Full Length Article

Vibration of a circular beam with variable cross sections using differential transformation method

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ABSTRACT
In this paper, an application of differential transformation method (DTM) is applied on free vibration analysis of Euler-Bernoulli beam. This beam has variable circular cross sections. Natural frequencies and corresponding mode shapes are obtained for three cases of cross section and boundary conditions. MATLAB program is used to solve the differential equation of the beam using DTM. Comparison of the obtained results with the previous solutions proves the accuracy and versatility of the presented problem.

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1. Introduction

Beams with abrupt changes of cross-section are used widely in engineering. It can be easily made in order to save weight or to satisfy various engineering requirements. However, in many theses, beams are treated as volumes, so their cross section is usually rectangular. This paper presents the results of a case study of a beam element with circular cross section. Circular shape of the cross section is chosen because it often occurs in practice.

In this paper, the vibration problems of circular Euler-Bernoulli beams have been solved analytically using DTM. The beam has variable cross sections and various end conditions. In order to calculate the fundamental natural frequencies and the corresponding mode shapes, variational techniques were applied in the past such as Rayleigh Ritz, differential quadrature (DQM) and Galerkin methods. Also, some numerical methods were also successfully applied to beam vibration analysis such as finite element method (FEM).

The differential transformation method leads to an iterative procedure for obtaining an analytic series solutions of functional equations. Pukhov (1982) have developed a so-called differential transformation method (DTM) for electrical circuits’ problems. In recent years, researchers Ayaz (2004), Moustafa (2008) and Qibo (2012) had applied the method to various linear and nonlinear problems. Comparison between (DTM) and (DQM) were applied by Attarnejad and Shahba (2008), and Rajasekaran (2009).
Mahmoud et al. (2013) used the differential transformation method for the free vibration analysis of rectangular beams with uniform and non-uniform cross sections. Akiji et al. (1982) discussed the importance of the geometrical injection efficiency of a neutral circular beam. Un-damped vibration of beams with variable cross-section was analyzed by Datta and Sil (1996). Au et al. (1999) used C’ modified beam vibration functions to study the vibration and stability of non-uniform beams with abrupt changes of cross-section. Li (2000) presented an exact approach for determining natural frequencies and mode shapes of non-uniform shear beams with arbitrary distribution of mass or stiffness. A three-dimensional method of analysis was presented for determining the free vibration frequencies and mode shapes of thick, tapered rods and beams with circular cross-section Kanga and Leissa (2004). Kisia and Arif Gurelb (2006) presented a numerical model that combines the finite element and component mode synthesis methods for the modal analysis of beams with circular cross section and containing multiple non-propagating open cracks. Vibration and bending analysis were applied for non-uniform beams, rods and tubes by Mehmet et al. (2007), AL Kaisy et al. (2008), De Rosaa et al. (2007), De Rosaa et al. (2008) and Shojaeifard et al. (2012). The beam elements, which are widely used in the absolute nodal coordinate formulation, were treated as iso-parametric elements by Grezegorz (2012).

In the present paper, an attempt is made to employ the differential transformation method to solve equations of motion for the free vibration of non-uniform circular beam. Three cases of boundary conditions are considered. Natural frequencies and corresponding mode shapes are obtained.

2. Basic equations

2.1. Free vibration of non-uniform circular beam

The governing differential equation for an Euler beam with a circular cross section with variable radius \( r_0 \) as shown in Fig. 1 is given by:

\[
ρA(X) \frac{∂^2 W(X, T)}{∂T^2} + \frac{∂^2}{∂X^2} \left( EI(X) \frac{∂^2 W(X, T)}{∂X^2} \right) = 0
\]

where \( ρ \) is the density of the beam material, \( A(X) \) is the cross sectional area of the beam, \( W(X, T) \) is displacement of the beam, \( E \) is young’s modulus of the beam and \( I(X) = \frac{π}{4} r_0^4 \) is the inertia of the beam.

2.2. Boundary conditions

Case a. Simply Supported Beam

\[
W(0, T) = 0; \quad \frac{∂^2 W(0, T)}{∂X^2} = 0; \quad W(L, T) = 0; \quad \frac{∂^2 W(L, T)}{∂X^2} = 0
\]

Case b. Clamped–Clamped Beam

\[
W(0, T) = 0; \quad \frac{∂ W(0, T)}{∂X} = 0; \quad W(L, T) = 0; \quad \frac{∂^2 W(L, T)}{∂X^2} = 0
\]

Case c. Clamped–Roller Beam

\[
W(0, T) = 0; \quad \frac{∂ W(0, T)}{∂X} = 0; \quad V(L, T) = 0; \quad \frac{∂^2 W(L, T)}{∂X^2} = 0
\]

where \( L \) is the beam length and \( T \) is the time.

