \( \sum_{i=1}^{m} P_i(t) \geq 1 \) and there exist a \( \tau > 0 \), natural numbers \( l_i \) \( (i = 1, 2, \ldots, m) \), and a \( t^* \geq t_0 \) such that

\[
\tau_i = l_i \tau, \quad (i = 1, 2, \ldots, m), \quad \sum_{i=1}^{m} P_i(t^* + l\tau) = 1, \quad l = 1, 2, \ldots
\]

If (1.1) and (1.2) has a solution \( x(t) \) such that \( x(t - \rho) > 0 \) for \( t \geq t_0 \), then there exists some \( T > t_0 \) such that the second-order impulsive differential inequality

\[
y''(t) + \rho^{-1} \left( \prod_{j=1}^{n} A_j^{\alpha_j} \right) Q(t) y(t) \leq 0, \quad t \geq T + \rho,
\]

\[
y(t_k^+) = y(t_k), \quad k = r, r + 1, \ldots,
\]

\[
y'(t_k^+) \leq b_k y'(t_k), \quad k = r, r + 1, \ldots
\]

has a solution \( y(t) \) such that \( y(t) > 0 \) and \( y'(t^+) > 0 \) for \( t > T + \rho \), where \( r = \min \{ k \geq 1 : t_k > T + \rho \} \) and \( y'(t^+) = y'(t) \) when \( t \neq t_k \).

**Proof.** As Condition L3 implies that (L1) holds, so the conditions of Lemma 2.1 are satisfied. By Lemma 2.1, we have \( z(t) > 0 \) for \( t \geq t_0 \) and (2.5) holds. Set \( M = 2^{-1} \min \{ x(t) : t_0 - \rho \leq t \leq t_0 \} \), then \( M > 0 \) and \( x(t) > M \) for \( t_0 - \rho \leq t \leq t_0 \). We claim that

\[
x(t) > M, \quad t \in (t_0, t_1].
\]
If (2.10) does not hold, then there exists a $t^* \in (t_0, t_1]$ such that $x(t^*) = M$ and $x(t) > M$ for $t_0 - \rho \leq t < t^*$. From (2.4), we have

$$M = x(t^*) = z(t^*) + \sum_{i=1}^{m} P_i(t^*) x(t^* - \tau_i) > \sum_{i=1}^{m} P_i(t^*) M \geq M,$$

which is a contradiction and so (2.10) holds. Noting that $z(t^+_1) \geq z(t_2) > 0$, we have

$$x(t^+_1) = z(t^+_1) + \sum_{i=1}^{m} P_i(t^+_1) x(t^+_1 - \tau_i)$$

$$> \sum_{i \in E_1} P_i(t^+_1) x(t^+_1 - \tau_i) + \sum_{i \notin E_2} P_i(t^+_1) x(t^+_1 - \tau_i)$$

$$\geq \sum_{i \in E_1} P_i(t_1)M + \sum_{i \notin E_2} P_i(t_1)M \geq M.$$

Repeating the above argument, by induction, we obtain

$$x(t) > M, \quad t \geq t_0 - \rho,$$

$$x(t^+_k) > M, \quad k = 1, 2, \ldots.$$

Because $z(t) > 0$ and $z(t)$ is nonincreasing, we can let $\lim_{t \to -\infty} z(t) = a$. There is two possible cases.

**Case I.** $a = 0$. There exists a $T_1 > t_0$ such that $z(t) \leq M/2$ for $t \geq T_1$. Then for any $t > T_1$, we have

$$\frac{1}{\rho} \int_{t}^{t+\rho} z(s) \, ds \leq M < x(t), \quad t \in [T_1, t + \rho].$$

**Case II.** $a > 0$. Then $z(t) \geq a$ for $t \geq t_0$. From (2.4), we get

$$x(t) \geq a + \sum_{i=1}^{m} P_i(t) x(t - \tau_i) \geq a + M, \quad t \geq t_0.$$

By induction, it is easy to see that $x(t) \geq na + M$ for $t \geq t_0 + (n - 1)\rho$, and so $\lim_{t \to -\infty} x(t) = \infty$, which implies that there exists a $T > T_1$ such that

$$\frac{1}{\rho} \int_{T}^{t+\rho} z(s) \, ds \leq 2z(T) < x(t), \quad t \in [T, T + \rho].$$

Combining Cases I and II we see that

$$x(t) > \frac{1}{\rho} \int_{T}^{t+\rho} z(s) \, ds, \quad t \in [T, T + \rho].$$

Let $l = \min \{k \geq 0 : t_k > T + \rho\}$, we claim that

$$x(t) > \frac{1}{\rho} \int_{T}^{t+\rho} z(s) \, ds, \quad t \in (T + \rho, t_l]. \quad (2.11)$$

