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On some spaces of lacunary convergent sequences derived by Nörlund-type mean and weighted lacunary statistical convergence

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Abstract. In this paper, we define some new sequence spaces of lacunary convergent sequences derived by Nörlund-type (Riesz) mean, which shall be denoted by $|\overline{N}, p_r, \theta|$ and $(\overline{N}, p_r, \theta)$, and investigate some relations between the sequence space $|\overline{N}, p_r, \theta|$ with the spaces $|\boldsymbol{w_\theta}|$ and $|\overline{N}, p_n|$. Further, we define a new concept, named weighted lacunary statistical convergence and examine some connections between this notion with the concept of lacunary statistical convergence and weighted statistical convergence. Also, some topological properties of these new sequence spaces are investigated.

Keywords: Nörlund-type mean; Weighted lacunary statistical convergence; Sequence space; Lacunary convergent sequences; Strong summability

Mathematics Subject Classification: Primary 40C05; Secondary 40A35; 46A45; 40A05; 40F05

1. Introduction

The idea of statistical convergence was given by Zygmund [51] in 1935, in order to extend convergence of sequences. The concept was formally introduced by Fast [25] and Steinhaus [49] and later on by Schoenberg [48], and also independently by Buck [13]. Many years later, it has been discussed in the theory of Fourier analysis, ergodic theory and number theory under different names. Further, it was investigated from varied points of view, see [11,15–17,19–24,27,28,33,34,37–40,44–47]. In 1993, Fridy and Orhan [30] introduced the concept of lacunary statistical convergence which has been studied by many researchers up to now (see [14,18,29]).

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Moricz and Orhan [36] have defined the concept of statistical summability $(\overline{N}, p_n)$. Later on, Karakaya and Chishti [31] have used $(\overline{N}, p_n)$-summability to generalize the concept of statistical convergence and have called this new method weighted statistical convergence. Mursaleen et. al. [41] have altered the definition of weighted statistical convergence and have found its relation with the concept of statistical summability $(\overline{N}, p_n)$. In [1,2], Altay and Başar have defined and studied some paranormed sequence spaces of non-absolute type derived by using the Nörlund-type (Riesz) mean. In the literature, some new sequence spaces are defined by using the Nörlund-type mean or the generalized weighted mean combining with the difference operator or the generalized $B-$ difference operator (see [3–10,43]). On the other hand, Şengönül and Kayaduman [50] have defined some N örlund-type almost convergent sequence spaces, Bhardwaj and Niranjan [12] have introduced some sequence spaces using $|\overline{N}, p_n|$-summability and a modulus function.

In this paper, we are interested in a new type of Nörlund-type mean by using a lacunary sequence. For this purpose, following Fridy and Orhan [29,30] and Mursaleen et. al. [41] we define some new spaces of lacunary convergent sequences derived by Nörlund-type mean, which shall be denoted by $|\overline{N}, p_r, \theta|$ and $(\overline{N}, p_r, \theta)$, and investigate some relations between the sequence space $|\overline{N}, p_r, \theta|$ with the spaces $|w_\theta|$ and $|\overline{N}, p_n|$. Further, we define a new concept, named weighted lacunary statistical convergence and examine some connections between this notion with the concept of lacunary statistical convergence and weighted statistical convergence. Also, some topological properties of these new sequence spaces are investigated.

2. Definitions and preliminaries

Let ω be the space of all real sequences. Any vector subspace of ω is called a sequence space. We denote the set of all natural numbers by $\mathbb{N}$. Throughout the paper, we mean the "Riesz transformation" by "Nörlund-type transformation" and write $(x_k - \lambda)$ instead of $(x_k - \lambda e)$, $e = (1, 1, 1, \ldots)$ for all $k \in \mathbb{N}$.

Let (p_k) be a sequence of positive real numbers and $P_n = p_1 + p_2 + \cdots + p_n$ for $n \in \mathbb{N}$. Then the Nörlund-type transformation of $x = (x_k)$ is defined as:

$$t_n := \frac{1}{P_n} \sum_{k=1}^{n} p_k x_k. \tag{1}$$

If the transformation sequence (t_n) has a finite limit λ then the sequence x is said to be Nörlund-type convergent to λ. We denote the set of all Nörlund-type convergent sequences by $(\overline{N}, p_n)$. Let us note that if $P_n \to \infty$ as $n \to \infty$then Nörlund-type mean is a regular summability method. Throughout the paper, let $P_n \to \infty$ as $n \to \infty$ and $P_0 = p_0 = 0$. If $p_k = 1$ for all $k \in \mathbb{N}$ in (1) then Nörlund-type mean reduces to Cesáro mean of order 1.

In addition, if $\frac{1}{P_n} \sum_{k=1}^{n} p_k |x_k - \lambda| \to 0$ as $n \to \infty$, then the sequence $x = (x_k)$ is said to be strongly Nörlund-type convergent to λ and in this case, we write $|\overline{N}, p_n| - \lim x = \lambda$. The set of these sequences is denoted by

$$|\overline{N}, p_n| = \left\{ x = (x_k) : \lim_{n \to \infty} \frac{1}{P_n} \sum_{k=1}^{n} p_k |x_k - \lambda| = 0, \textit{ for some } \lambda. \right\}. \tag{2}$$

If $p_k = 1$ for all $k \in \mathbb{N}$ in (2), the space $|\overline{N}, p_n|$ is reduced to the space of sequences of strongly Cesáro summable to λ which is denoted by $|C,1|$.