Table 1 – The first three non-dimensional frequencies \( (Ω_1, Ω_2, \text{and } Ω_3) \) of Simply Supported uniform Circular Euler beams for different number of terms.

<table>
<thead>
<tr>
<th>No. of terms ( N )</th>
<th>( Ω_1 )</th>
<th>( Ω_2 )</th>
<th>( Ω_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>9</td>
<td>9.8902098156</td>
<td>28.2140699048</td>
<td>–</td>
</tr>
<tr>
<td>10</td>
<td>9.8906683979</td>
<td>37.2824411319</td>
<td>–</td>
</tr>
<tr>
<td>11</td>
<td>9.8906683979</td>
<td>37.2824411319</td>
<td>–</td>
</tr>
<tr>
<td>12</td>
<td>9.8906683979</td>
<td>37.2824411319</td>
<td>–</td>
</tr>
<tr>
<td>13</td>
<td>9.8906683979</td>
<td>37.2824411319</td>
<td>–</td>
</tr>
<tr>
<td>14</td>
<td>9.8906683979</td>
<td>37.2824411319</td>
<td>–</td>
</tr>
<tr>
<td>15</td>
<td>9.8906683979</td>
<td>37.2824411319</td>
<td>–</td>
</tr>
<tr>
<td>16</td>
<td>9.8906683979</td>
<td>37.2824411319</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 2 – The first three non-dimensional frequencies \( (Ω_1, Ω_2, \text{and } Ω_3) \) of Clamped–Clamped uniform Circular Euler beams for different number of terms.

<table>
<thead>
<tr>
<th>No. of terms ( N )</th>
<th>( Ω_1 )</th>
<th>( Ω_2 )</th>
<th>( Ω_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>13</td>
<td>22.37732854481</td>
<td>61.6728228679</td>
<td>120.903394841</td>
</tr>
<tr>
<td>14</td>
<td>22.37732854481</td>
<td>61.6728228679</td>
<td>120.903394841</td>
</tr>
<tr>
<td>15</td>
<td>22.37732854481</td>
<td>61.6728228679</td>
<td>120.903394841</td>
</tr>
<tr>
<td>16</td>
<td>22.37732854481</td>
<td>61.6728228679</td>
<td>120.903394841</td>
</tr>
<tr>
<td>17</td>
<td>22.37732854481</td>
<td>61.6728228679</td>
<td>120.903394841</td>
</tr>
<tr>
<td>18</td>
<td>22.37732854481</td>
<td>61.6728228679</td>
<td>120.903394841</td>
</tr>
<tr>
<td>19</td>
<td>22.37732854481</td>
<td>61.6728228679</td>
<td>120.903394841</td>
</tr>
</tbody>
</table>
Assume that, the displacement of the beam is given by:

\[ W(X, T) = W(X) \exp(i\omega T) \]

where \( \omega \) is the natural frequency of the beam.

Equation (1) can be conveniently written as:

\[
-\rho A(X)\omega^2 W(X) + E I(X) \frac{d^4 W(X)}{dX^4} + 2E I(X) \frac{d^2 W(X)}{dX^2} = 0
\]

Equation (6) can be conveniently written in terms of dimensionless variables as:

\[
S(x) \frac{d^4 u(x)}{dx^4} + 2 \frac{dS(x)}{dx} \frac{d^3 u(x)}{dx^3} + \frac{d^2 S(x)}{dx^2} \frac{d^2 u(x)}{dx^2} = \Omega^2 (S(x))^{53} u(x)
\]

where, \( \frac{E I(x)}{E I_0} = S(x) = (1 - \beta x)^3 \), \( u = \frac{W}{L} \), \( x = \frac{X}{L} \), \( A(x) = S^{39}(x) \), \( S^{39}(x) = r(x) = (1 - \beta x) \), \( \Omega^2 = \frac{\omega^2 \rho A I_0}{E I_0} \).

\( \Omega \) is the non-dimensional frequency of the beam, \( r_0, A_0, I_0 \) are radius, cross sectional area and inertia at the left edge of the beam.

