INDEPENDENT DOMINATION IN HYPERCUBES

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Abstract—The use of hypercube graphs as the underlying architecture in many commercial parallel computers has stimulated interest in this family of graphs. We hope to further stimulate this interest by introducing a tantalizing unsolved problem that is based on dominating sets for this very regularly structured family.

Given a graph \( G = (V, E) \) and a subset \( S \subseteq V \), we say that \( S \) dominates its neighborhood \( N(S) \), the set of all nodes not in \( S \) which are adjacent to some node in \( S \). If \( S \) dominates \( V - S \), i.e., if \( N(S) = V - S \), then we call \( S \) a dominating set of \( G \). The domination number \( \alpha = \alpha(G) \) is the minimum cardinality of the dominating sets of \( G \). The independent domination number \( \alpha' = \alpha'(G) \) is the minimum cardinality of the independent dominating sets for \( G \) (\( S \) is independent if no two of its nodes are adjacent). We are interested in the numbers \( \alpha \) and \( \alpha' \) for the hypercube graphs and we found that the known values could be compiled from the literature of graph theory and coding theory. We refer to the book [1] for notation and terminology of graph theory.

Let \( Q_n = (V_n, E_n) \) denote the hypercube graph of dimension \( n \), and let \( \alpha_n = \alpha(Q_n) \) and \( \alpha'_n = \alpha'(Q_n) \). Although many alternative definitions of the hypercube graph may be offered [2], perhaps the easiest to describe is this: \( V_n \), the nodes of \( Q_n \), is the set of all binary \( n \)-tuples of zeros and ones; \( E_n \), the edges of \( Q_n \), is the set of pairs of nodes \( u = (u_1, u_2, \ldots, u_n) \) and \( v = (v_1, v_2, \ldots, v_n) \), where \( \sum_{i=1}^{n} |u_i - v_i| = 1 \). That is, two nodes of \( Q_n \) are adjacent if and only if their binary \( n \)-tuples differ in exactly one place. Figure 1 shows hypercubes of the first four dimensions.

We obviously have the following:

(a) \( \alpha_1 = \alpha'_1 = 1 \); select either node for singleton \( S_1 \).

(b) \( \alpha_2 = \alpha'_2 = 2 \); pick either pair of opposite nodes for \( S \), say \( S_2 = \{00, 11\} \).

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(c) $\alpha_3 - \alpha'_3 = 2$; choose any one of the four pairs of antipodal nodes for $S$, say $S_3 = \{000, 111\}$.

(d) $\alpha_4 = \alpha'_4 = 4$; choose two node-disjoint copies of $Q_3$ in $Q_4$ and select independent dominating sets for each of them so that the four nodes are also independent, say $S_4 = \{0000, 0111, 1011, 1100\}$.

Stanton and Kalbfleisch [3] proved that $Q_3$ has a unique (up to automorphism) minimum dominating set $S$ of seven nodes, which, in our notation, takes the form

$$S_3 = \{0000, 0100, 1100, 1011, 1010, 1111, 1101\}.$$

Since $S_3$ is not independent (as its first two nodes are adjacent), we see that $\alpha'_3 \geq 8$. It is easy to verify that the set

$$S'_3 = \{0000, 0011, 0111, 1011, 1001, 1001, 1111\}$$

is an independent dominating set for $Q_3$ and thus $\alpha'_3 = 8$.

In [3], it was also proved that any minimum dominating set for $Q_3$ has size 12, and although such a set was exhibited, it was not independent. A minimum independent dominating set for $Q_3$ consisting of 12 nodes was given by Stanton and Kalbfleisch in [4] and is shown below using our notation:

$$S'_3 = \{001000, 000100, 110000, 010010, 010001, 100011, 011100, 101110, 101101, 001111, 111011, 110111\}.$$

Results from coding theory can be applied to give information on $\alpha_n$ and $\alpha'_n$ for infinitely many values of $n$. For instance, the existence of perfect single-error-correcting codes [5, 6] shows that, for $n = 2^k - 1$,

$$\alpha_n - \alpha'_n = 2^{n-k}.$$

In addition, the results of van Wee [7] show that (1) holds also for $n = 2^k$. We summarize these results in Table 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>$2^k - 1$</th>
<th>$2^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>$2^{n-k}$</td>
<td>$2^{n-k}$</td>
</tr>
<tr>
<td>$\alpha'_n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>$2^{n-k}$</td>
<td>$2^{n-k}$</td>
</tr>
</tbody>
</table>

Although there are infinitely many values of $n$ for which $\alpha_n$ and $\alpha'_n$ are equal, we have found only one value, namely $n = 5$, for which they differ. We wonder whether

$$\alpha_n = \alpha'_n \quad \text{for } n \neq 5.$$

We also wonder whether the difference $\alpha_n - \alpha'_n$ is bounded. Barefoot, Harary and Jones [8] have investigated this difference for the class of cubic graphs.

The values of $\alpha$ and $\alpha'$ for other families of graphs constitutes an interesting topic for further study.

**References**