Let $\theta = (k_r)$ be the sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Then θ is called a lacunary sequence. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

The space of all lacunary strongly convergent sequences is denoted by $|w_\theta|$ and defined as follows:

$$|w_\theta| = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - \lambda| = 0, \ \textit{for some } \lambda. \right\}, \tag{3}$$

see [26].

A sequence $x = (x_k)$ is said to be statistically convergent to the number λ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : |x_k - \lambda| \geqslant \varepsilon\}| = 0, \tag{4}$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S - \lim x = \lambda$. We use S to denote the set of all statistically convergent sequences.

Let θ be a lacunary sequence, the sequence $x = (x_k)$ is lacunary statistically convergent to λ provided that for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - \lambda| \geqslant \varepsilon\}| = 0. \tag{5}$$

In this case, we write $S_\theta - \lim x = \lambda$. We denote the set of all lacunary statistically convergent sequences by S_θ, [30].

The weighted density of $K \subset \mathbb{N}$ is denoted by $\delta_{\overline{N}}(K) = lim_{n\to\infty} \frac{1}{P_n} |K_{P_n}(\varepsilon)|$ if the limit exists. The sequence $x = (x_k)$ is said to be weighted statistically convergent to λ if for every $\varepsilon > 0$, the set $K_{P_n}(\varepsilon) = \{k \leqslant P_n : p_k | x_k - \lambda| \geqslant \varepsilon\}$ has weighted density zero, i.e.

$$\lim_{n\to\infty} \frac{1}{P_n} |\{k \leqslant P_n : p_k |x_k - \lambda| \geqslant \varepsilon\}| = 0. \tag{6}$$

In this case, it is written as $S_{\overline{N}} - \lim x = \lambda$. The set of all weighted statistically convergent sequences is denoted by $S_{\overline{N}}$, [41].

3. Main results

We need some new notations, which will be used throughout the paper, by combining both the definitions of lacunary sequence and Nörlund-type mean:

Let $\theta = (k_r)$ be a lacunary sequence, (p_k) be a sequence of positive real numbers such that $H_r := \sum_{k \in I_r} p_k$, $P_{k_r} := \sum_{k \in (0,k_r]} p_k$, $P_{k_{r-1}} := \sum_{k \in (0,k_{r-1}]} p_k$, $Q_r := \frac{P_{k_r}}{P_{k_{r-1}}}$, $P_0 = 0$. The intervals determined by θ and (p_k) are denoted by $I'_r = (P_{k_{r-1}}, P_{k_r}]$. It is easy to see that $H_r = P_{k_r} - P_{k_{r-1}}$. If we take $p_k = 1$ for all $k \in \mathbb{N}$, then H_r, P_{k_r}, $P_{k_{r-1}}$, Q_r and I'_r reduce to h_r, k_r, k_{r-1}, q_r and I_r, respectively.

If $\theta = (k_r)$ is a lacunary sequence and $P_n \to \infty$ as $n \to \infty$, then $\theta' = (P_{k_r})$ is a lacunary sequence, that is, $P_0 = 0,\ 0 < P_{k_{r-1}} < P_{k_r}$ and $H_r = P_{k_r} - P_{k_{r-1}} \to \infty$ as $r \to \infty$.

Now, we define some new spaces of lacunary convergent sequences derived by Nörlund-type mean as follows:

$$|\overline{N}, p_r, \theta| = \left\{ x = (x_k) \in \omega : \lim_{r\to\infty} \frac{1}{H_r} \sum_{k\in I_r} p_k |x_k - \lambda| = 0,\ \textit{for some } \lambda. \right\},$$

$$(\overline{N}, p_r, \theta) = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{H_r} \sum_{k\in I_r} p_k (x_k - \lambda) = 0,\ \textit{for some } \lambda. \right\}$$

If the sequence $x = (x_k)$ is convergent to the limit λ in $|\overline{N}, p_r, \theta|$ or in $(\overline{N}, p_r, \theta)$, in this case we write $|\overline{N}, p_r, \theta| - \lim x = \lambda$ or $(\overline{N}, p_r, \theta) - \lim x = \lambda$, respectively. The proper inclusion $|\overline{N}, p_r, \theta| \subset (\overline{N}, p_r, \theta)$ holds and $|\overline{N}, p_r, \theta| - \lim x = (\overline{N}, p_r, \theta) - \lim x = \lambda$. In order to show that this inclusion is strict, let $\theta = (k_r) = (2^r)$, $p_k = 2$ for all $k \in \mathbb{N}$ and define $x = (x_k) = (1, -1, 1, -1, \ldots)$, then we have $\frac{1}{H_r} \sum_{k\in I_r} p_k (x_k - 0) \to 0$ as $r \to \infty$ i.e. $x \in (\overline{N}, p_r, \theta)$ with $\lambda = 0$. On the other hand, $\frac{1}{H_r} \sum_{k\in I_r} p_k |x_k - 0| \to 1 \neq 0$ as $r \to \infty$. That is, $x \notin |\overline{N}, p_r, \theta|$.

The following results are obtained for some special cases:

1. If we take $p_k = 1$ for all $k \in \mathbb{N}$, then the sequence space $|\overline{N}, p_r, \theta|$ is reduced to $|w_\theta|$, where $|w_\theta|$ is given in (3).
2. Let us choose $\theta = (k_r) = (2^r)$ for $r > 0$, then $|\overline{N}, p_n|$, which is given in (2), is the special case of $|\overline{N}, p_r, \theta|$.
3. If $p_k = 1$ for all $k \in \mathbb{N}$ and $\theta = (k_r) = (2^r)$ for $r > 0$, then $|\overline{N}, p_r, \theta|$ is reduced to the sequence space $|C, 1|$.
4. If we select $(p_k) = \frac{1}{k}$ for all $k \geqslant 1$ and $\theta = (k_r) = (2^r)$ for $r > 0$, then $(\overline{N}, p_r, \theta)$ is reduced to the sequence space $(H, 1)$ where $(H, 1)$ is given in [35].

