Journal of Algebra 324 (2010) 1887–1902

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the use of naturality in algorithmic resolution

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A R T I C L E   I N F O

Article history:
Received 11 December 2009
Available online 1 July 2010
Communicated by Steven Dale Cutkosky

MSC:
14E15
14F05
14B05
14B99
14E99

Keywords:
Marked ideals
Blowing-up
Permissible centers
Resolution
Equivalence

A B S T R A C T

This paper provides a simplified presentation of a known algorithm for resolution of singularities (in characteristic zero). It works in the context of marked ideals and uses naturality properties with respect to open restrictions and strong equivalence, to solve a delicate glueing problem that arises when induction on the dimension of the objects considered is applied.

Introduction

So far the most effective techniques to resolve singularities of algebraic varieties (in characteristic zero) have been the so-called algorithmic (or canonical, or constructive) methods. The goal of an algorithmic method is not simply to prove the existence of a proper, birational morphism $f : X' \to X$ that resolves the singularities of the variety X [8], but to accomplish this by means of specific blowing-ups whose centers are precisely described. Generally this is done by defining upper semicontinuous functions with values in a totally ordered set, the i-th center being the locus of points where the i-th function reaches a maximum.

Studies of algorithms of resolution include [1,2,5,6,12–14,16]. In [13] some results are extended to the case of schemes over (suitable) artinian rings. Those studies do not deal with the original problem directly; rather, they algorithmically resolve other auxiliary objects, seemingly more technical, that

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.06.008

receive different names in the literature (idealistic exponents, basic objects, presentations, marked ideals, etc.).

This paper presents a simple algorithm for resolution of *marked ideals*. Following [3], these ideals are 5-tuples $\mathcal{I} = (M, W, I, b, E)$ where M (the *ambient scheme*) is a regular variety (over a field of characteristic zero), E is a finite sequence of divisors of M with normal crossings, W is a closed subvariety of M transversal to E, I is a sheaf of $\mathcal{O}_W$-ideals, and b is a positive integer. The singular set of $\mathcal{I}$ is the set of points of W at which I has order at least b. There is a notion of permissible transformation of $\mathcal{I}$ (involving the blowing-up of a certain regular subscheme of W), which produces a new marked ideal. To resolve a marked ideal means to obtain, by means of iterated permissible transforms, a marked ideal with empty singular locus. If this is done in a reasonably "algorithmic fashion", it is not too difficult to obtain similar algorithmic methods to principalize ideals, resolve embedded subvarieties, or resolve abstract varieties.

The main contribution of our method, whose main ingredients are those of the resolution algorithm of O. Villamayor ([6] and [1]), is a simplification of the "inductive step". Indeed, so far, all the methods used to resolve algorithmically marked ideals (or its variants already mentioned) involve, at a crucial step, an inductive argument. The idea is to reduce the problem for a marked ideal (M, W, I, b, E) to a similar one for a suitable marked ideal of the form (M, Z, J, c, D), where Z is a regular divisor in W (usually called a maximal contact hypersurface). Following ideas of H. Hironaka, J. Giraud and other pioneers in the field, now it is not too difficult to do this *locally* (near a point $x \in W$). When the reduction is possible, it allows us to obtain locally (by induction of the dimension) a resolution for our marked ideal. But there are some serious "glueing" problems; namely, to show that the locally obtained resolutions are independent of the chosen hypersurface and that they match correctly, determining a resolution for the whole initial marked ideal. The first proposed algorithmic resolution methods solved these problems by means of rather complicated arguments, generally using some auxiliary constructions, like the *generalized basic objects* of [6], or the operation of *homogenization* of [16].

To justify the inductive step we try to use two naturality properties: compatibility with respect to restrictions to open sets, and compatibility with respect to equivalence. Concerning equivalence, following ideas of Hironaka, we say that marked ideals $\mathcal{I}$ and $\mathcal{J}$ are *equivalent* if they have the same singular loci and this property is preserved after performing any finite number of operations of either one of these types: (1) permissible transformation (using the same center for both), and (2) taking fiber product with an affine line. (See 1.9 for a more precise definition of equivalence). We think that the use of naturally properties simplifies substantially the presentation of the algorithm.

E. Bierstone and P. Milman were the first authors to use functoriality arguments in the construction of a resolution algorithm, in their very interesting article [3]. A difference between their paper and ours is that their method requires the use of a notion of equivalence stronger than the one we propose. Namely, the equivalence in the sense of [3] demands, aside from conditions (1) and (2) above, another condition, involving blowing-ups whose centers are the intersection of certain divisors. Our presentation, based on the t-function of [6], does not require the use of this new condition.

Moreover, we believe that the verification of the validity of the algorithmic resolution process given in [3] is not complete. In the crucial Claim 5.1 of [3], Section 5, Step I, it is not verified that if $\mathcal{I}$ and $\mathcal{J}$ are equivalent marked ideals (both of maximal order 3.1, with $E = \emptyset$), then the corresponding algorithmic resolution centers defined by induction (on the dimension) are the same.

The missing point would be a consequence of the following statement: assume $\mathcal{I} = (M, W, I, b, E)$ and $\mathcal{J} = (M, V, J, c, E)$ are equivalent marked ideals, and U is an open set in W, then the restrictions of $\mathcal{I}$ and $\mathcal{J}$ to U are equivalent. Since we cannot prove this statement, in our presentation we substitute the notion of equivalence described before by a stronger one, that we call *total equivalence*. In total equivalence, the operations of (1) and (2) above are not necessarily applied to a whole marked ideal, but possibly to its restriction to a suitable open set of its ambient scheme (see 1.9, 1.10 and 1.11). Using compatibility with respect to total equivalence and restrictions to open sets as our naturality conditions, it is possible to justify all the steps of the algorithm.

This paper is divided into four sections. Section 1 presents some basic concepts including those of marked ideals, permissible and open permissible transformations, and equivalence and total equivalence; the section also proves some theorems about equivalence. Section 2 introduces the notion of algorithmic resolution of marked ideals and discusses some numerical functions associated to those ideals (like the ω, or w-ord, and t functions). Section 3 addresses some concepts useful in the inductive step of our algorithm, such as adapted (or maximal contact) hypersurfaces and coefficient ideals; the section also studies some constructions essential to reduce the general situation to one where an argument, based on induction on the dimension, can be applied. Section 4 presents the algorithm and its proof, the main result being described in Theorem 4.1.

We do not address how algorithmic resolution of marked ideals implies similar results on principalization of ideals and resolution for embedded and abstract varieties, because excellent discussions of this topic abound (e.g., [6,1–3,16,12]).

There are several programs aimed at extending some form of algorithmic resolution to the case where one works over fields of positive characteristic (see [10,11,15,17]). It is hoped that a "naturality" approach similar to that of the present paper could play a role in this ongoing work.

We thank the referee for the numerous comments and corrections that greatly helped to improve this paper.

1. Basic notions

1.1. In general, we use the notation and terminology of [7] with a few exceptions. For instance, if W is a scheme, a W-ideal means a coherent sheaf of $\mathcal{O}_W$-ideals. If Y is a closed subscheme of a scheme W, the symbol $I(Y)$ denotes the W-ideal defining Y. If W is a reduced scheme, a never-zero W-ideal is a W-ideal I such that the stalk I_x is not zero for all $x \in W$. An algebraic variety over a field k is a reduced algebraic k-scheme, k a field.

The term *local ring* means *noetherian local ring*. The maximal ideal, or radical, of a local ring R usually is denoted by $r(R)$. Often, we write $(R, \mathcal{M})$ to denote the local ring R with maximal ideal $\mathcal{M}$. The order of an ideal I in the local ring $(R, \mathcal{M})$ is the largest integer s such that $I \subseteq \mathcal{M}^s$. If W is a noetherian scheme, I is a W-ideal and $x \in W$, then $\nu_x(I)$ denotes the order of the ideal I_x of $\mathcal{O}_{W,x}$.

A positive divisor in an algebraic variety X is called a hypersurface of X.

We work throughout with the class $\mathcal{V}$ of algebraic varieties defined over fields of characteristic zero (the base field is not fixed), but with minor changes we could work with the more general class of schemes $\mathcal{S}$ introduced in [1, 8.1]. Often we consider functions f from a set S to a totally ordered set Λ. We let $\max(f)$ denote the maximum value of f and $\mathrm{Max}(f)$ the set of points x where $f(x)$ is the maximum.

We denote the natural, rational, complex numbers and the integers by **N**, **Q**, **C** and **Z** respectively.

1.2. Let M be a regular variety, $E = (H_1, \dots, H_m)$ a sequence of regular hypersurfaces of M.

(a) E *has normal crossings* if, for all $x \in H_1 \cup \dots \cup H_m$, the ideal $(I(H_1) \dots I(H_m))_x \subset \mathcal{O}_{M,x}$ is generated by $a_1, \dots, a_r$, where $a_1, a_2, \dots, a_n$ is a suitable regular system of parameters of $\mathcal{O}_{M,x}$, $1 \leqslant r \leqslant n$.

(b) We say that a closed subscheme $V \subset M$ *has normal crossings with respect to* E (resp. *is transversal to* E) if, for all $x \in V$, there is a regular system of parameters $a_1, a_2, \dots, a_n$ of $\mathcal{O}_{M,x}$, such that $I(V)_x = (a_1, \dots, a_r)\mathcal{O}_{M,x}$, $1 \leqslant r \leqslant n$ and, if a divisor H_j contains x, then $I(H_j)_x = (a_i)\mathcal{O}_{M,x}$, for some index i (resp. for some index $i > r$). Such a subvariety V is necessarily regular.

