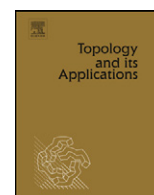


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A remark on proper partitions of unity [☆]

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ABSTRACT

In this paper we introduce, by means of the category of exterior spaces and using a process that generalizes the Alexandroff compactification, an analogue notion of numerable covering of a space in the proper and exterior setting. An application is given for fibrewise proper homotopy equivalences.

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0. Introduction

The notion of numerable covering of a space is a useful tool in order to obtain global results from local data. Among the most successful results in this sense for homotopy theory we can mention the papers of A. Dold [3] and T. tom Dieck [9], in which it is shown that being a homotopy equivalence or a fibration is a local property. The aim of this paper is to establish an analogous notion of numerable covering in the category \mathbf{P} of spaces and proper maps and to give an application. In order to do so we begin in Section 1 by giving some preliminary definitions and results that will be used throughout the paper. The most important tool is the category of *exterior spaces* [4]. An exterior space is nothing else but a topological space together with a distinguished collection of open subsets verifying certain natural conditions that capture the behavior of a neighborhood system ('at infinity'). Then the category \mathbf{E} of exterior spaces and exterior maps appears, containing \mathbf{P} as a full subcategory. Moreover, unlike \mathbf{P} , the category of exterior spaces has better categorical properties, such as having all limits and colimits. It is for all these reasons that we have chosen \mathbf{E} as a framework for our study in \mathbf{P} .

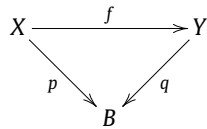
In Section 2 we establish the main notion of the paper, which is the one of proper (and exterior) numerable covering. In order to do this, we consider a more manageable description of the category of exterior spaces, based on the Alexandroff one-point compactification. Then it is shown that the notion of proper numerable (for short, p -numerable) covering is not very restrictive. Indeed, at the end of the section we obtain the following result:

Proposition. *Let X be a finite dimensional locally finite CW-complex (for instance, any open differentiable n -manifold or any open PL n -manifold). If $\{X_\alpha\}_{\alpha \in A}$ is a covering of X such that the family of their interiors $\{\text{int}(X_\alpha)\}_{\alpha \in A}$ also covers X and there exists $\alpha \in A$ such that the complement $X \setminus \text{int}(X_\alpha)$ is compact, then $\{X_\alpha\}_{\alpha \in A}$ is a p -numerable covering of X .*

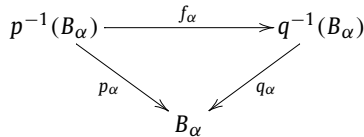
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Finally, using such notion of p -numerable covering, we establish in Section 3, an application. Consider $f : X \rightarrow Y$ any proper map over a fixed space B , that is, a commutative diagram in \mathbf{P} :



Theorem. *If $\{B_\alpha\}_{\alpha \in A}$ is a closed p -numerable covering of B and each restriction*



is a proper homotopy equivalence over B_α , then $f : X \rightarrow Y$ is a proper homotopy equivalence over B .

1. Preliminaries. Proper category and exterior spaces

Recall that a *proper* map is a continuous map $f : X \rightarrow Y$ such that $f^{-1}(K)$ is a compact subset of X , for every closed compact subset K of Y . We will denote by \mathbf{P} the category of spaces and proper maps. Proper homotopy is defined in a natural way.

As it is well known, the category \mathbf{P} has not good categorical properties such as limits and colimits. Therefore, many constructions cannot be considered in this setting. In order to palliate this problem in \mathbf{P} , exterior spaces were introduced in [4].

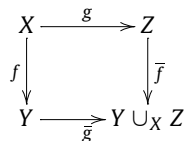
Definition 1. ([4]) An *exterior space* $(X, \mathcal{E} \subseteq \tau)$ consists of a topological space (X, τ) together with a non-empty family of open sets \mathcal{E} , called *externology* which is closed by finite intersections and, whenever $U \supseteq E$, $E \in \mathcal{E}$, $U \in \tau$, then $U \in \mathcal{E}$. We call *exterior open* subset, or in short, *e-open* subset, any element $E \in \mathcal{E}$. A map between exterior spaces $f : (X, \mathcal{E} \subseteq \tau) \rightarrow (X', \mathcal{E}' \subseteq \tau')$ is said to be *exterior* if it is continuous and $f^{-1}(E) \in \mathcal{E}$, for all $E \in \mathcal{E}'$.