Now, we give some theorems which will help us to see the relationships between the sequence spaces $|\overline{N}, p_n|$ and $|\overline{N}, p_r, \theta|$.

Theorem 1. *Let $\theta = (k_r)$ be a lacunary sequence and liminf$_r$ $Q_r > 1$. Then $|\overline{N}, p_n| \subseteq |\overline{N}, p_r, \theta|$ and $|\overline{N}, p_n| - \lim x = |\overline{N}, p_r, \theta| - \lim x = \lambda$.*

Proof. Suppose that *liminf$_r$* $Q_r > 1$, then there exists a $\delta > 0$ such that $Q_r \geqslant 1 + \delta$ for sufficiently large values of r, which implies that $\frac{H_r}{P_{k_r}} \geqslant \frac{\delta}{1+\delta}$. If $x = (x_k) \in |\overline{N}, p_n|$ with $|\overline{N}, p_n| - \lim x = \lambda$, then for sufficiently large values of r, we have

$$\begin{aligned}
\frac{1}{P_{k_r}} \sum_{k=1}^{k_r} p_k |x_k - \lambda| &= \frac{1}{P_{k_r}} \left(\sum_{k=1}^{k_{r-1}} p_k |x_k - \lambda| + \sum_{k=k_{r-1}+1}^{k_r} p_k |x_k - \lambda| \right) \\
&\geqslant \frac{H_r}{P_{k_r}} \left(\frac{1}{H_r} \sum_{k\in I_r} p_k |x_k - \lambda| \right) \\
&\geqslant \frac{\delta}{1+\delta} \cdot \frac{1}{H_r} \sum_{k\in I_r} p_k |x_k - \lambda|.
\end{aligned}$$

Then, it follows that $x = (x_k) \in |\overline{N}, p_r, \theta|$ with $|\overline{N}, p_r, \theta|-\lim x = \lambda$ by taking the limit as $r \to \infty$. This completes the proof. □

Theorem 2. *Let $\theta = (k_r)$ be a lacunary sequence with $\limsup_r Q_r < \infty$. Then $|\overline{N}, p_r, \theta| \subseteq |\overline{N}, p_n|$ and $|\overline{N}, p_r, \theta|-\lim x = |\overline{N}, p_n|-\lim x = \lambda$.*

Proof. Let $x = (x_k) \in |\overline{N}, p_r, \theta|$ with $|\overline{N}, p_r, \theta|-\lim x = \lambda$. Then for $\varepsilon > 0$ there exists a j_0 such that for every $j > j_0$

$$L_j = \frac{1}{H_j}\sum_{k\in I_j} p_k|x_k - \lambda| < \varepsilon, \tag{7}$$

that is, we can find some positive constant M such that

$$L_j \leqslant M \text{ for all } j. \tag{8}$$

$\limsup_r Q_r$ $< \infty$ implies that there exists some positive number K such that

$$Q_r \leqslant K \text{ for all } r \geqslant 1. \tag{9}$$

Therefore for $k_{r-1} < n \leqslant k_r$, we have by (7)–(9)

$$\begin{aligned}
\frac{1}{P_n}\sum_{k=1}^{n} p_k|x_k - \lambda| &\leqslant \frac{1}{P_{k_{r-1}}}\sum_{k=1}^{k_r} p_k|x_k - \lambda| \\
&= \frac{1}{P_{k_{r-1}}}\left(\sum_{k\in I_1} p_k|x_k - \lambda| + \sum_{k\in I_2} p_k|x_k - \lambda| + \cdots + \sum_{k\in I_{j_0}} p_k|x_k - \lambda|\right. \\
&\qquad \left. + \sum_{k\in I_{j_0+1}} p_k|x_k - \lambda| + \cdots + \sum_{k\in I_r} p_k|x_k - \lambda|\right) \\
&= \frac{1}{P_{k_{r-1}}}\left(L_1H_1 + L_2H_2 + \ldots + L_{j_0}H_{j_0} + L_{j_0+1}H_{j_0+1} + \cdots + L_rH_r\right) \\
&\leqslant \frac{M}{P_{k_{r-1}}}\left(H_1 + H_2 + \cdots + H_{j_0}\right) + \frac{\varepsilon}{P_{k_{r-1}}}\left(H_{j_0+1} + H_{j_0+2} + \cdots + H_r\right) \\
&= \frac{M}{P_{k_{r-1}}}(P_{k_1} - P_{k_0} + P_{k_2} - P_{k_1} + \cdots + P_{k_{j_0}} - P_{k_{j_0-1}}) \\
&\qquad + \frac{\varepsilon}{P_{k_{r-1}}}(P_{k_{j_0+1}} - P_{k_{j_0}} + P_{k_{j_0+2}} - P_{k_{j_0+1}} + \cdots + P_{k_r} - P_{k_{r-1}}) \\
&= M\frac{P_{k_{j_0}}}{P_{k_{r-1}}} + \varepsilon\frac{P_{k_r} - P_{k_{j_0}}}{P_{k_{r-1}}} \leqslant M\frac{P_{k_{j_0}}}{P_{k_{r-1}}} + \varepsilon K.
\end{aligned}$$

Since $P_{k_{r-1}} \to \infty$ as $r \to \infty$, we get $x \in |\overline{N}, p_n|$ with $|\overline{N}, p_n|-\lim x = \lambda$. This completes the proof. □

Corollary 1. *Let $1 < \liminf_r Q_r \leqslant \limsup_r Q_r < \infty$. Then $|\overline{N}, p_r, \theta| = |\overline{N}, p_n|$ and $|\overline{N}, p_r, \theta|-\lim x = |\overline{N}, p_n|-\lim x = \lambda$.*

Proof. It follows from Theorems 2 and 1. □

In the following theorem, we give the relations between the sequence spaces $|w_\theta|$ and $|\overline{N}, p_r, \theta|$.