Definition 1.3 *(Marked ideals).* A *marked ideal* is a 5-tuple $\mathcal{I} = (M, W, I, b, E)$ where M is a regular variety (in $\mathcal{V}$), $E = (H_1, \dots, H_m)$ is an ordered m-tuple of distinct hypersurfaces of M with *normal crossings*, W is an equidimensional subvariety of M, transversal to E (hence W is regular), I is a never-zero ideal of W and $b > 0$ is an integer.

Our terminology is borrowed from [3] but the "marked ideals" of [16] are the "basic objects" of [1] (or of 1.4).

1.4. If $\mathcal{I} = (M, W, I, b, E)$ is a marked ideal where $E = (H_1, \dots, H_m)$, then W is its *underlying scheme* and M its *ambient scheme*. They are denoted by $us(\mathcal{I})$ and $as(\mathcal{I})$ respectively. The *dimension* of $\mathcal{I}$ is the dimension of $us(\mathcal{I})$ and the a-dimension of $\mathcal{I}$ is the dimension of $as(\mathcal{I})$.

If U is an open set of M, the restriction of $\mathcal{I}$ to U is the marked ideal $\mathcal{I}_{|U} := (U, U \cap W, I_{|U}, b, E_{|U})$, where $E_{|U} := (H_1 \cap U, \dots, H_m \cap U)$.

A *basic object* is a 4-tuple (W, I, b, E) where W is a regular equidimensional variety, $E = (H_1, \dots, H_m)$ is an ordered m-tuple of distinct hypersurfaces of W with normal crossings, I is a never-zero ideal of W and $b > 0$ is an integer (see [1, 3.1]).

The *b*asic object associated to the marked ideal $\mathcal{I} = (M, W, I, b, E)$ is the 4-tuple $(W, I, b, E_{|W})$.

Here and in the sequel, if E is as above and N is a subscheme of M, then $E_{|N} := (H_1 \cap N, \dots, H_m \cap N)$.

1.5. If $\mathcal{I}$ is a marked ideal, its singular set (or locus) $\mathrm{Sing}(\mathcal{I})$ is $\{x \in W\colon\ \nu_x(I) \geqslant b\}$ (see 1.1).

For $i = 0, \dots, b-1$, there is a W-ideal $\Delta^i(I)$ whose stalk at a closed point x of W is the ideal of $\mathcal{O}_{W,x}$ generated by the derivatives (say, with respect to a regular system of parameters of $\mathcal{O}_{W,x}$) of elements of I_x of order up to i. See [1, 13.4–13.8] or [4, 6.1] for the definition and properties of $\Delta^i(I)$. We have the equality $\mathrm{Sing}(\mathcal{I}) = V(\Delta^{b-1}(I))$. It follows that $\mathrm{Sing}(\mathcal{I})$ is a closed subset of W.

We say that $\mathcal{I}$ is *nonsingular* or *resolved* if $\mathrm{Sing}(\mathcal{I}) = \emptyset$ and *singular* if $\mathrm{Sing}(\mathcal{I}) \neq \emptyset$.

1.6. We use the notation of 1.3.

(a) *Permissible centers and transformations*. We say that a closed subscheme C of W is *permissible* for the marked ideal $\mathcal{I}$ (or that it is $\mathcal{I}$-permissible) if C has normal crossings with E and $C \subseteq \mathrm{Sing}(\mathcal{I})$. A $\mathcal{I}$-permissible center is necessarily a regular subscheme of W.

If C is permissible and $W \leftarrow W_1$ is the blowing-up of W with center C, we define certain W_1-ideals, the *transforms* of the W-ideal I: (i) the *total transform* $I\mathcal{O}_{W_1}$; (ii) the *controlled transform* $I_1 := \mathcal{E}^{-b}\mathcal{O}_{W_1}$, where $\mathcal{E}$ defines the exceptional divisor of the blowing-up; and (iii) the *proper transform* $\bar{I}_1 := \mathcal{E}^{-a} I\mathcal{O}_{W_1}$, where the exponent a is as large as possible (it is locally constant, if C is irreducible then $a = \nu_y(I)$, where y is the generic point of C).

If C is $\mathcal{I}$-permissible, we define the transform of the marked ideal $\mathcal{I}$ with center C as the marked ideal $\mathcal{I}_1 = (M_1, W_1, I_1, b, E_1)$ where M_1 is the blowing-up of M with center C, W_1 is the strict transform of W_1 (identifiable to the blowing-up of W with center C), I_1 is the controlled transform of I, and $E_1 = (H'_1, \dots, H'_m, H'_{m+1})$ (with H'_i the strict transform of H_i, $i = 1, \dots, m$ and H'_{m+1} the exceptional divisor).

We write $\mathcal{I}_1 = \underline{\mathfrak{T}}(\mathcal{I}, C)$ and we denote a transformation of the marked ideal $\mathcal{I}$ by $\mathcal{I} \leftarrow \mathcal{I}_1$.

(b) *Pull-backs*. If $f : M' \to M$ is a smooth morphism, we define the *pull-back* of the marked ideal $\mathcal{I}$ as the marked ideal $f^*(\mathcal{I}) := (M', W', I\mathcal{O}_{W'}, b, E')$, where $W' = f^{-1}(W)$ and $E' = (f^{-1}(H_1), \dots, f^{-1}(H_m))$. If f is an isomorphism, we talk about an *isomorphism* of marked ideals. If $M' = U$ is an open subscheme of M and f is the inclusion, $f^*(\mathcal{I})$ will be called the restriction of $\mathcal{I}$ to U, usually denoted by $\mathcal{I}_{|U}$.

(c) *Extensions*. In the special case $M' = M \times_k \mathbf{A}^1$ (with $\mathbf{A}^1 = \mathbf{A}^1_k$, where M is defined over the field k) and f the first projection, there is another natural hypersurface in M', viz. $H' = M \times \{0\}$, where 0 is the origin of $\mathbf{A}^1$. Then the marked ideal $\mathcal{I}(e) := (M', W', I_1, b, E(e))$, where everything is as in $f^*(\mathcal{I})$, except that now $E(e) := (f^{-1}(H_1), \dots, f^{-1}(H_m), H')$, is called the *extension* of $\mathcal{I}$ (terminology of [6]). This extension will be denoted by $\underline{\mathcal{E}}(\mathcal{I})$.

(d) *Resolutions*. A *resolution* of a marked ideal $\mathcal{I}$ is a sequence of marked ideals and permissible transformations, $\mathcal{I} := \mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_r$, such that $\mathrm{Sing}(\mathcal{I}_r) = \emptyset$.

1.7. We shall need a slight generalization of concepts already introduced.

(a) *Open transformations*. An *open permissible transformation* of a marked ideal $\mathcal{I} = (M, W, I, b, E)$, relative to an open set $U \subseteq M$, is a permissible transformation of $\mathcal{I}_{|U}$ with an $\mathcal{I}_{|U}$-permissible center $C \subset U \cap W$. An *open permissible transformation* of a marked ideal $\mathcal{I} = (M, W, I, b, E)$ is one relative to some open set $U \subseteq M$ and a suitable $\mathcal{I}_{|U}$-center C. Then the marked ideal $\mathcal{I}_1 = \underline{\mathfrak{T}}(\mathcal{I}_{|U}, C)$ is called an *open transform* of $\mathcal{I}$. We shall denote this open transformation by $\mathcal{I} \dashleftarrow \mathcal{I}_1$ if there is no risk of confusion. The set U will be called the open (set) of definition of the open transformation.

A sequence $\mathcal{I}_0 \dashleftarrow \cdots \dashleftarrow \mathcal{I}_r$ of marked ideals is *open permissible* if each broken arrow stands for an open permissible transformation.

(b) *Open extensions.* An open extension of a marked ideal $\mathcal{I}=(M,W,I,b,E)$ is a marked ideal of the form $\underline{\mathcal{E}}(\mathcal{I}_{|U})$ where U is a dense open set of M (i.e., the extension of $\mathcal{I}_{|U}$, see 1.6(c)). The set U is called the *open of definition* of the open extension.

1.8. A sequence $W_0 \leftarrow \cdots \leftarrow W_r$ of algebraic varieties (in the class $\mathcal{V}$, 1.1) and morphisms is called an *open test sequence* (of varieties) if each morphism $W_i \leftarrow W_{i+1}$ is either the blowing-up of an open subvariety $U_i \subseteq W_i$ with a regular center $C_i \subset U_i$ or a projection $U_{i+1} = U_i \times \mathbf{A}^1 \to U_i$, where U_i is a dense open set in W_i. If always $U_i = W_i$, we talk about a *test sequence* (*of varieties*) (terminology borrowed from [3]).

A sequence

$$\mathcal{I}=\mathcal{I}_0 \dashleftarrow \cdots \dashleftarrow \mathcal{I}_r \tag{1}$$

of marked ideals ($\mathcal{I}_i=(M_i,W_i,I_i,b,E_i)$) where each arrow stands for either an open permissible transformation or an open extension 1.6, is called an *open trial sequence* of the marked ideal $\mathcal{I}$. It induces an open test sequence of varieties $W_0 \leftarrow \cdots \leftarrow W_r$. If all the open permissible transformations and extensions used in the sequence (1) are defined everywhere, we call it a *trial sequence*.

Definition 1.9 *(Equivalence and total equivalence).* (a) We say that marked ideals $\mathcal{I}=(M,W,I,b,E)$ and $\mathcal{I}'=(M,W',I',b',E)$ are *equivalent* if they induce the same test sequences of varieties. We write $\mathcal{I}\sim\mathcal{I}'$ to indicate equivalence. In [3], this notion is called *weak equivalence*.