The category of exterior spaces will be denoted by \mathbf{E} .

For a given topological space X we can consider its *cocompact externology* \mathcal{E}_{cc} which is formed by the family of the complements of all closed-compact subsets of X . The corresponding exterior space will be denoted by X_{cc} . The correspondence $X \mapsto X_{cc}$ gives rise to a full embedding [4, Theorem 3.2]:

$$(-)_{cc} : \mathbf{P} \hookrightarrow \mathbf{E}$$

Furthermore, the category \mathbf{E} is complete and cocomplete [4, Theorem 3.3]. For instance, the pushout in \mathbf{E} of $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, is the topological pushout



equipped with the *pushout* externology, given by those $E \subseteq Y \cup_X Z$ for which $\bar{g}^{-1}(E)$ and $\bar{f}^{-1}(E)$ are e-open. Another example is the *product* externology in $Y \times Z$, which consists of those open subsets which contain a product $E \times E'$ of e-open subsets of Y and Z respectively. More generally, the pullback in \mathbf{E} of $f : Y \rightarrow X$ and $g : Z \rightarrow X$ is the topological pullback $Y \times_X Z$ endowed with the relative externology induced by $Y \times Z$. (In general, the *relative* externology in $A \subseteq X$ is given by $\mathcal{E}_A := \{E \cap A, E \in \mathcal{E}_X\}$.)

Now we introduce the following functorial construction that can be made in this setting:

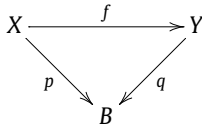
Definition 2. ([4]) Let X and Y be an exterior and a topological space respectively. On the product space $X \times Y$ consider the following externology: an open set E is exterior if for each $y \in Y$ there exists an open neighborhood of y , U_y , and an exterior open E_y such that $E_y \times U_y \subseteq E$. Then $X \bar{\times} Y$ will denote the resulting exterior space.

Remark 3. If Y is compact, then it is not difficult to check that E is an exterior open in $X \bar{\times} Y$ if and only if it is an open set and there exists $G \in \mathcal{E}_X$ for which $G \times Y \subseteq E$. In particular, if $\mathcal{E}_X = \mathcal{E}_{cc}^X$ and Y is compact, then $X_{cc} \bar{\times} Y = (X \times Y)_{cc}$.

There is a cylinder functor $-\bar{\times}I : \mathbf{E} \rightarrow \mathbf{E}$ with natural transformations $\iota_0, \iota_1 : id \rightarrow -\bar{\times}I$ and $\rho : -\bar{\times}I \rightarrow id$ obviously defined. This construction provides a natural way to define exterior homotopic maps ($f \simeq_e g$) in \mathbf{E} . The notion of exterior homotopy equivalence comes naturally.

Observe that, since $X_{cc}\bar{\times}I = (X \times I)_{cc}$ (by Remark 3 above), the cylinder functor may be restricted to the proper case, $-\bar{\times}I = -\times I : \mathbf{P} \rightarrow \mathbf{P}$. This functor is the one used in [1] to define an I -category structure (in the sense of Baues) on \mathbf{P}_∞ using proper cofibrations.

We can also consider the category \mathbf{E}_B of exterior spaces over a fixed object B . Its objects are exterior maps $X \rightarrow B$, called exterior spaces over B , and its morphisms, called exterior maps over B are commutative triangles in \mathbf{E} :



Remark 4. The exterior spaces and exterior maps over B are also called fibrewise exterior spaces and fibrewise exterior maps, respectively.

Given $f, g : X \rightarrow Y$ two exterior maps over B , then f is homotopic to g over B (or fibrewise homotopic to g), denoted $f \simeq_B g$, if there exists $H : X \times I \rightarrow Y$ a homotopy over B between f and g . This means that H is an exterior map such that $qH(x, t) = p(x)$ and $H(x, 0) = f(x)$, $H(x, 1) = g(x)$, for all $x \in X$ and $t \in I$. The homotopy over B is an equivalence relation, compatible with the composition of morphisms. The notion of homotopy equivalence over B (or fibrewise homotopy equivalence) is naturally defined. The fibrewise notions can be restricted to the category \mathbf{P} of spaces and proper maps.