Theorem 3. *The following statements are true:*

1. *If* $p_k < 1$ *for all* $k \in \mathbb{N}$, *then* $|w_\theta| \subset |\overline{N}, p_r, \theta|$ *and* $|w_\theta| - \lim x = |\overline{N}, p_r, \theta| - \lim x = \lambda$.
2. *If* $p_k > 1$ *for all* $k \in \mathbb{N}$ *and* $\left(\frac{H_r}{h_r}\right)$ *be upper-bounded, then* $|\overline{N}, p_r, \theta| \subset |w_\theta|$ *and* $|\overline{N}, p_r, \theta| - \lim x = |w_\theta| - \lim x = \lambda$.

Proof

1. If $p_k < 1$ for all $k \in \mathbb{N}$, then $H_r < h_r$ for all $r \in \mathbb{N}$. So, there exists an M_1 constant such that $0 < M_1 \leqslant \frac{H_r}{h_r} < 1$ for all $r \in \mathbb{N}$. Let $x = (x_k)$ be a sequence which converges to the limit λ in $|w_\theta|$, then for an arbitrary $\varepsilon > 0$ we have
$$\frac{1}{H_r}\sum_{k \in I_r} p_k |x_k - \lambda| \leqslant \frac{1}{M_1} \cdot \frac{1}{h_r} \sum_{k \in I_r} |x_k - \lambda|.$$
Therefore, we get the result by taking the limit as $r \to \infty$.
2. Let $p_k > 1$ for all $k \in \mathbb{N}$ and $\left(\frac{H_r}{h_r}\right)$ be upper-bounded, then $H_r > h_r$ for all $r \in \mathbb{N}$ and there exists an M_2 constant such that $1 < \frac{H_r}{h_r} \leqslant M_2 < \infty$ for all $r \in \mathbb{N}$. Assume that $x = (x_k)$ converges to the limit λ in $|\overline{N}, p_r, \theta|$. So the result is obtained by taking the limit as $r \to \infty$ from the following inequality:
$$\frac{1}{h_r}\sum_{k \in I_r} |x_k - \lambda| \leqslant M_2 . \frac{1}{H_r} \sum_{k \in I_r} p_k |x_k - \lambda|. \quad \square$$

Now, we define a new concept of statistical convergence which will be called weighted lacunary statistical convergence. Moreover, we investigate some connections between this notion with the concept of lacunary statistical convergence and the concept of weighted statistical convergence.

Definition 1. The weighted lacunary density of $K \subset \mathbb{N}$ is denoted by $\delta_{(\overline{N},\theta)}(K) = \lim_{r\to\infty} \frac{1}{H_r} |K_r(\varepsilon)|$ if the limit exists. We say that the sequence $x = (x_k)$ is said to be weighted lacunary statistically convergent to λ if for every $\varepsilon > 0$, the set $K_r(\varepsilon) = \{k \in I'_r : p_k |x_k - \lambda| \geqslant \varepsilon\}$ has weighted density zero, i.e.
$$\lim_{r\to\infty} \frac{1}{H_r} |\{k \in I'_r : p_k |x_k - \lambda| \geqslant \varepsilon\}| = 0.$$
In this case, we write $S_{(\overline{N},\theta)} - \lim x = \lambda$. We denote the set of all weighted lacunary statistically convergent sequences by $S_{(\overline{N},\theta)}$.

In the above definition, if we take $p_k = 1$ for all $k \in \mathbb{N}$, then we obtain the definition of lacunary statistical convergence which can be seen in (5). In case of $\theta = (k_r) = (2^r)$ for $r > 0$, the definition of weighted statistical convergence is obtained, which is given in (6). If we choose $\theta = (k_r) = (2^r)$ for $r > 0$ and $p_k = \frac{1}{k}$ for all $k \geqslant 1$, then weighted lacunary density reduces to logarithmic density (see in [32,42]). If $p_k = 1$ for all $k \in \mathbb{N}$ and $\theta = (k_r) = (2^r)$ for $r > 0$, then the definition of usual statistical convergence is obtained, see in (4).

Now, we give some theorems for inclusion relations between weighted statistical convergence and weighted lacunary statistical convergence.

Theorem 4. *For any lacunary sequence θ, if $\liminf_r Q_r > 1$ then $S_{\overline{N}} \subset S_{(\overline{N},\theta)}$ and $S_{\overline{N}}-\lim x = S_{(\overline{N},\theta)}-\lim x = \lambda$.*

Proof. Suppose that $\liminf_r Q_r > 1$, then there exists a $\delta > 0$ such that $Q_r \geqslant 1 + \delta$ for sufficiently large values of r, which implies that $\frac{H_r}{P_{k_r}} \geqslant \frac{\delta}{1+\delta}$. If $x = (x_k) \in S_{\overline{N}}$ with $S_{\overline{N}}-\lim x = \lambda$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned}\frac{1}{P_{k_r}}|\{k \leqslant P_{k_r} : p_k|x_k - \lambda| \geqslant \varepsilon\}| &\geqslant \frac{1}{P_{k_r}}|\{P_{k_{r-1}} < k \leqslant P_{k_r} : p_k|x_k - \lambda| \geqslant \varepsilon\}| \\ &= \frac{H_r}{P_{k_r}}\left(\frac{1}{H_r}|\{P_{k_{r-1}} < k \leqslant P_{k_r} : p_k|x_k - \lambda| \geqslant \varepsilon\}|\right) \\ &\geqslant \frac{\delta}{1+\delta}\left(\frac{1}{H_r}\left|\left\{k \in I'_r : p_k|x_k - \lambda| \geqslant \varepsilon\right\}\right|\right).\end{aligned}$$