In other words, $\mathcal{I}\sim\mathcal{I}'$ means: $\mathrm{Sing}(\mathcal{I})=\mathrm{Sing}(\mathcal{I}')$ (hence a center C is permissible for $\mathcal{I}$ if and only if C is permissible for $\mathcal{I}'$); if $\mathcal{I}_1$ (resp. $\mathcal{I}'_1$) is either a permissible transformation of $\mathcal{I}$ (resp. of $\mathcal{I}'$, with the same center) or an extension of $\mathcal{I}$ (resp. of $\mathcal{I}'$), then $\mathrm{Sing}(\mathcal{I}_1)=\mathrm{Sing}(\mathcal{I}'_1)$, etc. For any positive integer r, if we repeat this process $r-1$ times, we obtain trial sequences $\mathcal{I}\leftarrow\cdots\leftarrow\mathcal{I}_r$ and $\mathcal{I}'\leftarrow\cdots\leftarrow\mathcal{I}'_r$ respectively. Then it must be $\mathrm{Sing}(\mathcal{I}_r)=\mathrm{Sing}(\mathcal{I}'_r)$.

(b) We say that marked ideals $\mathcal{I}=(M,W,I,b,E)$ and $\mathcal{I}'=(M,W',I',b',E)$ (with the same ambient scheme and set E of hypersurfaces) are *totally equivalent* if they induce, as explained in (a), the same open trial sequences of varieties. We write $\mathcal{I}\overset{T}{\sim}\mathcal{I}'$ to indicate total equivalence.

Hence, to write $\mathcal{I}\overset{T}{\sim}\mathcal{J}$ means that any open trial sequence

$$\mathcal{I}=\mathcal{I}_0 \dashleftarrow \cdots \dashleftarrow \mathcal{I}_s \tag{1}$$

induces an open trial sequence

$$\mathcal{J}=\mathcal{J}_0 \dashleftarrow \cdots \dashleftarrow \mathcal{J}_s \tag{2}$$

with the same opens and centers and vice versa. More precisely if, in (1), $\mathcal{I}_i \dashleftarrow \mathcal{I}_{i+1}$ is the open transformation determined by the open $U_i \subseteq as(\mathcal{I}_i)$ and center $C_i \subset U_i$, then, in (2), $\mathcal{J}_i \dashleftarrow \mathcal{J}_{i+1}$ is the open transformation determined by the same open and center. Similar considerations apply for open extensions.

(c) It is easily seen that total equivalence implies equivalence.

Also, total equivalence is inherited by open restriction in the following sense.

Proposition 1.10. *Let $\mathcal{I}$ and $\mathcal{J}$ be totally equivalent marked ideals. Suppose $\mathcal{I}=\mathcal{I}_0\leftarrow\cdots\leftarrow\mathcal{I}_s$ and $\mathcal{J}=\mathcal{J}_0\leftarrow\cdots\leftarrow\mathcal{J}_s$ are permissible sequences of transformations, both obtained by using the same permissible centers (hence $M_i=as(\mathcal{I}_i)=as(\mathcal{J}_i)$, for all i). Let U be any open set in $W M_s$. Then, $\mathcal{I}_{s|U}\overset{T}{\sim}\mathcal{J}_{s|U}$.*

Proof. Consider an open trial sequence

$$\mathcal{I}_{s|U} \dashleftarrow \mathcal{I}_{s+1} \dashleftarrow \cdots \dashleftarrow \mathcal{I}_{s+q}. \tag{1}$$

Concatenating the first given sequence and (1), we get an open trial sequence

$$\mathcal{I} = \mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_s \dashleftarrow \mathcal{I}_{s+1} \dashleftarrow \cdots \dashleftarrow \mathcal{I}_{s+q}. \tag{2}$$

Since $\mathcal{I} \overset{T}{\sim} \mathcal{J}$, sequence (2) induces an open trial sequence (with the same opens and centers)

$$\mathcal{J} = \mathcal{J}_0 \leftarrow \mathcal{J}_s \leftarrow \mathcal{J}_s \dashleftarrow \mathcal{J}_{s+1} \dashleftarrow \cdots \dashleftarrow \mathcal{J}_{s+q}, \tag{3}$$

hence an open trial sequence

$$\mathcal{J}_s \dashleftarrow \mathcal{J}_{s+1} \dashleftarrow \cdots \dashleftarrow \mathcal{J}_{s+q}. \tag{4}$$

Since the open set of definition of $\mathcal{I}_s \dashleftarrow \mathcal{I}_{s+1}$ is contained in U, sequence (4) also determines a trial sequence

$$\mathcal{J}_{s|U} \dashleftarrow \mathcal{J}_{s+1} \dashleftarrow \cdots \dashleftarrow \mathcal{J}_{s+q}, \tag{5}$$

with the same opens and centers as (1). So, sequence (1) induces the open trial sequence (5) with the same opens and centers. The arguments are reversible, so that an open trial sequence (5) for $\mathcal{J}_{s|U}$ induces an open trial sequence (1) for $\mathcal{I}_{s|U}$. This means that $\mathcal{I}_{s|U}$ and $\mathcal{J}_{s|U}$ are totally equivalent. □

1.11. With the method of proof of Proposition 1.10 one easily obtains the following result:

If $\mathcal{I}_0$ and $\mathcal{J}_0$ are equivalent marked ideals, $\mathcal{I} = \mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_s$ and $\mathcal{J} = \mathcal{J}_0 \leftarrow \cdots \leftarrow \mathcal{J}_s$ are permissible sequences of transformations, both obtained by using the same permissible centers, then $\mathcal{I}_s$ and $\mathcal{J}_s$ are equivalent.

Notice that no reference to restrictions to an open set of $as(\mathcal{I}_s) = as(\mathcal{J}_s)$ is made. But we do not know whether the analog of Proposition 1.10, where *total equivalence* is replaced by *equivalence* (and we consider possible restrictions to an open set), is valid or not. Were permissible centers "extendable" in the following sense, then we could easily prove the latter statement.

Given a regular variety M and a dense open set $U \subseteq M$, we say that a marked ideal $\mathcal{I} = (U, V, I, b, E)$ is *M-extendable* if there is a marked ideal $\mathcal{I}'$ such that $as(\mathcal{I}') = M$ and $\mathcal{I}'_{|U} = \mathcal{I}$.

But not any marked ideal $\mathcal{I}$ with ambient scheme U, with U a dense open set of a regular variety M, is M-extendable, as the next example shows.

Example. We work with $k = \mathbf{C}$ (the complex numbers). Let $M = \mathbf{A}^2 = \operatorname{Spec}(k[x, y])$, $0 = (x, y)$ its origin, $U = M \setminus \{0\}$, $Y = V(y)$ (the x-axis), Z the parabola $V(x^2 - y)$, $Y' = Y \setminus \{0\}$, $Z' = Z \setminus \{0\}$. Consider $\mathcal{I} = (U, U, (xy), 2, (Y', Z'))$. Then $\mathcal{I}$ is not M-extendable. Indeed, the last entry of the extension would be $E = (Y, Z)$ and these hypersurfaces do not have normal crossings.

The following theorem, due to Hironaka [9], is very important. Sometimes this result (or the idea of its proof) is called *Hironaka's trick*.

Theorem 1.12. *Suppose $\mathcal{I} = (M, W, I, b, E)$ and $\mathcal{I}' = (M, W', I', b', E)$ are equivalent marked ideals with* $\dim W = \dim W'$, $S := \operatorname{Sing}(\mathcal{I}) = \operatorname{Sing}(\mathcal{I}')$. *Then for every point $x \in S$ we have $\nu_x(I)/b = \nu_x(I')/b'$.*

A proof, in the context of marked ideals, is found in [3, Theorem 6.1]; another one, working with basic objects, appears in [6, Proposition 7.3].

2. Some numerical functions

Definition 2.1 *(Algorithmic resolutions).* A resolution algorithm for marked ideals is a rule that assigns to each natural number d a totally ordered set $\Lambda^{(d)}$ and to each marked ideal $\mathcal{I}$ (with $\mathrm{Sing}(\mathcal{I}) \neq \emptyset$) functions $g_0, g_1, \dots, g_r$, with values in $\Lambda^{(d)}$, with the following properties.

The function g_0 is defined on $\mathrm{Sing}(\mathcal{I})$, is upper semicontinuous and (the closed set) $C_0 := \mathrm{Max}(g_0)$ is a permissible center for $\mathcal{I}$ (as in 1.1, $\mathrm{Max}(g_0)$ denotes the set of points where the function g_0 reaches its maximum). Let $\mathcal{I}_1$ be the transform of $\mathcal{I}$ with center C_0. If $\mathrm{Sing}(\mathcal{I}_1) = \emptyset$, then $r = 1$ and the asignment is complete. Otherwise, an upper semicontinuous function $g_1 : \mathrm{Sing}(\mathcal{I}_1) \to \Lambda^{(d)}$ is assigned, such that $C_1 := \mathrm{Max}(g_1)$ is a permissible center for $\mathcal{I}_1$. Continuing in this way, the algorithm defines $\Lambda^{(d)}$-valued upper semicontinuous functions g_j, $j = 0, \dots,$ such that the domain of g_j is $\mathrm{Sing}(\mathcal{I}_j)$, where (for each j) $\mathcal{I}_j = \underline{\mathcal{T}}(\mathcal{I}_{j-1}, C_{j-1})$, and $C_{j-1} = \mathrm{Max}(g_{j-1})$ is an $\mathcal{I}_{j-1}$ permissible center. This process terminates, that is there is an index r (depending on $\mathcal{I}$) such that $\mathrm{Sing}(\mathcal{I}_r) = \emptyset$.