Now we give a more manageable description of the category of exterior spaces. Such description will be crucial for our main notion, given in the next section. We shall consider what we call the category of ∞ -spaces.

Definition 5. An ∞ -space is a pointed space (X, x_0) such that $\{x_0\}$ is closed in X . An ∞ -map is a pointed map $f : (X, x_0) \rightarrow (Y, y_0)$ verifying $f^{-1}(\{y_0\}) = \{x_0\}$. We write \mathbf{Top}^∞ for the corresponding category of ∞ -spaces and ∞ -maps.

Proposition 6. There is an equivalence of categories $\mathbf{E} \simeq \mathbf{Top}^\infty$.

Proof. If $(X, \varepsilon_X \subset \tau_X)$ is an exterior space and ∞ is a point which does not belong to X , then we consider the pointed space $X^\infty = X \cup \{\infty\}$ with base point ∞ , equipped with the topology

$$\tau^\infty = \tau_X \cup \{E \cup \{\infty\} : E \in \varepsilon_X\}$$

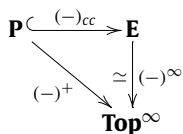
Given f any exterior map, f^∞ is obviously defined. Thus we obtain a functor $(-)^\infty : \mathbf{E} \rightarrow \mathbf{Top}^\infty$, which is an equivalence of categories. Indeed, the quasi-inverse of $(-)^\infty$ is the functor $\mathbf{Top}^\infty \rightarrow \mathbf{E}$ defined as follows:

Let (X, x_0) be any object in \mathbf{Top}^∞ ; then we take the exterior space $\bar{X} = X \setminus \{x_0\}$ whose topology and externology are given as

$$\tau_{\bar{X}} = \{A \setminus \{x_0\} : A \in \tau_X\}, \quad \varepsilon_{\bar{X}} = \{A \setminus \{x_0\} : A \in \varepsilon_X, x_0 \in A\}$$

Observe that $\tau_{\bar{X}} \subset \tau_X$ since $X \setminus \{x_0\} \in \tau_X$. The definition on morphisms is given by the obvious restriction. \square

The functor $(-)^\infty$ is closely related to the Alexandroff compactification functor. Indeed, if X is any topological space and we consider X_{cc} the cocompact externology, then it is clear that $(X_{cc})^\infty = X^+$ is the Alexandroff compactification of X . There is a commutative diagram of functors and categories



where $(-)^+ : \mathbf{P} \rightarrow \mathbf{Top}^\infty$ is the functor induced by the Alexandroff compactification construction. Consequently, the functor $(-)^+$ is a full embedding; furthermore, $(-)^+$ induces an equivalence

$$\mathbf{P}_{lch} \simeq \mathbf{Top}_{ch}^\infty$$

between the full subcategory \mathbf{P}_{lch} of \mathbf{P} whose objects are locally compact Hausdorff spaces and the full subcategory \mathbf{Top}_{ch}^∞ of \mathbf{Top}^∞ whose objects are based compact Hausdorff spaces. Here, the condition of being Hausdorff cannot be removed. For instance, if $\mathbf{2}_S$ denotes the space given by the set $\mathbf{2} = \{0, 1\}$ with the Sierpinski topology $\tau = \{\emptyset, \mathbf{2}, \{0\}\}$, then $\mathbf{2}_S$ does not come from the Alexandroff compactification.

2. Proper and exterior numerable coverings

In this section we will establish the notion of proper (more generally, exterior) numerable covering of an exterior space. In order to obtain such notion we need the functor $(-)^{\infty} : \mathbf{E} \rightarrow \mathbf{Top}^{\infty}$. Recall that a (not necessarily open) covering \mathcal{U} of a space X is said to be *numerable* if \mathcal{U} admits a refinement by a partition of unity (see [3] or [2] for more details). It is well known that any open covering of a paracompact space is numerable.

Definition 7. Let X be an exterior space and $\mathcal{U} = \{U_i\}_{i \in I}$ a covering of X . Then \mathcal{U} is said to be an *exterior numerable covering* (for short, *e-numerable covering*) of X if

$$\mathcal{U}^{\infty} = \{U_i \cup \{\infty\}\}_{i \in I}$$

is a numerable covering of the topological space X^{∞} . In particular, when X is a cocompact exterior space (i.e., X has the cocompact externology), then we say that \mathcal{U} is a *proper numerable covering* (or *p-numerable covering*) of X .