Hence, we get the result by taking the limit as $r \to \infty$. □

Theorem 5. *Let $\theta = (k_r)$ be a lacunary sequence with $\limsup_r Q_r < \infty$. Then $S_{(\overline{N},\theta)} \subset S_{\overline{N}}$ and $S_{(\overline{N},\theta)}-\lim x = S_{\overline{N}}-\lim x = \lambda$.*

Proof. If $\limsup_r Q_r < \infty$, then there is a $K > 0$ such that $Q_r \leqslant K$ for all $r \in \mathbb{N}$. Suppose that $x = (x_k) \in S_{(\overline{N},\theta)}$ with $S_{(\overline{N},\theta)}-\lim x = \lambda$ and let

$$N_r := \left|\left\{k \in I'_r : p_k|x_k - \lambda| \geqslant \varepsilon\right\}\right|. \tag{10}$$

By (10), given $\varepsilon > 0$, there is a $r_0 \in \mathbb{N}$ such that $\frac{N_r}{H_r} < \varepsilon$ for all $r > r_0$. Now, let

$$M := max\{N_r : 1 \leqslant r \leqslant r_0\} \tag{11}$$

and let n be any integer satisfying $k_{r-1} < n \leqslant k_r$, then we can write

$$\begin{aligned}\frac{1}{P_n}|\{k \leqslant P_n : p_k|x_k - \lambda| \geqslant \varepsilon\}| &\leqslant \frac{1}{P_{k_{r-1}}}|\{k \leqslant P_{k_r} : p_k|x_k - \lambda| \geqslant \varepsilon\}| \\ &= \frac{1}{P_{k_{r-1}}}(N_1 + N_2 + \cdots + N_{r_0} + N_{r_0+1} + \cdots + N_r) \\ &\leqslant \frac{M.r_0}{P_{k_{r-1}}} + \frac{1}{P_{k_{r-1}}}\varepsilon(H_{r_0+1} + \cdots + H_r) \\ &= \frac{M.r_0}{P_{k_{r-1}}} + \varepsilon\frac{(P_{k_r} - P_{k_{r_0}})}{P_{k_{r-1}}} \\ &\leqslant \frac{M.r_0}{P_{k_{r-1}}} + \varepsilon.Q_r \leqslant \frac{M.r_0}{P_{k_{r-1}}} + \varepsilon K\end{aligned}$$

which completes the proof by taking the limit as $r \to \infty$. □

Corollary 2. *Let $1 < \liminf_r Q_r \leqslant \limsup_r Q_r < \infty$. Then $S_{(\overline{N},\theta)} = S_{\overline{N}}$ and $S_{(\overline{N},\theta)}-\lim x = S_{\overline{N}}-\lim x = \lambda$.*

Proof. It follows from Theorems 4 and 5. □

In the following theorems, we find the relationship of $S_{(\overline{N},\theta)}$ with $|\overline{N}, p_r, \theta|$ and $(\overline{N}, p_r, \theta)$.

Theorem 6. *Let $\theta = (k_r)$ be a lacunary sequence. Then the inclusion $|\overline{N}, p_r, \theta| \subset S_{(\overline{N},\theta)}$ is proper and $|\overline{N}, p_r, \theta|-\lim x = S_{(\overline{N},\theta)}-\lim x = \lambda$.*

Proof. Let the sequence $x = (x_k) \in |\overline{N}, p_r, \theta|$ with $|\overline{N}, p_r, \theta|-\lim x = \lambda$ and

$$K_r(\varepsilon) := \{k \in I'_r : p_k|x_k - \lambda| \geqslant \varepsilon\}. \tag{12}$$

Then for a given $\varepsilon > 0$ we have

$$\frac{1}{H_r}\sum_{k\in I_r} p_k|x_k - \lambda| \geqslant \frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\in K_r(\varepsilon)}} p_k|x_k - \lambda| \geqslant \varepsilon.\frac{1}{H_r}|K_r(\varepsilon)|,$$

which yields the result by taking the limit as $r \to \infty$.

In order to establish that the inclusion $|\overline{N}, p_r, \theta| \subset S_{(\overline{N},\theta)}$ is proper, let $\theta = (k_r)$ be given and define $x = (x_k)$ to be $1, 2, \ldots, \sqrt{h_r}$ at the first $\sqrt{h_r}$ integers in I_r, and $x_k = 0$ otherwise. Note that x is not bounded. Let $(p_k) = 1^2, 2^2, \ldots, h_r$ for $k \in I_r$, and $p_k = 0$ otherwise. We have for every $\varepsilon > 0$,

$$\frac{1}{H_r}|\{k \in I'_r : p_k|x_k - 0| \geqslant \varepsilon\}| = \frac{\sqrt{h_r}}{H_r} = \frac{6\sqrt{h_r}}{\sqrt{h_r}(\sqrt{h_r}+1)(2\sqrt{h_r}+1)} \to 0$$

as $r \to \infty$, i.e., $x \in S_{(\overline{N},\theta)}$ with $S_{(\overline{N},\theta)}-\lim x = \lambda = 0$. On the other hand,

$$\frac{1}{H_r}\sum_{k\in I_r} p_k|x_k - 0| = \frac{1}{H_r}\left(1^3 + 2^3 + \cdots + \left(\sqrt{h_r}\right)^3\right)$$

$$= \frac{6}{\sqrt{h_r}(\sqrt{h_r}+1)(2\sqrt{h_r}+1)}.\left[\frac{\sqrt{h_r}(\sqrt{h_r}+1)}{2}\right]^2 \to \infty$$

as $r \to \infty$, hence $x \notin |\overline{N}, p_r, \theta|$ and $|\overline{N}, p_r, \theta|-\lim x \neq 0$. □