<table>
<thead>
<tr>
<th>( \frac{E I(x)}{E I_0(x)} = (1 - \beta X)^3 )</th>
<th>( \Omega_1 )</th>
<th>( \Omega_2 )</th>
<th>( \Omega_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta = 0 )</td>
<td>( \beta = 0.25 )</td>
<td>( \beta = 0.5 )</td>
<td>( \beta = 0 )</td>
</tr>
<tr>
<td>Rectangular [10, 16]</td>
<td>9.8696</td>
<td>8.5772</td>
<td>7.1215</td>
</tr>
<tr>
<td>Circular</td>
<td>9.8696</td>
<td>8.3847</td>
<td>6.8020</td>
</tr>
</tbody>
</table>

Table 3 – Simply supported non-uniform Euler Beams with (Rectangular and Circular) cross sections, for \( N = 33 \).

Fig. 2 – The first three mode shapes of uniform circular beam (a. Simply Supported, b. Clamped–Clamped and c. Clamped-Roller) boundary.
the beam, and $\beta$ is a constant – equals zero for uniform beam respectively.

The non-dimensional boundary conditions are,

Case a. Simply Supported Beam

$$u(0) = 0; \quad \frac{d^2u(0)}{dx^2} = 0; \quad u(1) = 0; \quad \frac{d^2u(1)}{dx^2} = 0$$  \hspace{1cm} (8)

Case b. Clamped-Clamped Beam

$$u(0) = 0; \quad \frac{du(0)}{dx} = 0; \quad u(1) = 0; \quad \frac{du(1)}{dx} = 0$$  \hspace{1cm} (9)

Case c. Clamped-Roller Beam

$$u(0) = 0; \quad \frac{du(0)}{dx} = 0; \quad u(1) = 0; \quad \frac{d^2u(1)}{dx^2} = 0$$  \hspace{1cm} (10)

3. Differential transformation method

The DTM is a technique that uses Taylor series for the solution of differential equations in the form of a polynomial. The Taylor series method is computationally tedious for high order equations. Following Ayaz (2004) we can obtain the idea of the DTM, the differential transformation of function $y(x)$ is defined as follows;

$$Y(k) = \frac{1}{k!} \left( \frac{d^k y(x)}{dx^k} \right)_{x=0}$$  \hspace{1cm} (11)

In Equation (1), $y(x)$ is the original function and $Y(k)$ is the transformed function. Differential inverse transform of $Y(k)$ is defined as follows;

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k)$$  \hspace{1cm} (12)

Fig. 3 – Mode shapes of non-uniform Simply Supported circular beam (a. first, b. second and c. third modes) for ($\beta = 0, 0.4$ and $0.8$).
from (11) and (12), we obtain
\[
y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left( \frac{d^k y(x)}{dx^k} \right)_{k=0} \tag{13}
\]

From the definitions (11) and (12), it is easy to obtain the following mathematical operations; ref. AL Kaisy et al. (2007), Moustafa (2008), Qibo (2012) and Attarnejad and Shahba (2008)

1. If \( f(x) = g(x) \pm h(x) \), then \( F(k) = G(k) \pm H(k) \).
2. If \( f(x) = cg(x) \), then \( F(k) = cG(k) \) \( c \) is a constant.
3. If \( f(x) = \frac{d^n g(x)}{dx^n} \), then \( F(k) = \frac{(k+n)}{k!} C(k+n) \).
4. If \( f(x) = g(x) h(x) \), then \( F(k) = \sum_{l=0}^{k} G(l) H(k-l) \).
5. If \( f(x) = x^n \), then \( F(k) = \delta(k-n) \) \( \delta \) is the Kronecker delta.
7. If \( f(x) = a(1-bx)^n \frac{d^n g(x)}{dx^n} \) then \( F(k) = a \sum_{r=0}^{n} C_r(k+n-r)! \) \( (k-r)! \)

\((-b)^r G(k+n-r)\), where \( a \) and \( b \) are constants.

In solving the problem, governing differential equation must be solved together with applied boundary conditions. Applying the (DTM) to the non-dimensional governing Equation (7) yield,

\[
U_{k+4} = \frac{1}{(k+1)(k+2)(k+3)(k+4)} \left[ \Omega^2 U_k - \beta^2 [(k-1)(k+1)(k+2) + 6(k+1)(k+2) + 6(k+1)(k+2)] U_{k+2} + 4\beta[k(k+1)(k+2)(k+3) + 3(k+1)(k+2)(k+3)] U_{k+3} \right] \tag{14}
\]

Then apply the DTM to the non-dimensional boundary conditions equations. The solution here will written for simply supported beam only,

DT of Equation (8) is written as

\[
u(0) = \sum_{k=0}^{\infty} U[k] x^k
\]

which leads to \( U[0]=0 \)

\[
u'[0] = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{U[k+2]} x^k
= \frac{1}{(1)(2)U[1]} x^1 + \frac{2(2)(3)U[2]}{U[3]} x^2 + \frac{3(4)U[2]}{U[3]} x^3 + \frac{4(5)U[2]}{U[5]} x^4 + \frac{5(6)U[4]}{U[5]} x^5 + \frac{6(7)U[4]}{U[5]} x^6 + \ldots = 0 \tag{15.b}
\]

which leads to \( U[2]=0 \)