Although at first sight the notion of e-numerable covering might seem too restrictive we will see that it turns out to be far from the case. Indeed, we will prove that for a wide class of exterior spaces X , namely the exterior CW-complexes, we have that X^{∞} is a paracompact space. Therefore, any open covering of X is an e-numerable covering, as long as there exists a member of the covering which is an exterior open subset.

Definition 8. ([5]) An *exterior CW-complex* consists of an exterior space X together with a filtration $\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \dots \subset X_n \subset \dots$, such that X is the colimit of this filtration and for each $n \geq 0$, X^n is obtained from X^{n-1} by an exterior pushout of the form

$$\begin{array}{ccc} \sqcup_{\gamma \in \Gamma} \mathfrak{S}_{\gamma}^{n-1} & \xrightarrow{\sqcup_{\gamma \in \Gamma} \varphi_{\gamma}} & X^{n-1} \\ \downarrow & & \downarrow \\ \sqcup_{\gamma \in \Gamma} \mathfrak{D}_{\gamma}^n & \xrightarrow{\sqcup_{\gamma \in \Gamma} \psi_{\gamma}} & X^n \end{array}$$

Here \mathfrak{S}^k denotes either the k -dimensional sphere S^k (with its topology as externology) or the k -dimensional \mathbb{N} -sphere $\mathbb{N}\bar{\times}S^k$. Analogously \mathfrak{D}^k denotes either the classical disc D^k or $\mathbb{N}\bar{\times}D^k$, the \mathbb{N} -disc. We point out that the inclusion $\mathfrak{S}^{k-1} \hookrightarrow \mathfrak{D}^k$ means either $S^{k-1} \hookrightarrow D^k$ or $\mathbb{N}\bar{\times}S^{k-1} \hookrightarrow \mathbb{N}\bar{\times}D^k$.

Observe that every classical CW-complex X with its topology as externology is an exterior CW-complex. Moreover, the class of exterior CW-complexes contains many spaces in \mathbf{P} ; for instance, if X is any finite dimensional locally finite CW-complex, then X_{cc} has the structure of an exterior CW-complex. In order to see this fact one has just to take into account the following result:

Lemma 9. ([6, Proposition 2.3]) Consider the pushout in \mathbf{E} of $f : A \rightarrow X_{cc}$ and $g : A \rightarrow Y_{cc}$

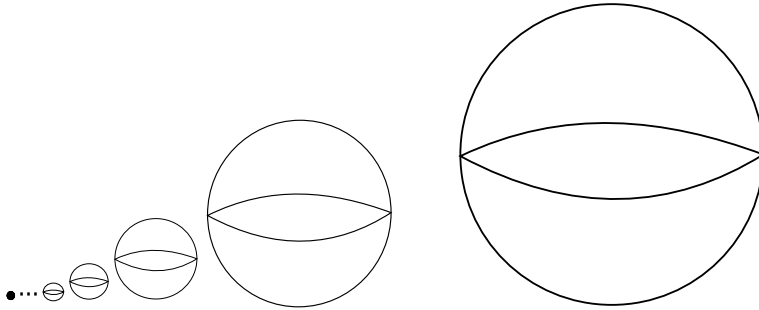
$$\begin{array}{ccc} A & \xrightarrow{g} & Y_{cc} \\ f \downarrow & & \downarrow \\ X_{cc} & \longrightarrow & X_{cc} \cup_A Y_{cc} \end{array}$$

in which X, Y are Hausdorff, locally compact spaces, and A is a Hausdorff, locally compact exterior space. Then, the pushout externology is contained in the cocompact externology in $X \cup_A Y$. Furthermore, if f and g are proper maps and f (or g) is injective, then $X_{cc} \cup_A Y_{cc} = (X \cup_A Y)_{cc}$. \square

In particular, suppose that M is any open differentiable n -manifold or any open PL n -manifold. As a differentiable manifold M admits a triangulation and therefore a structure of a finite dimensional locally finite CW-complex. Consequently M_{cc} can be seen as an exterior CW-complex.

Remark 10. If X is a strongly locally finite CW-complex (not necessarily finite dimensional), then each skeleton X^n equipped with its cocompact externology is an exterior CW-complex. Furthermore, X together with the colimit externology (i.e., the externology given by those open subsets $E \subset X$ such that $E \cap X^n$ is cocompact in X^n , for all n) has the structure of an exterior CW-complex. Note that, in the non-finite dimensional case, the colimit externology in X need not agree with the cocompact externology and therefore X might not be considered as a space in \mathbf{P} .