Theorem 7. *Let $p_k|x_k - \lambda| \leqslant M$ for all $k \in \mathbb{N}$. Then the inclusion $S_{(\overline{N},\theta)} \subset |\overline{N}, p_r, \theta|$ is proper and $S_{(\overline{N},\theta)}-\lim x = |\overline{N}, p_r, \theta|-\lim x = \lambda$.*

Proof. Let $x = (x_k) \in S_{(\overline{N},\theta)}$ with $S_{(\overline{N},\theta)}-\lim x = \lambda$. Since $H_r \to \infty$ as $r \to \infty$ and $p_k|x_k - \lambda| \leqslant M$ for all $k \in \mathbb{N}$. Then for a given $\varepsilon > 0$ we have

$$\frac{1}{H_r}\sum_{k\in I_r} p_k|x_k - \lambda| \frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\in K_r(\varepsilon)}} p_k|x_k - \lambda| + \frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\notin K_r(\varepsilon)}} p_k|x_k - \lambda|$$

$$\leqslant M.\frac{1}{H_r}|K_r(\varepsilon)| + \frac{h_r}{H_r}.\varepsilon$$

$$\leqslant M.\frac{1}{H_r}|K_r(\varepsilon)| + \varepsilon,$$

where $K_r(\varepsilon)$ is given in (12). Since ε is arbitrary, we get $x=(x_k)\in|\overline{N},p_r,\theta|$ with $|\overline{N},p_r,\theta|-\lim x=\lambda$.

For the converse, let $p_k=2$ for all $k\in\mathbb{N}$ and $\theta=(k_r)=(2^r-1)$ for all $r>0$. Consider the sequence $x=(x_k)=(1,0,1,0,\ldots)$, of course the inequality $p_k|x_k-\lambda|\leqslant M$ holds for all $k\in\mathbb{N}$. The sequence $x=(x_k)$ is not a weighted lacunary statistically convergent. On the other hand, $x\in|\overline{N},p_r,\theta|$ with the limit $\lambda=\frac{1}{2}$. □

Corollary 3. *Let $p_k|x_k-\lambda|\leqslant M$ for all $k\in\mathbb{N}$. Then $S_{(\overline{N},\theta)}\subset(\overline{N},p_r,\theta)$ is proper and $S_{(\overline{N},\theta)}-\lim x=(\overline{N},p_r,\theta)-\lim x=\lambda$.*

Proof. Since $|\overline{N},p_r,\theta|\subset(\overline{N},p_r,\theta)$ with $|\overline{N},p_r,\theta|-\lim x=(\overline{N},p_r,\theta)-\lim x=\lambda$, then it can be seen clearly from Theorem 7.

For the converse, let $p_k=\frac{1}{2}$ for all $k\in\mathbb{N}$ and $\theta=(k_r)=(2^r+1)$ for all $r>0$. Consider the sequence $x=(x_k)=(1,-1,1,-1,\ldots)$, then we get $p_k|x_k-\lambda|\leqslant M$ for all $k\in\mathbb{N}$, clearly. Of course this sequence is not a weighted lacunary statistically convergent. On the other hand, $x\in(\overline{N},p_r,\theta)$ with the limit $\lambda=0$. □

Theorem 8. *The following statements are true:*

1. *If $p_k\leqslant 1$ for all $k\in\mathbb{N}$, then $S_\theta\subset S_{(\overline{N},\theta)}$ and $S_\theta-\lim x=S_{(\overline{N},\theta)}-\lim x=\lambda$.*
2. *If $p_k\geqslant 1$ for all $k\in\mathbb{N}$ and $\frac{H_r}{h_r}$ is upper bounded, then $S_{(\overline{N},\theta)}\subset S_\theta$ and $S_{(\overline{N},\theta)}-\lim x=S_\theta-\lim x=\lambda$.*

Proof.

1. If $p_k\leqslant 1$ for all $k\in\mathbb{N}$, then $H_r\leqslant h_r$ for all $r\in\mathbb{N}$. So, there exist M_1 and M_2 constants such that $0<M_1\leqslant\frac{H_r}{h_r}\leqslant M_2\leqslant 1$ for all $r\in\mathbb{N}$. Let $x=(x_k)$ be a sequence which converges to the limit λ in S_θ, then for an arbitrary $\varepsilon>0$ we have
$$\begin{aligned}&\frac{1}{H_r}|\{k\in I'_r:p_k|x_k-\lambda|\geqslant\varepsilon\}|\\&=\frac{1}{H_r}|\{P_{k_{r-1}}<k\leqslant P_{k_r}:p_k|x_k-\lambda|\geqslant\varepsilon\}|\\&\leqslant\frac{1}{M_1}\cdot\frac{1}{h_r}|\{P_{k_{r-1}}\leqslant k_{r-1}<k\leqslant P_{k_r}\leqslant k_r:|x_k-\lambda|\geqslant\varepsilon\}|\\&=\frac{1}{M_1}\cdot\frac{1}{h_r}|\{k_{r-1}<k\leqslant k_r:|x_k-\lambda|\geqslant\varepsilon\}|\\&=\frac{1}{M_1}\cdot\frac{1}{h_r}|\{k\in I_r:|x_k-\lambda|\geqslant\varepsilon\}|.\end{aligned}$$
Hence, we obtain the result by taking the limit as $r\to\infty$.
2. Let $\frac{H_r}{h_r}$ be upper bounded and $p_k\geqslant 1$ for all $k\in\mathbb{N}$, then $H_r\geqslant h_r$ for all $r\in\mathbb{N}$ and there exist M_1 and M_2 constants such that $1\leqslant M_1\leqslant\frac{H_r}{h_r}\leqslant M_2<\infty$ for all $r\in\mathbb{N}$. Assume that $x=(x_k)$ converges to the limit λ in $S_{(\overline{N},\theta)}$ with $S_{(\overline{N},\theta)}-\lim x=\lambda$, then for an arbitrary $\varepsilon>0$ we have