In other words, the resulting permissible sequence $\mathcal{I} \leftarrow \mathcal{I}_1 \leftarrow \cdots \leftarrow \mathcal{I}_r$ of marked ideals is a resolution of $\mathcal{I}$.

We are interested in algorithms which, additionally, satisfy the following compatibility conditions:

(a) *Compatibility with open immersions*: if $\mathcal{I} = (M, W, I, b, E)$ and U is an open set in M, then the algorithmic resolution functions of $\mathcal{I}$ induce those of $\mathcal{I}_{|U}$ (ignoring situations where there is an induced isomorphism).

(b) *Compatibility with total equivalence*: if $\mathcal{I} = (M, W, I, b, E)$ and $\mathcal{J} = (M, V, J, c, E)$ are totally equivalent marked ideals with $\dim(W) = \dim(V)$, then the algorithmic resolutions functions for $\mathcal{I}$ and $\mathcal{J}$ are the same.

2.2. *The ω and t-functions*. We present two numerical functions associated to marked ideals. These (in the context of basic objects) are studied in [6], where proofs of the facts that we mention can be found.

(a) Assume

$$\mathcal{I} = \mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_s \tag{1}$$

is a sequence of marked ideals, where each arrow stands for either a permissible transformation or an isomorphism (we write $\mathcal{I}_i = (M_i, W_i, I_i, b, E_i)$). For $x \in \mathrm{Sing}(\mathcal{I}_i)$, we define $\omega_i(x) := \nu_x(\bar{I}_i)/b$ (see 1.6(a)). The function ω_i is denoted by w-ord$_i$ in [6] and [1], where the authors work with basic objects.

The sequence (1) is called a *ω-sequence* if each center C_i used in a permissible transformation satisfies $C_i \subseteq \mathrm{Max}(\omega_i)$. If (1) is an ω-sequence then, if $x \in \mathrm{Sing}(\mathcal{I}_{i+1})$ and x' is the image of x in W_i, we have $\omega_{i+1}(x) \leqslant \omega_i(x')$ (see [6, Remark 4.12]).

(b) Given an ω-sequence (1), we define functions $t_j : \mathrm{Sing}(\mathcal{I}_j) \to \mathbf{Q} \times \mathbf{Z}$ as follows. If $x_j \in \mathrm{Sing}(\mathcal{I}_j)$, $i \leqslant j$, let $x_i \in \mathrm{Sing}(\mathcal{I}_i)$ be the image of x_j via the natural morphism $\mathrm{Sing}(\mathcal{I}_j) \to \mathrm{Sing}(\mathcal{I}_i)$ determined by (1) (so $\omega(x_i) \leqslant \omega(x_j)$ for all i) and q the smallest index such that $\omega(x_q) = \omega(x_j)$ (so if $j = 0$ then $q = 0$). Let $n_j(x_j)$ denote the number of hypersurfaces in E_j which are strict transforms of hypersurfaces in E_q and contain x_j. We set $t_j(x) := (\omega_j(x_j), n_j(x_j))$.

2.3. The sequence 2.2(1) is called a *t-sequence* if, whenever an arrow $\mathcal{I}_i \leftarrow \mathcal{I}_{i+1}$ represents a permissible transformation of marked ideals, its center C_i satisfies $C_i \subseteq \mathrm{Max}(t_i)$. Then we have $\max(t_{i+1}) \leqslant \max(t_i)$ for all i [6, 4.15].

It is clear that if U is an open set in $W_0 = us(\mathcal{I})$ then the t-permissible sequence above induces a permissible sequence starting from $\mathcal{I}_{|U}$ (ignoring those arrows which correspond to isomorphisms). We express this property by saying "t is compatible with open immersions or inclusions".

2.4. *Monomial objects*. A marked ideal $\mathcal{I} = (M, W, I, b, E)$, $E = (H_1, \dots, H_m)$ is *monomial* if for each $z \in W$ we have: $I_z = I(H_1)^{\alpha_1(z)} \dots I(H_m)^{\alpha_m(z)}$, where each function $\alpha_i : W \to \mathbf{Z}$ is constant on each irreducible component of H_i and zero outside H_i. If $\mathcal{I}$ is monomial, one may introduce a function $\Gamma = \Gamma_{\mathcal{I}}$ from $S := \mathrm{Sing}(\mathcal{I})$ to $\mathbf{Z} \times \mathbf{Q} \times \mathbf{Z}^{\mathbf{N}}$, by the formula

$$\Gamma(z) = \big(-\Gamma_1(z), \Gamma_2(z), \Gamma_3(z)\big),$$

where Γ_i, $i = 1, 2, 3$ are defined as follows:

If $z \in S$, $\Gamma_1(z)$ is the smallest integer p such that there are indices $i_1, \ldots, i_p$ such that $\alpha_{i_1}(z) + \cdots + \alpha_{i_p}(z) \geqslant b$.

Consider, for $z \in S$, the set $P'(z)$ of sequences $i_1, \ldots, i_p$ satisfying this inequality, then $\Gamma_2(z)$ is the maximum of the rational numbers $(\alpha_{i_1}(z) + \cdots + \alpha_{i_p}(z))/b$, for $(i_1, \ldots, i_p) \in P'(z)$.

If $z \in S$, let $P(z)$ be the set of all sequences $(i_1, \ldots, i_p, 0, 0, \ldots)$ such that $(\alpha_{i_1}(z) + \cdots + \alpha_{i_p}(z))/b = \Gamma_2(z)$, lexicographically ordered; then $\Gamma_3(z) \in \mathbf{Z}^{\mathbf{N}}$ is the maximum of $P(z)$.

Next we list some important properties of the function Γ. Proofs are found in [6, Section 5], [4, 6.4], or [3, Section 5], where the authors work with basic objects, but the arguments in the context of marked ideals are practically the same.

(a) When the target is lexicographically ordered the function Γ is upper semicontinuous.

(b) If max $(\Gamma_3) = (i_1, \ldots, i_p, 0, 0, \ldots)$ and $C = H_{i_1} \cap \cdots \cap H_{i_p}$, then C is a permissible center for the marked ideal $\mathcal{I}$, called the *canonical monomial center*. The transform $\mathcal{I}_1$ of $\mathcal{I}$ is again monomial, satisfying $\max(\Gamma_{\mathcal{I}_1}) < \max(\Gamma_{\mathcal{I}})$ (see [4, 6.17]).

Note that, in fact, $\Gamma_{\mathcal{I}}$ takes values on a well-ordered subset $V_{\mathcal{I}}$ of $\mathbf{Z} \times \mathbf{Q} \times \mathbf{Z}^{\mathbf{N}}$. Thus, $\{\alpha \in V_{\mathcal{I}}\colon \alpha < \max(\Gamma)\}$ is finite. Hence, iterating this process, after a finite number of steps we reach a situation where the singular locus is empty.

It is clear that $\Gamma_{\mathcal{I}}$ is compatible with open restrictions in the sense that if U is open in M then $(\Gamma_{\mathcal{I}})_{|U \cap S} = \Gamma_{\mathcal{I}_{|U}}$ (with $S = \mathrm{Sing}(\mathcal{I})$).

2.5. A permissible sequence $\mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_r$ of marked ideals is called:

(a) a γ*-sequence* if each $\mathcal{I}_i$ is monomial and each center used is a canonical monoidal center.
(b) a ρ*-sequence* if it is a t-sequence 2.3, or a γ-sequence, or there is an index s such that $\mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_s$ is a t-sequence and $\mathcal{I}_{s+1} \leftarrow \cdots \leftarrow \mathcal{I}_r$ is a γ-sequence.

2.6. Notice that, in terms of the ω-functions of 2.2, Theorem 1.12 says that if $\mathcal{I}$ and $\mathcal{J}$ are equivalent marked ideals of the same dimension, then $\omega_0(\mathcal{I}) = \omega_0(\mathcal{J})$. This result generalizes, as we explain next.

Consider a permissible sequence of marked ideals: $\mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_r$ where, for all i, $\mathcal{I}_i = (M_i, W_i, I_i, b, E_i)$. In the notation of 1.6 we write $E_i = (H_{i,1}, \ldots, H_{i,m}, H_{i,m+1}, \ldots, H_{i,m+i})$, where $H_{i,1}, \ldots, H_{i,m}$ are the strict transforms of the hypersurfaces in E_0, $H_{i,m+j}$, $m < j < i$ are the strict transforms of the exceptional divisors that appear when we take the appropriate permissible transforms and $H_{i,m+i}$ is the last exceptional divisor. Then we have an expression

$$I_r = \bar{I}_r I(H_{r,m+1})^{a_1} \ldots I(H_{r,m+r})^{a_r} \tag{1}$$

where each exponent a_j is a function with non-negative integral values, constant on each irreducible component of $H_{r,m+j}$, and zero outside $H_{r,m+j}$.

The functions $\alpha_j := a_j/b$, viewed as functions from $\mathrm{Sing}(\mathcal{I}_r)$ to $\mathbf{Q}$, will be called the α*-functions of* $\mathcal{I}_r$. Of course, they depend on the chosen permissible sequence and not just on $\mathcal{I}_r$.