For an exterior CW-complex X , X^∞ need not be a classical CW-complex. For instance, if $X = \mathbb{N} \bar{\times} S^k$ we have that $X^\infty = X^+$ is the Alexandorff compactification, which is not a CW-complex since the property of being locally contractible fails at $\infty \in X^+$ (see the figure below)



What is clear is that $(\mathfrak{E}^k)^\infty = (\mathfrak{E}^k)^+$ and $(\mathfrak{D}^k)^\infty = (\mathfrak{D}^k)^+$ are paracompact Hausdorff spaces (and therefore normal spaces). Next we state some technical results that will help us to reach our aim. Given m any infinite cardinal number, a topological space X is said to be m -paracompact if for any open covering with cardinal $\leq m$ it admits a locally finite refinement. Then X is paracompact if and only if X is m -paracompact for any infinite cardinal number m .

A topological space X is said to have the *weak topology* with respect to a closed covering $\{A_\lambda\}_{\lambda \in \Lambda}$ if for any $\Lambda' \subset \Lambda$ we have that every subset $C \subset \bigcup_{\lambda \in \Lambda'} A_\lambda$ which verifies that $C \cap A_\lambda$ is closed for all $\lambda \in \Lambda'$ then C is closed in X . Every topological space has the weak topology with respect to any locally finite closed covering.

Lemma 11. ([8, Theorem 3.1]) *If a topological space X has the weak topology with respect to a closed covering $\{A_\alpha\}_{\alpha \in A}$ where each A_α is m -paracompact and normal, then X is also m -paracompact and normal. \square*

Lemma 12. ([8, Theorem 3.4]) *Let X be a topological space and $\{A_n\}_{n=0}^\infty$ a countable closed covering such that if $C \subset X$ verifies that $C \cap A_n$ is closed for all n implies that C is closed in X . If each A_n is m -paracompact and normal, then X is also m -paracompact and normal. \square*

Now recall that given any map $f : C \rightarrow Y$, where C is a closed subspace of a space X , the *adjunction space* $X \cup_f Y$ is the pushout

$$\begin{array}{ccc} C & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \cup_f Y \end{array}$$

Lemma 13. ([7, Corollary 1]) *Consider X, Y m -paracompact and normal spaces, C a closed subspace of X and $f : C \rightarrow Y$ any map. Then the adjunction space $X \cup_f Y$ is m -paracompact and normal. \square*

Using the above results one can prove the following proposition.

Proposition 14. *If X is an exterior CW-complex, then X^∞ is a paracompact space.*

Proof. Since $(-)^\infty : \mathbf{E} \rightarrow \mathbf{Top}^\infty$ is an equivalence of categories, in particular it preserves all small limits and colimits. This fact implies that:

- (i) $X^\infty = \text{colim}(X^n)^\infty$; that is X^∞ is the union of all $(X^n)^\infty$ with the hypothesis of Lemma 12.
- (ii) $(X^{n-1})^\infty$ and $(X^n)^\infty$ are related through the topological pushout

$$\begin{array}{ccc} \bigvee_{\gamma \in \Gamma} (\mathfrak{E}_\gamma^{n-1})^+ & \longrightarrow & (X^{n-1})^\infty \\ \downarrow & & \downarrow \\ \bigvee_{\gamma \in \Gamma} (\mathfrak{D}_\gamma^n)^+ & \longrightarrow & (X^n)^\infty \end{array}$$

That is, every $(X^n)^\infty$ is an adjunction space.

The combination of Lemmas 11, 12 and 13 together with an easy induction argument permits us to prove the result. The details are left to the reader. \square

As corollaries we obtain the following results.