Otherwise, there exists a $t^* \in (T + \rho, t_l]$ such that

$$x(t^*) = \frac{1}{\rho} \int_{T}^{t^*+\rho} z(s) \, ds; \quad x(t) > \frac{1}{\rho} \int_{T}^{t+\rho} z(s) \, ds, \quad \text{for } t \in (T + \rho, t^*).$$
Then, from (2.4), we have

\[ \frac{1}{\rho} \int_{t^*}^{t^*+\rho} z(s) \, ds = x(t^*) = z(t^*) + \sum_{i=1}^{m} P_i(t^*) (t^* - \tau_i) \]

\[ > \frac{1}{\rho} \int_{t^*}^{t^*+\rho} z(s) \, ds + \left( \sum_{i=1}^{m} P_i(t^*) \right) \frac{1}{\rho} \min_{1 \leq i \leq m} \left\{ \int_{t}^{t^*+\rho-\tau_i} z(s) \, ds \right\} \]

\[ \geq \frac{1}{\rho} \int_{t^*}^{t^*+\rho} z(s) \, ds. \]

This is a contradiction and so (2.11) holds. Thus, from (2.2), (2.3), and (2.11) we have

\[ x(t^+_i) = z(t^+_i) + \sum_{i=1}^{m} P_i(t^+_i) (t^+_i - \tau_i) \]

\[ = z(t^+_i) + \sum_{i \in E_{t_i}} P_i(t^+_i) x(t^+_i - \tau_i) + \sum_{i \in E_{E_t}} P_i(t^+_i) x(t^+_i - \tau_i) \]

\[ \geq z(t^+_i) + \sum_{i \in E_{t_i}} P_i(t_i) x(t_i - \tau_i) + \sum_{i \in E_{E_t}} P_i(t^+_i) b_i x(t_i - \tau_i) \]

\[ > \frac{1}{\rho} \int_{t_i}^{t^+_i+\rho} z(s) \, ds + \left( \sum_{i=1}^{m} P_i(t_i) \right) \frac{1}{\rho} \min_{1 \leq i \leq m} \left\{ \int_{t}^{t^+_i+\rho-\tau_i} z(s) \, ds \right\} \]

\[ > \frac{1}{\rho} \int_{t_i}^{t^+_i+\rho} z(s) \, ds. \]

Repeating the above procedure, by induction, we can see that

\[ x(t) > \frac{1}{\rho} \int_{t_i}^{t^+_i+\rho} z(s) \, ds, \quad t \geq T. \]

Thus, for \( t > T + \rho \), we obtain

\[ x(t - \sigma_j) > \frac{1}{\rho} \int_{t}^{t+\rho-\sigma_j} z(s) \, ds \geq \frac{1}{\rho} \int_{t_i}^{t} z(s) \, ds, \quad j = 1, 2, \ldots, n. \]  \hspace{1cm} (2.12)

Substituting (2.12) into (2.5) and using the assumptions leads to

\[ z'(t) + \left( \prod_{j=1}^{n} A^{0}_j \right) Q(t) \left( \frac{1}{\rho} \int_{t}^{t} z(s) \, ds \right) \leq 0, \quad t \geq T + \rho. \]

Let

\[ y(t) = \frac{1}{\rho} \int_{t}^{t} z(s) \, ds. \]

Then \( y(t^+_k) = y(t_k) \), \( y'(t^+_k) = \rho^{-1} z(t^+_k) \leq \rho^{-1} b_k z(t_k) = b_k y'(t_k) \) for \( k = r, r+1, \ldots \). Thus, \( y(t) > 0 \), \( y'(t^+) > 0 \) for \( t > T + \tau \) and \( y(t) \) satisfies (2.9). The proof is complete.

The following Lemma 2.3 follows from the similar arguments to that in [4, Theorem 1] by letting \( \varphi(x) = x \). We omit the details.

**Lemma 2.3.** Consider the impulsive differential inequality

\[ y''(t) + G(t)y(t) \leq 0, \quad t \geq t_0, \quad t \neq t_k, \]

\[ y(t^+_k) \geq y(t_k), \quad k = 1, 2, \ldots, \]

\[ y'(t^+_k) \leq c_k y'(t_k), \quad k = 1, 2, \ldots, \]  \hspace{1cm} (2.13)

where \( 0 \leq t_0 < t_1 < \cdots < t_k < t_{k+1} \rightarrow \infty \) as \( k \rightarrow \infty \), \( G(t) \in PC([t_0, \infty), R^+) \) and \( c_k > 0 \). If

\[ \sum_{i=0}^{t} \int_{t_i}^{t_{i+1}} \frac{1}{c_0 c_1 \cdots c_i} G(t) \, dt = \infty, \]

where \( c_0 = 1 \). Then inequality (2.13) has no solution \( y(t) \) such that \( y(t) > 0 \) and \( y'(t) > 0 \) for \( t \geq t_0 \).
THEOREM 2.1. Assume that (L2) and (L3) hold and \( x f_j(x) \geq A_j x^2 \), \( A_j > 0 \). If
\[
\int_{t_0 + \rho}^{t_r} Q(t) \, dt + \sum_{i=0}^{\infty} \frac{1}{b_m \cdots b_{m+i}} \int_{t_{m+i}}^{t_{r+i+1}} Q(t) \, dt = \infty. \tag{2.14}
\]
Then all solutions of equations (1.1) and (1.2) oscillate.