Then we have:

Proposition 2.7. *Let $\mathcal{I}_0$ and $\mathcal{J}_0$ be equivalent marked ideals of the same dimension. Suppose $\mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_r$ and $\mathcal{J}_0 \leftarrow \cdots \leftarrow \mathcal{J}_r$ are ω-permissible sequences, obtained by using the same centers. Then both the ω- and the α-functions of $\mathcal{I}_r$ and $\mathcal{J}_r$ coincide.*

Proof. It is by induction on the length r of the ω-sequences involved. If $r=0$, the statement concerning α is trivially true since there are no a-functions and that about ω is true by Theorem 1.12. For the inductive step consider, as in (1) of 2.6, the expressions:

$$I_r = \bar{I}_r I(H_{r,m+1})^{a_1} \dots I(H_{r,m+r})^{a_r}, \tag{1}$$

$$J_r = \bar{J}_r I\big(H'_{r,m+1}\big)^{a'_1} \dots I\big(H'_{r,m+r}\big)^{a'_r} \tag{2}$$

(we write $\mathcal{I}_i = (M_i, W_i, I_i, b, E_i)$, $\mathcal{J}_i = (M_i, W'_i, J_i, c, E_i)$). By induction, the only points for which the claimed result is non-trivial are those in $H_{r,m+r}$ (the exceptional divisor of the last blowing-up $p_i : W_r \to W_{r-1}$, with center C_{r-1}). So, let z be in $H_{r,m+r}$ and $z' = p(z) \in W_{r-1}$. Denote by y the generic point of the irreducible component of C_{r-1} containing z'. Then:

$$a_r(z)/b = \omega_{r-1}(y) - 1. \tag{3}$$

This equality follows from the formula $a_{j+1}(z) = \nu(\bar{I}_j, C_j) - b$, an easy consequence of the definitions.

Let ω'_j denotes the j-th ω function determined by $\mathcal{J}_0 \leftarrow \cdots \leftarrow \mathcal{J}_r$ and $\alpha'_i = a'_i/c$ (see (2)). Then, similarly, working with J_r, we obtain:

$$a'_r(z)/c = \omega'_{r-1}(y) - 1. \tag{4}$$

By induction, $\omega_{r-1}(y) = \omega'_{r-1}(y)$. So, (3) and (4) imply $\alpha_r(z) = \alpha'_r(z)$. Also by induction, $\alpha_i(z) = \alpha'_i(z)$, $i = 1, \dots, r-1$.

Now, by taking order in (1) and dividing by b, we obtain:

$$\omega_r(z) = \nu_z(I_r)/b - \alpha_1(z) - \cdots - \alpha_r(z). \tag{5}$$

Similarly, using (2) we get

$$\omega'_r(z) = \nu_z(J_r)/c - \alpha'_1(z) - \cdots - \alpha'_r(z). \tag{6}$$

But I_r and J_r are equivalent 1.11; so by 1.12 $\nu_z(I_r)/b = \nu_z(J_r)/c$. We just saw that $\alpha_i(z) = \alpha'_i(z)$ for all i. Therefore, (5) and (6) imply $\omega_r(z) = \omega'_r(z)$, completing the inductive step. □

Theorem 2.8. *If $\mathcal{I}$ and $\mathcal{I}'$ are equivalent marked ideals, then their t and Γ functions coincide.*

Proof. The statement means: if $\mathcal{I} = \mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_r$ is a ρ-sequence and $\mathcal{J}$ is equivalent to $\mathcal{I}$ then, using the same centers, we get an induced ρ-sequence $\mathcal{J} = \mathcal{J}_0 \leftarrow \cdots \leftarrow \mathcal{J}_r$ so that $\mathcal{I}_i$ is monomial if and only if $\mathcal{J}_i$ is monomial; moreover in the non-monomial (resp. monomial) case the t-functions (resp. Γ-functions) of $\mathcal{I}_i$ and $\mathcal{J}_i$ are equal. This result is an immediate consequence of the definitions, of the fact that the sequence of hypersurfaces E_i is the same both for $\mathcal{I}_i$ and $\mathcal{J}_i$ and of Proposition 2.7. □

Remark 2.9. Some of the notions and results discussed above admit a generalization that will be useful later. The proofs are practically the same, although the notation becomes a little more complicated. We omit them.

First, the ω and t functions can be defined practically in the same way, when the sequence (1) of 2.2 is substituted by an *open* permissible sequence $\mathcal{I}_0 \dashleftarrow \cdots \dashleftarrow \mathcal{I}_s$. Again, when each center C_i is contained in $\mathrm{Max}(t_i)$ (resp. $\mathrm{Max}(\omega_i)$), we say that the sequence is an open t-permissible sequence (resp. open ω-permissible sequence), or just an open t-sequence (resp. open ω-sequence). If the sequence is open t-permissible (resp. open ω-permissible), and if $x \in \mathrm{Sing}(\mathcal{I}_{i+1})$ and x' is its image in

$\mathrm{Sing}(\mathcal{I}_i)$, then $t_i(x') \leqslant t_{i+1}(x)$ (resp. $\omega_i(x') \leqslant \omega_{i+1}(x)$). These notions are compatible with inclusions, as explained in 2.3.

Also, Proposition 2.7 extends to the case of open ω-sequences, and leads to:

Theorem 2.10. *If $\mathcal{I}$ and $\mathcal{I}'$ are totally equivalent marked ideals, then their t and Γ functions coincide.*

This is shown, with minor modifications, with the method of the proof of 2.8.

3. Inductive tools

In this section we present further results about marked ideals, taken primarily from [6] (see also [1] and [4]). In these references the authors work with basic objects, but the transition to our context of marked ideals is straightforward. The proofs are practically the same in both settings, and in general will be omitted. We shall use this material in Section 4, when we introduce resolutions functions using induction on the dimension of the marked ideal.

3.1. *Good marked ideals*. A marked ideal $\mathcal{I} = (M, W, I, b, E)$ is *good* if $\max\{\nu_x(I)\colon x \in \mathrm{Sing}(\mathcal{I})\} = b$ ($\mathcal{I}$ is called *of maximal order* in [3]).

If $\mathcal{I}$ is good, there is a standard way to produce, locally on W, smooth hypersurfaces. Namely, if $x \in \mathrm{Sing}(\mathcal{I})$ then the stalk $\Delta^{b-1}(I)_x$ contains elements f such that $\nu_x(f) = 1$. Hence f defines a smooth hypersurface Z, on an open neighborhood U of x (in W). In general, it won't be true that Z is transversal to E. So, we introduce the next definitions.

3.2. Following the terminology of [13], a marked ideal $\mathcal{I} = (M, W, I, b, E)$ is said to be *nice* if there is a regular hypersurface Z of W such that $\mathrm{Sing}(\mathcal{I}) \subseteq Z$ and Z is transversal to E (see 1.6(b)); that Z will be called an *adapted hypersurface* for $\mathcal{I}$ or an $\mathcal{I}$-adapted hypersurface. Then Z must be smooth and the marked ideal $\mathcal{I}$ must be good.

$\mathcal{I}$ is *locally nice* if for all $x \in W$ there is an open neighborhood U of x (in M) such that the restriction $\mathcal{I}_{|U}$ is nice.

For the remainder of the article, if $\mathcal{I}$ is a marked ideal as above and $F \subseteq W$ is a closed subscheme, by *the codimension of F* we mean the codimension of F in W.

Lemma 3.3. *Let $\mathcal{J} = (U, V, J, c, E)$ be a nice marked ideal, C the union of the one-codimensional irreducible components of* $\mathrm{Sing}(\mathcal{J})$ *and assume $C \neq \emptyset$. Then,*

(a) *C is a permissible $\mathcal{J}$-center.*
(b) *If $\mathcal{J}_1 = \underline{\mathcal{T}}(\mathcal{J}, C)$, then* $\mathrm{Sing}(\mathcal{J}_1)$ *has no one-codimensional irreducible component.*

Proof. (a) Let Z be an adapted hypersurface for the nice marked ideal $\mathcal{J}$. Then, $\mathrm{Sing}(\mathcal{J}) \subseteq Z$. If D is an irreducible component of $\mathrm{Sing}(\mathcal{J})$ of codimension one, for reasons of dimension D must be also an irreducible component of Z. Since Z is adapted, D must be transversal to E, hence it has normal crossings with E.

(b) Since C is a divisor of V, if $p : U' \to U$ is the blowing-up of U with center C and V' is the strict transform of V, then p induces an isomorphism from V' onto V. We identify V and V' by means of this isomorphism. With this identification we may write $\mathcal{J}_1 = (U', V, J_1, c, E')$. Note that by the definition of controlled transform, if $x \in C$ then $(J_1)_x = \mathcal{O}_{V,x}$ and thus $x \notin \mathrm{Sing}(\mathcal{J}_1)$. So, the components of $\mathrm{Sing}(\mathcal{J}_1)$ are those components of $\mathrm{Sing}(\mathcal{J})$ of codimension > 1. $\square$

3.4. *Coefficient ideals, inductive objects*. Given a marked ideal $\mathcal{I} = (M, W, I, b, E)$, its *coefficient ideal* is the W-ideal

$$C(I) := \sum_{i=0}^{b-1} \left[\Delta^i(I)\right]^{b!/b-i}.$$

Assume now that $\mathcal{I}$ is nice with adapted hypersurface Z, and that $C(I)_{|Z}$ (the restriction of $C(I)$ to Z) is a never-zero Z-ideal. Then, the five-tuple $\mathcal{I}_{|Z} = (M, Z, C(I)_{|Z}, b!, E)$ is a new marked ideal, called the *inductive object* (induced by $\mathcal{I}$ on Z). Always $\dim(\mathcal{I}_{|Z}) = \dim(\mathcal{I}) - 1$. Note that if $\dim(\mathrm{Sing}(\mathcal{I})) \leqslant \dim(\mathcal{I}) - 2$, then $C(I)_{|Z}$ is a never-zero Z-ideal. Hence the *inductive marked ideal* $\mathcal{I}_{|Z}$ is defined.