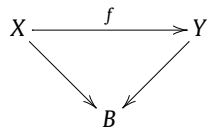
Proposition 15. *If \mathcal{U} is an open covering of an exterior CW-complex X where at least one of its members is an exterior open subset, then \mathcal{U} is an e-numerable covering of X . \square*

Proposition 16. *Let X be a finite dimensional locally finite CW-complex (for instance, any open differentiable n -manifold or any open PL n -manifold). Then, its Alexandroff compactification X^+ is paracompact. As a consequence, if $\{X_\alpha\}_{\alpha \in A}$ is a covering of X such that the family of their interiors $\{\text{int}(X_\alpha)\}_{\alpha \in A}$ also covers X and there exists $\alpha \in A$ such that the complement $X \setminus \text{int}(X_\alpha)$ is compact, then $\{X_\alpha\}_{\alpha \in A}$ is a p-numerable covering of X . \square*

3. An application: The fibrewise proper homotopy equivalences

In this section we will give an application. Namely, we will prove that the fibrewise proper homotopy equivalences satisfy a local to global type theorem. For this aim we first translate the corresponding notions to the framework of exterior spaces, or ∞ -spaces. First of all, we note the following important fact, connecting the (fibrewise) proper homotopy equivalences and their exterior counterpart. Its proof is routine and left to the reader.

Proposition 17. *Let B is a fixed topological space and $f : X \rightarrow Y$ a proper map over B , that is a commutative diagram in \mathbf{P}*

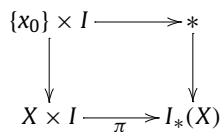


Then f is a proper homotopy equivalence over B if and only if f_{cc} is an exterior homotopy equivalence over B_{cc} . \square

We want to connect this result with the category of ∞ -spaces.

3.1. Fibrewise homotopy equivalences in \mathbf{Top}^∞

Let (X, x_0) be an ∞ -space. Its pointed cylinder $I_*(X)$ is just the quotient space $I_*(X) = (X \times I) / (\{x_0\} \times I)$ coming from the topological pushout



Observe that $I_*(X)$ is again an ∞ -space since $\pi^{-1}(*) = \{x_0\} \times I$ is closed in $X \times I$ and therefore $*$ is closed in $I_*(X)$. This construction gives rise to a functor

$$I_* : \mathbf{Top}^\infty \rightarrow \mathbf{Top}^\infty$$

The pointed cylinder is also equipped with natural transformations $i_0, i_1 : X \rightarrow I_*(X)$ and $p : I_*(X) \rightarrow X$ in \mathbf{Top}^∞ . This way a notion of homotopy comes naturally: Given $f, g : X \rightarrow Y$ ∞ -maps we say that f is ∞ -homotopic to g ($f \simeq_\infty g$) if there exists $F : I_*(X) \rightarrow Y$ an ∞ -map such that $F i_0 = f$ and $F i_1 = g$. However we may also use the non-pointed cylinder; it is straightforward to check that $f \simeq_\infty g$ if and only if there exists a continuous map $F : X \times I \rightarrow Y$ such that $F^{-1}(\{y_0\}) = \{x_0\} \times I$ (in particular is a classical pointed homotopy) and $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, for all $x \in X$. The notion of ∞ -homotopy equivalence (respectively ∞ -homotopy equivalence over B) comes naturally.

The next result explores the connection between the cylinder construction in \mathbf{Top}^∞ and \mathbf{E} .

Lemma 18. *Let X be any exterior space. Then there exists a natural isomorphism in \mathbf{Top}^∞*

$$(X \bar{\times} I)^\infty \cong I_*(X^\infty)$$

Proof. Recall that $I_*(X^\infty) = (X^\infty \times I)/(\{\infty\} \times I)$ and $\pi : X^\infty \times I \rightarrow I_*(X^\infty)$ denotes the canonical projection. We define the ∞ -map $\varphi : (X \bar{\times} I)^\infty \rightarrow I_*(X^\infty)$ by $\varphi(x, t) = \pi(x, t)$, for $(x, t) \in X \times I$, and $\varphi(\infty) = *$. Then we have that φ is an isomorphism. Indeed, the map $h : X^\infty \times I \rightarrow (X \bar{\times} I)^\infty$ given by $h(x, t) = (x, t)$ and $h(\infty, t) = \infty$, satisfies $h(\{\infty\} \times I) = \{\infty\}$ (moreover, $h^{-1}(\{\infty\}) = \{\infty\} \times I$) and it induces an ∞ -map $\psi : I_*(X^\infty) \rightarrow (X \bar{\times} I)^\infty$ such that $\psi\pi = h$. One can straightforwardly check that $\psi\varphi = id$ and $\varphi\psi = id$. \square

Remark 19. In the proper case, the above result can be read as

$$(X \times I)^+ \cong I_*(X^+)$$

That is, given any space X , the Alexandroff compactification of the cylinder $X \times I$ is, up to isomorphism in \mathbf{Top}^∞ , the pointed cylinder of the Alexandroff compactification of X .