PROOF. Suppose that equations (1.1) and (1.2) has a nonoscillatory solution \( x(t) \). Without loss of generality, we assume that \( x(t - \rho) > 0 \), \( t \geq t_0 \). Then, by Lemma 2.2, the second-order impulsive differential inequality
\[
y''(t) + \rho^{-1} \left( \prod_{j=1}^{n} A_j^{\alpha_j} \right) Q(t) y(t) \leq 0, \quad t \geq T + \rho,
\]
\[
y \left( t_k^+ \right) = y(t_k), \quad k = r, r + 1, \ldots,
\]
\[
y' \left( t_k^+ \right) \leq b_k y'(t_k), \quad k = r, r + 1, \ldots,
\]
has a solution \( y(t) \) such that \( y(t) > 0 \) and \( y'(t^+) > 0 \) for \( t > T + \rho \).

On the other hand, by Lemma 2.3, the second-order impulsive differential inequality (2.15) has no solution \( y(t) \) such that \( y(t) > 0 \) and \( y'(t^+) > 0 \). Which is a contradiction. The proof is complete.

COROLLARY 2.1. Let (L2) and (L3) hold. Assume that \( x f_j(x) \geq A_j x^3 \), \( A_j > 0 \) and there exists a constant \( \beta > 0 \) such that
\[
\frac{1}{b_k} \geq \left( \frac{t_{k+1}}{t_k} \right)^\beta, \quad k = 1, 2, \ldots. \tag{2.16}
\]
If
\[
\int_{t_r}^{\infty} t^\beta Q(t) \, dt = \infty. \tag{2.17}
\]
Then all solutions of equations (1.1) and (1.2) oscillate.

PROOF. From (2.16), we have
\[
\int_{t_0 + \rho}^{t_r} Q(t) \, dt + \sum_{i=0}^{l} \frac{1}{b_{r_i} \cdots b_{r_{i+l}}} \int_{t_{r_i}}^{t_{r_{i+l+1}}} Q(t) \, dt \gtrless \frac{1}{b_r} \int_{t_r}^{t_{r+1}} Q(t) \, dt + \cdots + \frac{1}{b_{r_{l-1}} \cdots b_{r_{l+l}}} \int_{t_{r_{l-1}}}^{t_{r_{l+l+1}}} Q(t) \, dt \gtrless \frac{1}{t_r^\beta} \left( \int_{t_r}^{t_{r+1}} t^\beta Q(t) \, dt + \cdots + \int_{t_{r_{l-1}}}^{t_{r_{l+l+1}}} t^\beta Q(t) \, dt \right) \gtrless \frac{1}{t_r^\beta} \left( \int_{t_r}^{t_{r+l+1}} t^\beta Q(t) \, dt \right) = \frac{1}{t_r^\beta} \int_{t_r}^{t_{r+l+1}} t^\beta Q(t) \, dt.
\]
Let \( l \to \infty \), from (2.17), we see that (2.14) holds. According to Theorem 3.1, all solutions of (1.1) and (1.2) oscillate. The proof is complete.

EXAMPLE 4.1. Consider the equation
\[
\left[ x(t) - \frac{1}{2} x(t-1) - \frac{1}{2} x(t-2) \right]' + Q(t)x^{1/3}(t-1)x^{1/3}(t-2)x^{1/3}(t-3) = 0, \quad t \geq 5. \tag{2.18}
\]
where $t_k = ke$ and
\[ Q(t) = \frac{3t - 4}{2t(t - 1)(t - 2) \ln^{1/3}(t - 1) \ln^{1/3}(t - 2) \ln^{1/3}(t - 3)}. \]

It is easy to see
\[ P_1(t) + P_2(t) = 1, \quad \frac{1}{b_k} = \frac{k + 1}{k} = \frac{t_{k+1}}{t_k}. \]

Thus,
\[ \int_5^\infty t^3 Q(t) \, dt = \int_5^\infty t \cdot \frac{3t - 4}{2t(t - 1)(t - 2) \ln^{1/3}(t - 1) \ln^{1/3}(t - 2) \ln^{1/3}(t - 3)} \, dt = \infty. \]

By Corollary 2.1, all solutions of (2.18) and (2.19) oscillate.

**Remark 2.1.** We note that equation (2.18) has an eventually positive solution $z(t) = \ln t$. Therefore, the oscillatory properties of all solutions of equation (2.18) and (2.19) are caused by the impulsive perturbations.

**REFERENCES**