Lemma 3.5. *Let $\mathcal{I} = (M, V, I, b, E)$ be a nice marked ideal, Z an $\mathcal{I}$-adapted hypersurface, C an $\mathcal{I}$-permissible center, $\mathcal{I}_1 = \underline{\mathcal{T}}(\mathcal{I}, C)$, and $Z_1 \subset W_1 = us(\mathcal{I}_1)$ the strict transform of Z. Then, Z_1 is an $\mathcal{I}_1$-adapted hypersurface.*

This result is often called "Giraud's lemma". A proof, in the context of basic objects, but still valid in the present one, can be seen in [6, 9.1, 9.2], and [4, 6.20, 6.21].

Proposition 3.6. *Let*

$$\mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_s$$

be a permissible sequence of marked ideals, where $\mathcal{I}_s$ is nice and admits an inductive hypersurface Z. Then, $\mathcal{I}_s \overset{T}{\sim} (\mathcal{I}_s)_Z$.

Proof. First, suppose that

$$\mathcal{I}_s = (\mathcal{I}_s)_0 \dashleftarrow (\mathcal{I}_s)_1 \dashleftarrow \cdots \dashleftarrow (\mathcal{I}_s)_q, \tag{1}$$

$$(\mathcal{I}_s)_Z = \big[(\mathcal{I}_s)_Z\big]_0 \dashleftarrow \big[(\mathcal{I}_s)_Z\big]_1 \dashleftarrow \cdots \dashleftarrow \big[(\mathcal{I}_s)_Z\big]_q \tag{2}$$

are permissible sequences of open transformations (where (1) involves open sets $U_i \subseteq as((\mathcal{I}_s)_i)$ and centers $C_i \subset U_i \cap Z_i$, with Z_i the strict transform of Z to $us((\mathcal{I}_s)_i)$; and (2) involves the same opens $U_i \cap Z_i$ and centers C_i). Then we have

$$\mathrm{Sing}\big((\mathcal{I}s)_q\big) = \mathrm{Sing}\big(\big[(\mathcal{I}_s)_Z\big]_q\big). \tag{3}$$

Indeed, the proof of [6, 9.4], which uses (for $x \in Z_q$ and x' its image in Z_{q-1}) calculations in the completions of local rings at x and x' respectively, applies without changes to our situation involving open transformations.

Second, note that dealing with extensions we have, for a marked ideal $\mathcal{I}$, an isomorphism $\underline{\mathcal{E}}(\mathcal{I}_Z) \cong \underline{\mathcal{E}}(I)_{Z \times \mathbf{A}^1}$.

Our lemma follows from these observations. □

As in [6], we shall discuss how a useful nice marked ideal may be associated (locally, under suitable hypotheses) to an arbitrary marked ideal $\mathcal{I} = (M, V, I, b, E)$. This process will be useful later.

3.7. *The marked ideal $\mathcal{I}''_r$.* Consider a t-permissible sequence of marked ideals

$$\mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_r \tag{1}$$

where $\mathcal{I}_j = (M_j, W_j, I_j, b, E_j)$, let $x \in \mathrm{Max}(\mathcal{I}_r)$. Then, there is an open neighborhood U of x (in M_r) and a nice marked ideal $(\mathcal{I}_{r|U})''$ with ambient scheme U admitting an adapted hypersurface Z.

To construct it, introduce first a W_r-ideal J_r as follows. Write max(t_r)=($b_r/b, \bar{n}$), $\bar{I}_r$ the proper transform of I_0 to W_r. If $b_r \geqslant b$ let $J_r = \bar{I}_r$.

If $b_r < b$, write $E_r = (H_1, \ldots, H_m, \ldots, H_{m+r})$ (where $H_1, \ldots, H_m$ are the strict transforms of the hypersurfaces that appear in E_0). Then, as in the proof of Proposition 2.7, there is an expression $I_r = \mathcal{C}_r \bar{I}_r$, where $\mathcal{C}_r = I(H_{m+1})^{a_1} \ldots I(H_{m+r})^{a_r}$. Define $J_r = \bar{I}_r^{b-b_r} + \mathcal{C}^{b_r}$.

Now suppose q be the smallest index such that $\max(\omega_q) = \max(\omega_r)$ (note that (1) is an ω-sequence), and let E_j^- denote the set of hypersurfaces in E_j which are strict transforms of hypersurfaces in E_q. Let $H_{j(1)}, \ldots, H_{j(\bar{n})}$ be the hypersurfaces in E_r^- containing the point x and take a neighborhood U of x such that

$$\operatorname{Max}(t_{r|U}) = \operatorname{Max}(\omega_{r|U}) \cap I(H_{j(1)}) \cap \cdots \cap I(H_{j(\bar{n})}).$$

Write $b'_r = b_r$ if $b_r \geqslant b$ and $b'_r = b_r(b - b_r)$ if $b_r < b$. Define the ideal:

$$I''_r := J_r + I(H_{j(1)})^{b'_r} + \cdots + I(H_{j(\bar{n})})^{b'_r}.$$

Let E''_r be the sequence of hypersurfaces in E_r which are *not* in E_r^-.

Finally, define:

$$(\mathcal{I}_r)''_{|U} = \big(U, U \cap W_r, I''_{r\,|U}, b'_r, E''_{r\,|U}\big).$$

We call an open set U as above an *amenable* open set (at x). If $U = W_r$, we simply write $\mathcal{I}''_r = (\mathcal{I}_r)''_{|U}$ and say that $\mathcal{I}''_r$ is *globally defined.*

3.8. Assuming, to simplify the notation, that $U = W_r$, we list next some useful properties of this marked ideal $\mathcal{I}''_r$. This material, in the context of basic objects, is discussed in [6, 9.5] (specially 9.5.7) and [4, 6.32].

(i) $\operatorname{Sing}(\mathcal{I}''_r) = \operatorname{Max}(t_r)$.
(ii) If $C \subset \operatorname{Max}(t_r)$ is an $\mathcal{I}_r$-center (hence, by (i), an $\mathcal{I}''_r$-center), $(\mathcal{I}''_r)_1 = \mathcal{T}(\mathcal{I}''_r, C)$, $\mathcal{I}_{r+1} = \underline{\mathcal{T}}(\mathcal{I}_r, C)$, and $\max(t_r) = \max(t_{r+1})$, then $(\mathcal{I}''_r)_1 = (\mathcal{I}_{r+1})''$.
(iii) Suppose that

$$\mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_s \tag{1}$$

is a t permissible sequence of marked ideals. Let

$$\mathcal{I}''_s \leftarrow \big[\mathcal{I}''_s\big]_1 \leftarrow \cdots \leftarrow \big[\mathcal{I}''_s\big]_q \tag{2}$$

be a permissible sequence of marked ideals, with centers $C_0, \ldots, C_{q-1}$ Then, there is a sequence

$$\mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_s \leftarrow \mathcal{I}_{s+1} \leftarrow \cdots \leftarrow \mathcal{I}_{s+q} \tag{3}$$

extending (1), where the center of the transformation $\mathcal{I}_{s+j} \leftarrow \mathcal{I}_{s+j+1}$ is C_j, $j = 0, \ldots, q-1$. Moreover, for $j = 0, \ldots, q-1$, $\operatorname{Sing}([\mathcal{I}''_s]_j) = \operatorname{Max}(t_{s+j})$, $\max(t_{s+j}) = \max(t_s)$, and if $\max(t_{s+q}) = \max(t_s)$, then again $\operatorname{Max}(t_{s+q}) = \operatorname{Sing}([\mathcal{I}''_s]_q)$. We also have $\operatorname{Sing}([\mathcal{I}''_s]_q) = \emptyset$ if and only if either $\operatorname{Sing}(\mathcal{I}_{s+q}) = \emptyset$ or $\max(t_s) > \max(t_{s+q})$.

For the proofs of (i) and (ii) see the cited references, property (iii) is obtained by repeated application of (ii).

Proposition 3.9. *Let*

$$\mathcal{I} = \mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_r \tag{1}$$

be a t-permissible sequence of marked ideals, C the union of the one-codimensional irreducible components of $\operatorname{Max}(t_r)$*, and assume $C \neq \emptyset$. Then,*

(a) *C is t-permissible center for $\mathcal{I}_r$ (i.e., it is regular, with normal crossings with E_r).*
(b) *If we extend* (1) *to another t-permissible sequence by using $\mathcal{I}_{r+1} := \underline{\mathcal{T}}(\mathcal{I}_r, C) = (M_{r+1}, W_{r+1}, I_{r+1}, b, E_{r+1})$, then* $\mathrm{Max}(t_{r+1})$ *has no irreducible components of codimension one.*

Proof. Both (a) and (b) are consequences of the following observation. If $x \in C$, we may take an amenable open set U such that the nice marked ideal $(\mathcal{I}_{|U})''$ is defined; hence Lemma 3.3, applied to $(\mathcal{I}_{|U})''$, ensures that $\mathrm{Sing}((\mathcal{I}_{|U})'')$ satisfies (a) and (b) of 3.3. □

Proposition 3.10. *Let $\mathcal{I}$ and $\mathcal{J}$ be marked ideals, $\mathcal{I} \overset{T}{\sim} \mathcal{J}$, $\mathcal{I} = \mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_s$, $\mathcal{J} = \mathcal{J}_0 \leftarrow \cdots \leftarrow \mathcal{J}_s$ t-permissible sequences, obtained by using the same t-permissible centers $C_0, \ldots, C_{s-1}$. Suppose that $U \subseteq W_s = us(\mathcal{I}_s) = us(\mathcal{J}_s)$ is an open set, amenable for both $\mathcal{I}_s$ and $\mathcal{J}_s$. Then $(\mathcal{I}_{s|U})'' \overset{T}{\sim} (\mathcal{J}_{s|U})''$.*

Proof. We must study what happens under: (a) permissible open transformations, (b) extensions.