As an immediate result, given $f, g : X \rightarrow Y$ exterior maps, we have that $f \simeq_e g$ if and only if $f^\infty \simeq_\infty g^\infty$. Moreover,

Proposition 20. Let B be a fixed exterior space and $f : X \rightarrow Y$ an exterior map over B . Then f is an exterior homotopy equivalence over B if and only if f^∞ is an ∞ -homotopy equivalence over B^∞ . \square

Now we establish our result. But first we need the following notions and results related to the classical topological case. Their proofs can be found in [3,2] or [10].

Recall that given B a topological space, a halo around $A \subset B$ is a subset $V \subset B$ such that there a continuous map $\tau : B \rightarrow [0, 1]$ with $A \subset \tau^{-1}(1)$ and $B \setminus V \subset \tau^{-1}(0)$. A continuous map $p : E \rightarrow B$ is said to have the Section Extension Property (SEP) if for every $A \subset B$ and every section s over A which admits an extension as a section to a halo V around A , there exists a section $S : B \rightarrow E$ over B with $S|_A = s$. (In particular, if p has the SEP then p always has a section by taking $A = \emptyset = V$.)

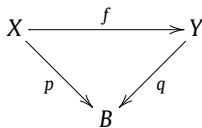
Consider the category \mathbf{Top}_B of topological spaces over a fixed space B and maintain the same notation and terminology as the ones given for the exterior case. That is, we shall deal with spaces and maps over B (or fibrewise spaces and maps) and fibrewise homotopies, also denoted as \simeq_B . Then it is said that $p : E \rightarrow B$ is dominated by $p' : E' \rightarrow B$ if there exist fibrewise maps $f : E \rightarrow E'$ and $g : E' \rightarrow E$ such that $gf \simeq_B id_E$.

By a shrinkable space over B we mean any space over B which has the same fibrewise homotopy type as the identity $id_B : B \rightarrow B$.

Proposition 21. ([3, Proposition 2.3]) Suppose that $p : E \rightarrow B$ is dominated by $p' : E' \rightarrow B$. If p' has the SEP, then so does p . In particular, every shrinkable space has the SEP. \square

Lemma 22. ([3, Corollary 2.7]) Let $p : E \rightarrow B$ a continuous map and $\{V_\lambda\}_{\lambda \in A}$ a numerable covering of B . If each restriction $p_\lambda : p^{-1}(V_\lambda) \rightarrow V_\lambda$ is shrinkable (over V_λ) then p is also shrinkable. \square

And the last technical previous result. Let



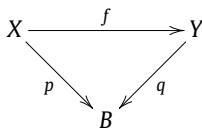
be a fibrewise map. We can consider the following subspace of $X \times Y^I$:

$$R = \{(x, \gamma) \in X \times Y^I : p(x) = q\gamma(t), \forall t \in I, \text{ and } \gamma(1) = f(x)\}$$

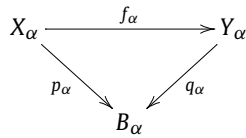
together with the map $q : R \rightarrow Y$ defined as $q(x, \gamma) = \gamma(0)$.

Lemma 23. ([3, Lemma 3.4]) If $f : X \rightarrow Y$ is a homotopy equivalence over B , then $q : R \rightarrow Y$ is shrinkable. \square

Finally, we establish our theorem as application. Consider $f : X \rightarrow Y$ any exterior map over a fixed exterior space B



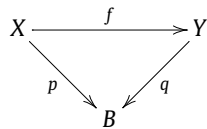
If $\{B_\alpha\}_{\alpha \in A}$ is covering of B we will write, for each α



where $X_\alpha = p^{-1}(B_\alpha)$, $Y_\alpha = q^{-1}(B_\alpha)$ and f_α, p_α and q_α denote the natural restrictions.