$$\frac{1}{h_r}|\{k \in I_r : |x_k - \lambda| \geqslant \varepsilon\}|$$
$$= \frac{1}{h_r}|\{k_{r-1} < k \leqslant k_r : |x_k - \lambda| \geqslant \varepsilon\}|$$
$$\leqslant \frac{M_2}{H_r}.|\{k_{r-1} \leqslant P_{k_{r-1}} < k \leqslant k_r \leqslant P_{k_r} : p_k|x_k - \lambda| \geqslant \varepsilon\}|$$
$$= M_2.\frac{1}{H_r}|\{P_{k_{r-1}} < k \leqslant P_{k_r} : p_k|x_k - \lambda| \geqslant \varepsilon\}|$$
$$= M_2.\frac{1}{H_r}\left|\left\{k \in I_r' : p_k|x_k - \lambda| \geqslant \varepsilon\right\}\right|.$$

Hence, the result is obtained by taking the limit as $r \to \infty$. □

A paranormed space (X,g) is a topological linear space with the topology given by the paranorm g. It may be recalled that a paranorm g is a real subadditive function on X such that $g(\theta) = 0$, $g(x) = g(-x)$and scalar multiplication is continuous, i.e. $\lambda_r \to \lambda$, $g(x^r - x) \to 0$ as $r \to \infty$ implies that $g(\lambda_r x^r - \lambda\, x) \to 0$as $r \to \infty$ where λ_r, λ are scalars and (x^r), $x \in X$.

Now, we introduce a new sequence space as follows:

$$|\,\overline{N}, p_r, \theta, q| = \left\{x = (x_k) : \lim_{r\to\infty} \frac{1}{H_r}\sum_{k\in I_r} p_k|x_k - \lambda|^{q_k} = 0,\ \textit{for some } \lambda.\right\}$$

where (q_k) is a bounded sequence of strictly positive real numbers. If (q_k) is constant, then $|\,\overline{N}, p_r, \theta, q\,|$ reduces to $|\,\overline{N}, p_r, \theta\,|_q$. If we take $q_k = 1$ for all $k \in \mathbb{N}$, then we get the sequence space $|\,\overline{N}, p_r, \theta\,|$ which is defined at the beginning of this section.

Theorem 9. *Let (q_k) be a bounded sequence of strictly positive real numbers with $h' = \sup_k\ q_k < \infty$ and $M = \max(1,h')$. Then $|\,\overline{N}, p_r, \theta, q\,|$ is a complete linear topological space total paranormed by*

$$g(x) = \sup_r\left(\frac{1}{H_r}\sum_{k\in I_r} p_k|x_k|^{q_k}\right)^{\frac{1}{M}}.$$

If we take $q_k = q \geqslant 1$ for all $k \in \mathbb{N}$ then the sequence space $|\,\overline{N}, p_r, \theta\,|_q$ is a Banach space normed by

$$\|x\| = \sup_r\left(\frac{1}{H_r}\sum_{k\in I_r} p_k|x_k|^{q}\right)^{\frac{1}{q}}.$$

Proof. It is easy to see that $|\,\overline{N}, p_r, \theta, q\,|$ is a linear space with coordinate wise addition and scalar multiplication and $g(x)$ is a paranorm on $|\,\overline{N}, p_r, \theta, q\,|$, and the case $q_k = q \geqslant 1$ for all $k \in \mathbb{N}$ in which $\|x\|$ is a norm, so we omit them. To prove that $|\,\overline{N}, p_r, \theta, q\,|$ is complete, suppose that (x^i) is any Cauchy sequence in $|\,\overline{N}, p_r, \theta, q\,|$ where $(x^i) = \left(x_1^i, x_2^i, \ldots\right)$. Then for a given $\varepsilon > 0$ there exists a positive integer $i_0(\varepsilon)$ such that $g(x^i - x^j) < \varepsilon$ for every $i, j > i_0(\varepsilon)$. By using the definition of g we obtain

$$\left(\frac{1}{H_r}\sum_{k\in I_r} p_k\left|x_k^i - x_k^j\right|^{q_k}\right)^{\frac{1}{M}} \leqslant g(x^i - x^j) < \varepsilon, \tag{13}$$

for each fixed r and for every $i, j > i_0(\varepsilon)$. From (13) we have

$$\frac{1}{H_r}\sum_{k\in I_r} p_k|x_k^i - x_k^j|^{q_k} \leqslant g(x^i - x^j)^M < \varepsilon^M \tag{14}$$