We examine case (a) first. Write $\mathcal{I}_{sU} := (\mathcal{I}_{s|U})''$ and $\mathcal{J}_{sU} := (\mathcal{J}_{s|U})''$, to simplify, let $\tau = \max(t_s)$, $N = \mathrm{Max}(t_s) = \{x\colon\ t_s(x) = \tau\}$, where t_s denotes the t-function of $\mathcal{I}_s$. But, since $\mathcal{I}_s \sim \mathcal{J}_s$, by Theorem 2.8, the t-function of $\mathcal{J}_s$ is again t_s. So, $\mathrm{Sing}(\mathcal{I}_{sU}) = \mathrm{Sing}(\mathcal{J}_{sU})$ because, by 3.8(i), both are equal to $N \cap U$.

Now take an open set $V \subseteq U$ and a common permissible center C for both $\mathcal{I}_{sU|V}$ and $\mathcal{J}_{sU|V}$. Assume $V \cap N \neq \emptyset$ (the only nontrivial case). Note that $\mathcal{I}_{sU|V} = (\mathcal{I}_{s|V}) := \mathcal{I}_{sV}$ and $\mathcal{J}_{sU|V} = (\mathcal{J}_{s|V}) := \mathcal{J}_{sV}$. Transform $\mathcal{I}_{s|V}$, $\mathcal{J}_{s|V}$, $\mathcal{I}_{sV}$ and $\mathcal{J}_{sV}$, using C as center in all cases. We get marked ideals $(\mathcal{I}_{s|V})_1$, $(\mathcal{J}_{s|V})_1$, $(\mathcal{I}_{sV})_1$, $(\mathcal{J}_{sV})_1$ respectively. Since $\mathcal{I} \overset{T}{\sim} \mathcal{J}$, by 2.8, the t-functions of both $(\mathcal{I}_{|V})_1$ and $(\mathcal{J}_{|V})_1$ agree, say they are $= t'_1$. So, $\max(t'_1) = \tau$ and $\mathrm{Sing}(\mathcal{I}_{sV}) = \mathrm{Sing}(\mathcal{J}_{sV})$ because both are equal to $N_1 = \{x\colon\ t'_1(x) = \tau\}$. If N_1 is empty, we are done because both ideals are trivial (equal to the structure sheaf). If N_1 is non-empty, we may repeat the argument. By iteration, we have the result in this case.

Case (b) is a consequence of the fact that a t-sequence $\mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_s$ of marked ideals induces a t-sequence $\underline{\mathcal{E}}(\mathcal{I})_0 \leftarrow \cdots \leftarrow \underline{\mathcal{E}}(\mathcal{I})_s$ of extensions 1.6(c), and then there is an isomorphism $\underline{\mathcal{E}}(\mathcal{I}_s'') \cong (\underline{\mathcal{E}}(\mathcal{I})_s)''$. □

4. A resolution algorithm

In this section we prove the following result:

Theorem 4.1. *To each marked ideal $\mathcal{I}$ we may attach algorithmic resolution functions g_i, $i = 0, \ldots, r-1$ (where r depends on $\mathcal{I}$); this process satisfies conditions* (a), (b) *of* 2.1 *and, moreover, the following condition*:

(c) *The resolution $\mathcal{I} = \mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_r$ determined by the functions g_i (i.e., the i-th center is* $\mathrm{Max}(g_i)$*) is a ρ-sequence* (*see* 2.5)*.*

To prove the theorem we shall introduce, for each positive integer d, a totally ordered set $\Lambda^{(d)}$ and for each d-dimensional marked ideal $\mathcal{I}$ functions g_i (with domains as in Definition 2.1 and with values in $\Lambda^{(d)}$), which will be algorithmic resolution functions as needed. We discuss separately the cases $d = 1$ and d arbitrary. The details will be presented in Sections 4.2 through 4.5.

4.2. *The functions g_i when $d = 1$.* If $\dim(\mathcal{I}_0) = 1$, we set $\Lambda^{(1)} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\infty_1\}$, where if $a \in \mathcal{S}_2$ and $b \in \mathcal{S}_1$, then $a > b$ and ∞_1 is the largest element of the set. Next, we define for $w \in \mathrm{Sing}(\mathcal{I}_0)$, $g_0(w) = t_0(w)$. If g_i is defined for $i < s$, determining a t-permissible sequence $\mathcal{I}_0 \leftarrow B_1 \leftarrow \cdots \leftarrow \mathcal{I}_s$, then we set, for $w \in \mathrm{Sing}(\mathcal{I}_s)$, $g_s(w) = t_s(w)$ if $\omega_s(w) > 0$, while $g_s(w) = \Gamma_{B_s}(w)$ in case $\omega_s(w) = 0$. Since in this one-dimensional situation $C_i = \mathrm{Max}(g_i)$ is always a finite collection of closed points, it follows that these are permissible centers.

Now we examine conditions (a), (b) and (c) in Theorem 4.1. Clearly the functions g_i satisfy condition (a) (compatibility with open immersions) and they also satisfy (b) by Theorem 2.8. Concerning (c), if for certain s the marked ideal $\mathcal{I}_s$ is monomial, then all the successive marked ideals will be so. Proposition 3.9 implies that, for some s, $\mathcal{I}_s$ is either nonsingular or monomial, hence (c) is also satisfied.

Proposition 3.9 and 2.4(b) show that these are indeed resolution functions.

4.3. *The functions g_i in general.* Now assuming that the resolution functions are defined when the dimension of the marked ideal is $< d$, we define resolution functions g_j for objects of dimension d. If $d > 1$, the totally ordered set of values will be: $\Lambda^{(d)} = (\mathcal{S}_1 \times \Lambda^{(d-1)}) \cup \mathcal{S}_2 \cup \{\infty_d\}$, where $\mathcal{S}_1 \times \Lambda^{(d-1)}$ is lexicographically ordered, any element of $\mathcal{S}_2$ is larger than any element of $\mathcal{S}_1 \times \Lambda^{(d-1)}$, and ∞_d is the largest element of $\Lambda^{(d)}$.

Given a marked ideal $\mathcal{I}_0$, we shall define the corresponding function g_0.

For a point $x \in \mathrm{Sing}(\mathcal{I}_0)$, we necessarily have $\omega_0(x) > 0$. Let N_1 be the union of the 1-codimensional components of $\mathrm{Max}(t_0)$; there are two cases: (i) $N_1 \neq \emptyset$; (ii) $N_1 = \emptyset$. Let $x \in \mathrm{Sing}(\mathcal{I}_0)$.

In case (i),

$$g_0(x) = \begin{cases} \infty_d & \text{if } x \in N_1, \\ (t_0(x), \infty_{d-1}) & \text{otherwise.} \end{cases}$$

So, the 0-th center C_0 is N_1. By Proposition 3.9, $C_0 = N_1$ is a permissible center.

In case (ii), pick up an amenable open neighborhood U of x with adapted hypersurface $Z \subset U$, and take the associated nice marked ideal $(\mathcal{I}_{0|U})''$. Consider the inductive marked ideal $\mathcal{I}^*_Z := ((\mathcal{I}_{0|U})'')_Z$. By induction on the dimension, for $\mathcal{I}^*_Z$ there are defined resolution functions $\widetilde{g_{Z,i}}$; then set $g_0(x) := (t_0(x), \widetilde{g_{Z,0}}(x))$. We claim that if a different amenable open set and adapted hypersurface were chosen, then the result would be the same. First of all, since t_0 and, by induction, the resolution functions in dimension $d-1$ are compatible with restrictions to open sets, we may assume that the open set U is the same in both cases. Let Z' be the new adapted hypersurface. Now, by Proposition 3.10 (in the special case $\mathcal{I} = \mathcal{J}$), $\mathcal{I}^*_Z \overset{T}{\sim} \mathcal{I}^*_{Z'}$. Since $\dim(Z) = \dim(Z') < d$, by induction (b) is satisfied, hence $\widetilde{g_{Z,0}}(x) = \widetilde{g_{Z',0}}(x)$. So, the value $g_0(x)$ is independent of the choices, and g_0 is well defined.

Suppose now that the resolutions functions g_i, $i = 0, \dots, j-1$ (satisfying (a), (b) and (c) of 4.1) have been defined, determining centers $C_i = \mathrm{Max}\,(g_i)$, $i = 0, \dots, j-1$, and leading to a permissible sequence $\mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_j$, $\mathcal{I}_i = (M_i, W_i, I_i, b, E_i)$, $i = 0, \dots, j$, $j \geqslant 0$. We assume that if $\mathcal{I}_{j-1}$ is not a monomial object, then this is a t-sequence. There are two possible cases: (A) $\max(\omega_j) = 0$, and (B) $\max(\omega_j) > 0$.

In case (A), $\mathcal{I}_j$ is monomial. For $x \in \mathrm{Sing}(\mathcal{I}_j)$ let Γ_j be its Γ-function and set $g_j(x) := \Gamma_j(x)$.

In case (B), letting $N_1(j)$ denote the union of the one-codimensional components of $\mathrm{Max}(t_j)$, there are two subcases: (B_1) $N_1(j) \neq \emptyset$, (B_2) $N_1(j) = \emptyset$.