Fig. 4 – Mode shapes of non-uniform Clamped–Clamped circular beam (a. first, b. second and c. third modes) for (\( \beta = 0, 0.3 \) and 0.6).
\[ u(1) = \sum_{k=0}^{\infty} U[k] x^k \]
(15.c)

which leads to \( \sum_{k=0}^{\infty} U[k] = 0 \)  
(15.d)

and taking \( U[1] = C_1 \)

\[ u'(1) = \sum_{k=0}^{\infty} (k+1)(k+2) U[k+2] x^k \]
(15.e)

which leads to \( \sum_{k=0}^{\infty} k(k+1) U[k] = 0 \)

and taking \( U[3] = D_1 \)  
(15.f)

Equations (15.c) and (15.e) may be written as

\[
\begin{bmatrix} A_{11} & B_{11} & C_1 \\ C_1 & D_{11} & D_1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} 
\]
(16)

Since, \( C_1 \) and \( D_1 \) are not zero, for a non-trivial solution to exist the determinant of the matrix must be zero, i.e.

\[ A_{11} \times D_{11} - C_{11} \times B_{11} = 0 \]  
(17)

where, \( A_{11}, B_{11} \) are the coefficients of \( C_1 \) and \( D_1 \) in the Equation (15.c) and \( C_{11}, D_{11} \) are the coefficients of \( C_1 \) and \( D_1 \) in the Equation (15.e). The root of Equation (17) is the solution for case a of the problem.

4. Cases study and discussions

Numerical results are presented in this section for uniform and non-uniform circular beam. Also, a comparison with a rectangular cross section is applied. The boundary conditions are assumed to be simple–simple, clamped-clamped and clamped-roller supports.

Table 1 compares the accuracy of the first three non-dimensional frequencies of simply supported uniform Euler beams – \( \beta \) equals zero – with circular cross section for different number of terms \( N \). The table shows that, when we use

![Fig. 5 - Mode shapes of non-uniform Clamped-Roller circular beam (a. first, b. second and c. third modes) for (\( \beta = 0, 0.5 \) and 0.75).](image-url)
the differential transformation method, we need 23 terms only to reach exact solution of the first frequency. While, the second and third frequency needs 33 and 43 terms respectively. Table 2 gives results of the first three non-dimensional frequencies of uniform circular clamped–clamped beam. It is observed that the first frequency of the clamped circular beam needs 33 terms, while, the second and the third frequency needs 39 terms to reach exact solution. From Tables 1 and 2 the reader can observe that the convergence speed increases when the frequency order decreases, i.e. Table 3 gives a comparison of simply supported non-uniform beams with rectangular and circular cross sections for ($\beta = 0, 0.25$ and $0.5$). It can be demonstrated that the first three frequencies of the simply non-uniform beam decreases with increasing $\beta$ due to decreasing the beam cross section. Also, Table 3 shows that variation of the beam cross section is effective in the case of circular cross sections.

Fig. 2 shows the mode shapes of uniform circular Euler beams i.e $\beta$ equals zero with three cases of boundary conditions. It can be seen that the first three mode shapes drawn using DTM is highly agreement with the mode shaped drawn using exact methods. Fig. 3 shows, respectively, the first, the second and the third mode shapes of non-uniform simple-simple Euler beams with different circular cross sections ($\beta = 0, 0.4$ and $0.8$). Fig. 4 shows, respectively, the first, the second and the third mode shapes of non-uniform clamped–clamped Euler beams with different circular cross sections ($\beta = 0, 0.3$ and $0.6$). Fig. 5 shows, respectively, the first, the second and the third mode shapes of non-uniform clamped-roller Euler beams with different circular cross sections ($\beta = 0, 0.5$ and $0.75$).

From Figs. (3–5), it can be seen that the value of beta $\beta$ has a significant effect on the mode shapes and the deflection values of circular beams with different boundary conditions. Deflection of non-uniform circular beams increases as cross section decreases ($\beta$ values increases).

5. Conclusion

Based on the previous results, it can be demonstrated that the differential transformation method is an efficient method to solve the vibrations problems of the non-uniform circular beams with good accuracy using a few terms. It can be seen that the variation of cross section has a significant effect on the mode shapes and the deflection values of circular beams comparing with rectangular beams. It is also possible to extend this method to the use for other cross section shapes.

REFERENCES