for each r and for every i, $j > i_0(\varepsilon)$. Since $H_r \to \infty$ as $r \to \infty$, it follows that $|x_k^i - x_k^j| \to 0$ as $i, j \to \infty$ for each k. Hence (x^i) is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is a Banach space, there exists $x \in \mathbb{C}$ such that $x^i \to x$ as $i \to \infty$. It follows from (14) that given $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that $\frac{1}{H_r}\sum_{k\in I_r} p_k|x_k^i - x_k^j|^{q_k} < \varepsilon^M$ for every i, $j > i_0(\varepsilon)$ and for each r. Let us pass to the limit first as $j \to \infty$, so we have

$$\left(\frac{1}{H_r}\sum_{k\in I_r} p_k|x_k^i - x_k|^{q_k}\right)^{\frac{1}{M}} < \varepsilon \tag{15}$$

for every $i > i_0(\varepsilon)$ and for each r. Next take the supremum with respect to r in (15) to obtain $g(x^i - x) < \varepsilon$ for every $i > i_0(\varepsilon)$. Since $g(x^i - x) < \varepsilon$ for every $i > i_0(\varepsilon)$, then $x^i \to x$ as $i \to \infty$. It follows that $(x^i - x) \in |\overline{N}, p_r, \theta, q|$. Since $(x^i) \in |\overline{N}, p_r, \theta, q|$ and $|\overline{N}, p_r, \theta, q|$ is a linear space, so we have $x = x^i - (x^i - x) \in |\overline{N}, p_r, \theta, q|$ and this concludes the proof. □

Now, we give the following theorems to demonstrate the relations between the sequence spaces $|\overline{N}, p_r, \theta|_q$ and $S_{(\overline{N},\theta)}$.

Theorem 10. *If the following conditions hold, then* $|\overline{N}, p_r, \theta|_q \subset S_{(\overline{N},\theta)}$ *and* $|\overline{N}, p_r, \theta|_q - \lim x = S_{(\overline{N},\theta)} - \lim x = \lambda$.

1. $0 < q < 1$ and $0 \leqslant |x_k - \lambda| < 1$.
2. $1 \leqslant q < \infty$ and $1 \leqslant |x_k - \lambda| < \infty$.

Proof. Let a sequence $x = (x_k)$ be $|\overline{N}, p_r, \theta|_q$–convergent to the limit λ. Since $p_k|x_k - \lambda|^q \geqslant p_k|x_k - \lambda|$ for case (1) and (2), then we have

$$\begin{aligned}\frac{1}{H_r}\sum_{k\in I_r} p_k|x_k - \lambda|^q &\geqslant \frac{1}{H_r}\sum_{k\in I_r} p_k|x_k - \lambda| \\ &\geqslant \frac{1}{H_r}\sum_{\substack{k\in I_r \\ k\in K_r(\varepsilon)}} p_k|x_k - \lambda| \\ &\geqslant \varepsilon.\frac{1}{H_r}|K_r(\varepsilon)|.\end{aligned}$$

We get the result if we take the limit as $r \to \infty$. That is, $\lim_{r\to\infty}\frac{1}{H_r}|K_r(\varepsilon)| = 0$ where $K_r(\varepsilon)$ is given in (12). Hence $x = (x_k)$ is convergent to λ in $S_{(\overline{N},\theta)}$. This completes the proof. □

Theorem 11. *Let* $p_k|x_k - \lambda| \leqslant M$ *for all* $k \in \mathbb{N}$. *If the following conditions hold, then* $S_{(\overline{N},\theta)} \subset |\overline{N}, p_r, \theta|_q$ *and* $S_{(\overline{N},\theta)} - \lim x = |R, p_r, \theta|_q - \lim x = \lambda$.

1. $0 < q < 1$ and $1 \leqslant |x_k - \lambda| < \infty$.
2. $1 \leqslant q < \infty$ and $0 \leqslant |x_k - \lambda| < 1$.

Proof. Assume that $x = (x_k)$ is convergent to the λ in $S_{(\overline{N},\theta)}$. Then for $\varepsilon > 0$, we have $\delta(K_r(\varepsilon)) = 0$ where $K_r(\varepsilon)$ is given in (12). Since $p_k|x_k - \lambda| \leqslant M$ for all $k \in \mathbb{N}$, then we have

$$\frac{1}{H_r}\sum_{k\in I_r} p_k|x_k - \lambda|^q \leqslant \frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\notin K_r(\varepsilon)}} p_k|x_k - \lambda|^q + \frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\in K_r(\varepsilon)}} p_k|x_k - \lambda|^q = T_r + T'_r$$

where $T_r = \frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\notin K_r(\varepsilon)}} p_k|x_k - \lambda|^q$ and $T'_r = \frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\notin K_r(\varepsilon)}} p_k|x_k - \lambda|^q$.

For $k \notin K_r(\varepsilon)$, we have

$$T_r = \frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\notin K_r(\varepsilon)}} p_k|x_k - \lambda|^q \leqslant \frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\notin K_r(\varepsilon)}} p_k|x_k - \lambda| \leqslant \frac{h_r}{H_r}\varepsilon \leqslant \varepsilon.$$

If $k \in K_r(\varepsilon)$, then

$$T'_r = \frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\notin K_r(\varepsilon)}} p_k|x_k - \lambda|^q \leqslant \frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\notin K_r(\varepsilon)}} p_k|x_k - \lambda| \leqslant \frac{M}{H_r}|K_r(\varepsilon)|.$$

If we take the limit as $r \to \infty$, since $\delta\,(K_r(\varepsilon)) = 0$ then $x = (x_k)$ is convergent to λ in $|\overline{N}, p_r, \theta|_q$. This completes the proof. ☐

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