In case (B_1), set $g_j(x) = \infty_d$, if $x \in N_1(j)$, and set $g_j(x) = (t_j(x), \infty_d)$, if $x \in \mathrm{Sing}(\mathcal{I}_j)$ but $x \notin M_1(j)$.

4.4. *The case (B_2) (the inductive situation).* In this case, if $x \in \mathrm{Sing}(\mathcal{I}_j) \setminus \mathrm{Max}(t_j)$, set $g_s(x) = (t_j(x), \infty_d)$. If $x \in \mathrm{Max}(t_s)$, let s be the smallest index such that $\max(t_s) = \max(t_j)$ and $N_1(s)$ (the union of one-codimensional components of $\mathrm{Max}(t_s)$) is empty. Let x_s be the image of x in W_s. Proceed as in case $j = 0$: pick up an amenable open neighborhood U of x_s, with adapted hypersurface $Z_s \subset U$, and take the associated nice marked ideal $(\mathcal{I}_{s|U})''$. Consider the inductive marked ideal $\mathcal{I}^*_{s\,Z_s} := ((\mathcal{I}_{0|U})'')_{Z_s}$. By induction on the dimension, for $\mathcal{I}^*_{s\,Z}$ there are defined resolution functions $\widetilde{g_{Z_s,i}}$, $i = s, \dots$. Set $g_j(x) := (t_j(x), \widetilde{g_{Z_s,j}}(x))$.

We assert that the final result is not affected by a different choice of the amenable open set and adapted hypersurface. To see this fact, we may assume the open set U to be the same in both cases, because t_j and, by induction, the resolution functions in dimension $d-1$ are compatible with restrictions to opens. Let Z'_s be the new adapted hypersurface. By Proposition 3.10, $\mathcal{I}^*_{s\,Z_s} \overset{T}{\sim} \mathcal{I}^*_{s\,Z'_s}$. Since by induction on the dimension condition (b) is satisfied, $\widetilde{g_{Z_s,j}}(x) = \widetilde{g_{Z'_s,j}}(x)$. So, the value $g_j(x)$ is independent of the choices and thus g_j is well defined.

With this definition, if $\mathcal{I}_j$ is not monomial then the center $C_j = \mathrm{Max}(g_j)$ is contained in $\mathrm{Max}(t_j)$, hence condition (c) is valid.

In this situation (B_2), the center C_j locally coincides with one of the algorithmic resolution centers of $\mathcal{I}^*_{s\,Z_s}$ and, moreover, $C_j \subseteq \mathrm{Max}(t_j)$. It follows that C_j is t-permissible.

By induction, the functions $g_0, \dots, g_{j-1}$ satisfy conditions (a), (b), and (c). Concerning g_j, the discussion above shows that condition (c) is valid and (a) is clear.

Let us prove that g_j also satisfies condition (b). Suppose that $\mathcal{I} = (M, W, I, b, E)$ and $\mathcal{J} = (M, V, J, c, E)$ are totally equivalent marked ideals with $\dim(W) = \dim(V)$, and that g_i (resp. g'_i), $i = 0, \dots, j$, are the resolution functions of $\mathcal{I}$ (resp. $\mathcal{J}$), constructed as above. By induction, $g_i = g'_i$, $i < j$; so by using centers $C_i = \mathrm{Max}(g_i) = \mathrm{Max}(g'_i)$ $(0 \leqslant i < j)$, we obtain permissible ρ-sequences

$$\mathcal{I} = \mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_j, \tag{1}$$

$$\mathcal{J} = \mathcal{J}_0 \leftarrow \cdots \leftarrow \mathcal{J}_j. \tag{2}$$

We shall see that for every point $x \in \mathrm{Sing}(\mathcal{I}_j) = \mathrm{Sing}(\mathcal{J}_j)$,

$$g_i(x) = g'_i(x), \tag{3}$$

this will show that g_j also satisfies (b).

We proceed by induction on the $d = \dim W = \dim V$. If $\dim W = 1$, the statement follows from 2.8.

Suppose now the equality (3) valid when the dimension is less than d, let $\dim \mathcal{I} = d$. The only case worth considering is (B_2), the other cases being a consequence of 2.8 and 3.9. Since $\mathcal{I} \sim \mathcal{J}$, by 2.8 the t-functions $t_0, t_1, \dots, t_j$ of the sequences (1) and (2) are the same. For the same reason, the index s used in (B_1) is the same. Now, the functions g_i and g'_i satisfy (a), and we know that $g_i = g'_i$ for $i < j$. Then, from 3.10 and the fact that $\mathcal{I}$ and $\mathcal{J}$ are totally equivalent (with the notation used in the discussion of case (B_1)) we may find a common open neighborhood U of x_s (the image of x in M_s) and adapted hypersurfaces Z (for $\mathcal{I}_s$) and Z' (for $\mathcal{J}_s$) such that $(\mathcal{I}_{|U})''_{|Z} \overset{T}{\cong} (\mathcal{J}_{|U})''_{|Z'}$. By inductive hypothesis, the resolution functions $\tilde{g}_i$ and $\tilde{g}'_i$ ($s \leqslant i \leqslant r'$, where $j \leqslant r'$) corresponding to $(\mathcal{I}_{|U})''_{|Z}$ and $(\mathcal{J}_{|U})''_{|Z'}$ respectively, agree. Hence $g_j(x) = (t_j(x), \tilde{g}_j(x)) = (t_j(x), \tilde{g}'_j(x)) = g'_j(x)$, proving (3).

4.5. *The process terminates*. So far, we have obtained well-defined functions g_i satisfying conditions (a), (b), and (c). Once $g_0, \dots, g_{q-1}$ have been defined, by taking centers $C_i = \mathrm{Max}(g_i)$ we obtain a permissible sequence which, moreover, is a ρ-sequence:

$$\mathcal{I} = \mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_q, \tag{σ_q}$$

called an *algorithmic sequence*. We assert that, for a suitable index $q = r$, the sequence (σ_r) is a resolution, i.e., $\mathrm{Sing}(\mathcal{I}_r) = \emptyset$.

This is done in [6], in the context of basic objects. For completeness, we review the main steps.

We proceed by induction on the dimension d of the marked ideal. The case $d = 1$ has been already established.

Note that if in an algorithmic sequence (σ_q), for some index j the marked ideal $\mathcal{I}_j$ is monomial, then all the terms $\mathcal{I}_i$, $i \geqslant j$ are also monomial, obtained as transforms with monomial canonical centers. As remarked in 2.4, for a suitable $r \geqslant j$, $\mathcal{I}_r$ will be resolved. So, it suffices to show that for a certain index q, $\mathcal{I}_q$ is monomial. This will be the case if $\max(\omega_q) = 0$ (a consequence of formula (1) in 2.6 and the definitions).

Now, if $\mathcal{I}_{j-1}$ in (σ_q) is not monomial, then we have a t-sequence. So, the sequence $\{\max(t_i)\}$ is non-increasing. Moreover, since the functions t_i take values in $(1/b)\mathbf{N}_0 \times \mathbf{N}_0$, this sequence takes on finitely many values.

So, to show that eventually we get either a resolved or monomial marked ideal, it suffices to prove the following assertion:

$(*)$ If $\tau = \max(t_s) = (a, n)$, $a > 0$, for some index s in (σ_q), then, for j large enough, in the sequence σ_j we have $\max(t_j) < \tau$.

We prove $(*)$. By 3.9 we may assume that $N = \mathrm{Max}(t_s)$ has codimension > 1 and s is minimal with this property.

Using compactness, cover N with amenable open sets $U_1, \ldots, U_k$, with inductive hypersurfaces $Z_1, \ldots, Z_k$ respectively. Consider for each $\alpha \in \{1, \ldots, k\}$, the inductive marked ideal $\mathcal{I}^*_{\alpha Z_\alpha} := (\mathcal{I}_{|U_\alpha})''_{Z_\alpha}$ and its algorithmic resolution

$$\mathcal{I}^*_{\alpha Z_\alpha} = \left(\mathcal{I}^*_{\alpha Z_\alpha}\right)_0 \leftarrow \left(\mathcal{I}^*_{\alpha Z_\alpha}\right)_1 \leftarrow \cdots \leftarrow \left(\mathcal{I}^*_{\alpha Z_\alpha}\right)_{r(\alpha)}$$

(that we have by induction on the dimension), with algorithmic centers $C^*_{\alpha i}$. Let $m = \max\{r(\alpha)\colon 1 \leqslant \alpha \leqslant k\}$. According to 3.8(iii), in the algorithmic sequence (σ_{s+m}):

$$\mathcal{I} = \mathcal{I}_0 \leftarrow \cdots \leftarrow \mathcal{I}_s \leftarrow \cdots \leftarrow \mathcal{I}_{s+m},$$

if $U_{\alpha i}$ is the pre-image of U_α in $us(\mathcal{I}_{s+i})$, then the algorithmic centers $C_s, \ldots, C_{s+m}$ satisfy $C_{s+i} \cap U_{\alpha i} = C^*_{\alpha i}$, provided that $U_{\alpha i} \cap \{x \in \mathrm{Sing}(\mathcal{I}_{U_{\alpha i}})\colon t_i(x) = \tau\} \neq \emptyset$. By the choice of m, $\{x \in \mathrm{Sing}(\mathcal{I}_{s+m})\colon t_{s+m}(x) = \tau\}$ is empty, i.e., $\max(t_{s+m}) < \tau$. This proves assertion $(*)$ and hence Theorem 4.1.

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