Theorem 24. *If $\{B_\alpha\}_{\alpha \in A}$ is an e -numerable covering of B and each f_α is an exterior homotopy equivalence over B_α , then $f : X \rightarrow Y$ is an exterior homotopy equivalence over B .*

Proof. By Proposition 20 we can assume that we are working in \mathbf{Top}^∞ , and by abuse of language we can use the same notation for the corresponding objects and maps. That is, we can think that



is a commutative diagram in \mathbf{Top}^∞ and $\{B_\alpha\}_{\alpha \in A}$ is a classical numerable covering of B such that the base point $b_0 \in B$ belongs to each B_α . Moreover, each f_α is an ∞ -homotopy equivalence over B_α . We shall denote by x_0 and y_0 the respective base points of X and Y .

Now, for every $\alpha \in A$, let $f_\alpha^- : Y_\alpha \rightarrow X_\alpha$ denote a fibrewise ∞ -homotopy inverse. As the fibrewise homotopy equivalences in \mathbf{Top}^∞ are, in particular, classical fibrewise homotopy equivalences, we can use the previous results. By Lemma 23 we have that each $q_\alpha : R_\alpha \rightarrow X_\alpha$ is shrinkable, where

$$R_\alpha = \{(x, \gamma) \in X_\alpha \times Y_\alpha^I : p_\alpha(x) = q_\alpha \gamma(t), \forall t \in I \text{ and } \gamma(1) = f_\alpha(x)\}$$

Therefore, by Lemma 22 we have that $q : R \rightarrow Y$ is also shrinkable. In particular q has the SEP so there exists a section $S = (f', \theta) : Y \rightarrow R$.

First of all we observe that $q : R \rightarrow Y$ is an ∞ -map, being (x_0, C_{y_0}) the base point of R (here C_{y_0} denotes the constant path). Indeed, if $(x, \gamma) \in R$ satisfies $q(x, \gamma) = \gamma(0) = y_0$ then we have

$$p(x) = q(\gamma(0)) = q(y_0) = b_0$$

so that $x = x_0$. On the other hand, for any $t \in I$ we have

$$q(\gamma(t)) = p(x) = p(x_0) = b_0$$

so $\gamma = C_{y_0}$ is the constant path and we conclude that $(x, \gamma) = (x_0, C_{y_0})$.

Being S a section of the ∞ -map q we also observe that, necessarily $S : Y \rightarrow R$ must be an ∞ -map. Indeed, if $S(y) = (x_0, C_{y_0})$ then $y = qS(y) = q(x_0, C_{y_0}) = y_0$, so that $S^{-1}(\{(x_0, C_{y_0})\}) = \{y_0\}$.

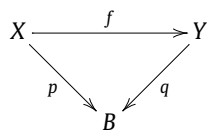
Recall that S is of the form $S(y) = (f'(y), \theta(y))$ where f' is a map $f' : Y \rightarrow X$ over B and θ is a map $\theta : Y \rightarrow I$, which induces, in a natural way, a homotopy $\Theta : Y \times I \rightarrow X$ over B , given by $\Theta(y, t) = \theta(y)(t)$. Since S is an ∞ -map it is straightforward to check that f' is also an ∞ -map and that $\Theta^{-1}(\{x_0\}) = \{y_0\} \times I$. Moreover, $\Theta : id_Y \simeq_B ff'$ in $(\mathbf{Top}^\infty)_B$.

It only remains to prove that there exists $\Theta' : id_X \simeq_B f'f''$ in $(\mathbf{Top}^\infty)_B$. Applying the above reasoning to f' we have that there exist $f'' : X \rightarrow Y$ over B and $\Theta' : id_X \simeq_B f'f''$ in $(\mathbf{Top}^\infty)_B$. But

$$f'f'' = f'id_Y f'' \simeq_B f'ff'f'' \simeq_B f'fid_X = f'f$$

so $id_X \simeq_B f'f$. \square

This result has a very interesting proper counterpart. Indeed, consider $f : X \rightarrow Y$ any proper map over a fixed space B , that is, a commutative diagram of proper maps



Then we obtain as a corollary the corresponding theorem in the proper setting:

Theorem 25. *If $\{B_\alpha\}_{\alpha \in A}$ is a closed p -numerable covering of B and each restriction*

$$\begin{array}{ccc} p^{-1}(B_\alpha) & \xrightarrow{f_\alpha} & q^{-1}(B_\alpha) \\ & \searrow p_\alpha & \swarrow q_\alpha \\ & B_\alpha & \end{array}$$

is a proper homotopy equivalence over B_α , then $f : X \rightarrow Y$ is a proper homotopy equivalence over B . \square

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